Condition spectrum and Almost multiplicative function

Dr. D. Sukumar

Department of Mathematics
Indian Institute of Technology Hyderabad

Department of Mathematics
Texas A&M University
Almost multiplicative functions

Comparative results with spectrum and pseudo-spectrum
Condition spectrum
Let $A$ be a complex commutative Banach algebra with unit.

$$\sigma(a) := \{ \lambda \in \mathbb{C} : \lambda - a \in Sing(A) \}$$
**Condition spectrum**

Let $A$ be a complex commutative Banach algebra with unit.

$$\sigma(a) := \{ \lambda \in \mathbb{C} : \lambda - a \in \text{Sing}(A) \}$$

**Definition (\(\varepsilon\)-Condition spectrum (0 < \(\varepsilon\) < 1))**

$$\sigma_\varepsilon(a) := \left\{ \lambda \in \mathbb{C} : \|\lambda - a\| \|(\lambda - a)^{-1}\| \geq \frac{1}{\varepsilon} \right\}$$
Condition spectrum

Let $A$ be a complex commutative Banach algebra with unit.

$$\sigma(a) := \{ \lambda \in \mathbb{C} : \lambda - a \in \text{Sing}(A) \}$$

**Definition ($\varepsilon$- Condition spectrum ($0 < \varepsilon < 1$))**

$$\sigma_\varepsilon(a) := \left\{ \lambda \in \mathbb{C} : \| \lambda - a \| \| (\lambda - a)^{-1} \| \geq \frac{1}{\varepsilon} \right\}$$

1. $\sigma_\varepsilon(a)$ is non-empty, compact subset of $\mathbb{C}$.
2. $\sigma(a) \subseteq \sigma_\varepsilon(a)$, for every $a \in A$ and for every $\varepsilon > 0$. The two spectrum coincides if and only if $a$ is a scalar multiple of identity.
3. If $\lambda \in \sigma_\varepsilon(a)$ then $|\lambda| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \|a\|$. 
Almost multiplicative function

A linear map $\phi : A \rightarrow \mathbb{C}$ is said to be **multiplicative** if

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all} \quad a, b \in A.$$ 

**Definition (almost multiplicative function)**

A linear map $\phi : A \rightarrow \mathbb{C}$ is said to be **amf** if there exists a $\delta > 0$ such that

$$|\phi(ab) - \phi(a)\phi(b)| \leq \delta \|a\| \|b\| \quad \text{for all} \quad a, b \in A.$$ 

- From perturbation theory.
- AMNM Algebra
Theorem

Let $A$ be complex commutative Banach algebra with unit. If $\phi : A \to \mathbb{C}$ be a $\delta$-amf and $\phi(1) = 1$, then

$$\phi(a) \in \sigma_\delta(a), \quad \forall a \in A.$$ 

• amf are continuous.
Theorem

Let $A$ be a complex commutative unital Banach algebra and $\phi : A \to \mathbb{C}$ be a linear function. If, for every $a$ in $A$

$$\phi(a) \in \sigma_{\varepsilon}(a),$$

then $\phi$ is $\delta$-amf, where

$$\delta = \frac{4}{\ln(1/\varepsilon)} \left( 1 + \frac{2}{(\ln(2)/3)^2} \right) \text{ with } 0 < \varepsilon < 1/3.$$
Theorem

Let $A$ be a complex commutative unital Banach algebra and $\phi : A \to \mathbb{C}$ be a linear function. If, for every $a$ in $A$

$$\phi(a) \in \sigma_\varepsilon(a),$$

then $\phi$ is $\delta$-amf, where

$$\delta = \frac{4}{\ln(1/\varepsilon)} \left(1 + \frac{2}{(\ln(2)/3)^2}\right) \quad \text{with} \quad 0 < \varepsilon < 1/3.$$

Theorem (GKŻ Theorem)

Let $A$ be complex unital Banach algebra and $\phi : A \to \mathbb{C}$ be a linear map with $\phi(1) = 1$. If, for every $a \in A$,

$$\phi(a) \in \sigma(a)$$

then $\phi$ is multiplicative.
1. If $\phi$ is $\delta$-amf, then $\phi(a) \in \sigma_\delta(a)$ for all $a$ in $A$.

2. If $\phi$ is linear and

$$\phi(a) \in \sigma_\varepsilon(a) \quad \forall a \in A,$$

then $\phi$ is $\delta$-amf for some $\delta(\varepsilon)$. 
Definition (Spectrum)

The spectrum of $A$ is defined as

$$\sigma(A) := \{z \in \mathbb{C} : z - A \notin \text{inv}(A)\}$$

Definition (Pseudospectrum)

Let $\varepsilon > 0$. The $\varepsilon$-pseudospectrum of a matrix $A$ is defined as

$$\Lambda_\varepsilon(A) := \{z \in \mathbb{C} : z - A \notin \text{inv}(A) \text{ or } \|(z - A)^{-1}\| \geq \varepsilon\}.$$

Definition (Condition spectrum)

Let $0 < \varepsilon < 1$. The $\varepsilon$-condition spectrum of a $A$ is defined as

$$\sigma_\varepsilon(A) := \left\{ z \in \mathbb{C} : z - A \notin \text{inv}(A) \text{ or } \|(z - A)^{-1}\| \|z - A\| \geq \frac{1}{\varepsilon} \right\}.$$
Consider solving the system of equations

\[ Ax - \lambda x = b. \]

- Spectrum \( \sigma(A) \leftrightarrow \) uniqueness of the solution.
- Pseudospectrum of \( \Lambda_\varepsilon(A) \leftrightarrow \) computational aspects of the solution.
- Condition spectrum \( \sigma_\varepsilon(A) \leftrightarrow \) computational stability aspect of deriving the solution.
Lemma

1. For every $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ is non empty.
2. For every $0 < \varepsilon < 1$, $\sigma(A) \subseteq \sigma_\varepsilon(A)$.
3. For every $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ is compact.
4. For every $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ has no isolated points.
Lemma

1. For every $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ is non empty.
2. For every $0 < \varepsilon < 1$, $\sigma(A) \subseteq \sigma_\varepsilon(A)$.
3. For every $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ is compact.
4. For every $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ has no isolated points.

Theorem

The following sets are equivalent.

1. $\sigma_\varepsilon(A) = \{z \in \mathbb{C} : \|(z - A)^{-1}\| \|z - A\| \geq \varepsilon^{-1}\}$
2. $\{z \in \mathbb{C} : \exists u \in \mathbb{C}^n, \|u\| = 1 \text{ with } \|(z - A)u\| \leq \varepsilon \|(z - A)\|\}$
3. $\{z \in \mathbb{C} : z \in \sigma(A + E) \text{ for some } E \text{ with } \|E\| \leq \varepsilon \|(z - A)\|\}$
Singularity

**Theorem**

\[ A \text{ is singular} \iff 0 \in \sigma(A) \]

\[ \|A^{-1}\| \geq \varepsilon^{-1} \iff 0 \in \Lambda_{\varepsilon}(A) \]

\[ \|A\| \|A^{-1}\| \geq \varepsilon^{-1} \iff 0 \in \sigma_{\varepsilon}(A) \]
Almost multiplicative funtions

Comparative results with spectrum and pseudo-spectrum

Bounds for the spectral radius

**Theorem**

\[\lambda \in \sigma(A) \Rightarrow |\lambda| \leq \|A\|\]

\[\lambda \in \Lambda_\varepsilon(A) \Rightarrow |\lambda| \leq \|A\| + \varepsilon\]

\[\lambda \in \sigma_\varepsilon(A) \Rightarrow |\lambda| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \|A\|\]
Digonalization

Theorem

\( A \) has \( N \) distinct eigenvalues \( \Rightarrow \) \( A \) is diagonalizable.

\( \Lambda_\epsilon(A) \) has \( N \) distinct components \( \Rightarrow \) \( A \) is diagonalizable.

\( \sigma_\epsilon(A) \) has \( N \) distinct components \( \Rightarrow \) \( A \) is diagonalizable.
### Lower bound of the inverse

**Theorem**

\[
\| (z - A)^{-1} \| \geq \frac{1}{d(z, \sigma(A))} \\
\| (z - A)^{-1} \| \geq \frac{1}{d(z, \Lambda_\epsilon(A)) + \epsilon} \\
\| (z - A)^{-1} \| \geq \frac{1}{d(z, \sigma_\epsilon(A)) + \frac{2\epsilon}{1 - \epsilon} \|A\|}
\]
Preserving similarity

Theorem

\[ A = SBS^{-1} \Rightarrow \sigma(A) = \sigma(B) \]

\[ A = SBS^{-1} \Rightarrow \Lambda_\varepsilon(A) \subseteq \Lambda_{\kappa(S)\varepsilon}(B) \]

\[ A = SBS^{-1} \Rightarrow \sigma_\varepsilon(A) \subseteq \sigma_{\kappa(S)^2\varepsilon}(B) \]

Similarity transformation through orthogonal and unitary matrices preserves the condition spectrum.
Gerschgorin’s theorem

Theorem

\[ \sigma(A) \subseteq \bigcup_{j=1}^{N} D(d_j, r_j) \]

\[ \Lambda_\varepsilon(A) \subseteq \bigcup_{j=1}^{N} D(d_j, r_j + \sqrt{N\varepsilon}) \]

\[ \sigma_\varepsilon(A) \subseteq \bigcup_{j=1}^{N} D \left( d_j, r_j + \sqrt{N} \frac{2\varepsilon}{1 - \varepsilon} \|A\| \right) \]
Connection with numerical range

Theorem

\[ W(A) \supseteq \text{conv}(\sigma(A)) \]

\[ W(A) \supseteq \text{conv}(\Lambda_\varepsilon(A)) \setminus \varepsilon - \text{border} \]

\[ W(A) \supseteq \text{conv}(\sigma_\varepsilon(A)) \setminus \varepsilon_1 - \text{border}, \text{ here } \varepsilon_1 = \frac{2\varepsilon}{1 - \varepsilon} \|A\| \]

The notion \( S \setminus \varepsilon\)-border means the set of points \( z \in \mathbb{C} \) such that \( D(z, \varepsilon) \subseteq S \)
Almost multiplicative functions

Comparative results with spectrum and pseudo-spectrum

Change under linear transformation

**Theorem**

For all \( \alpha, \beta \in \mathbb{C} \)

\[
\sigma(\alpha + \beta A) = \alpha + \beta \sigma(A)
\]

\[
\Lambda_{\varepsilon|\beta|}(\alpha + \beta A) = \alpha + \beta \Lambda_{\varepsilon}(A)
\]

\[
\sigma_{\varepsilon}(\alpha + \beta A) = \alpha + \beta \sigma_{\varepsilon}(A)
\]
T. J. Ransford.
Generalised spectra and analytic multivalued functions.

B. E. Johnson.
Approximately multiplicative functionals.

Krzysztof Jarosz.
Almost multiplicative functionals.

Stuart J. Sidney.
Are all uniform algebras AMNM?

Mark Embree and Lloyd N. Trefethen.
Generalizing eigenvalue theorems to pseudospectra theorems.


Thank you
• The class of Banach algebras with this property

\[ \forall a \in \text{Inv}(A), \exists b \in \text{Sing}(A) \text{ such that } \|a - b\| = \frac{1}{\|a^{-1}\|}. \]

• Does condition spectrum determine the norm behaviour of the matrices?