Almost multiplicative functions on a class of Banach algebras

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Almost multiplicative function

A linear map $\phi : A \to \mathbb{C}$ is said to be multiplicative if

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all} \quad a, b \in A.$$

Definition (almost multiplicative function)

A linear map $\phi : A \to \mathbb{C}$ is said to be amf if there exists a $\delta > 0$ such that

$$|\phi(ab) - \phi(a)\phi(b)| \leq \delta \, \|a\| \, \|b\| \quad \text{for all} \quad a, b \in A.$$

- Originated from perturbation theory.
- amf are continuous.
- AMNM Algebra

Condition spectrum

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda - a \in Sing(A)\},$$

Definition (ϵ- Condition spectrum ($0 < \epsilon < 1$))

$$\sigma_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \|\lambda - a\| \left\|(\lambda - a)^{-1}\right\| \geq \frac{1}{\epsilon}\right\}$$

1. $\sigma(a) \subseteq \sigma_\epsilon(a)$, for every $a \in A$ and for every $\epsilon > 0$. The two spectrum coincides if and only if a is a scalar multiple of identity.
2. If $\lambda \in \sigma_\epsilon(a)$ then $|\lambda| \leq \frac{1+\epsilon}{1-\epsilon}\|a\|$.
3. If $\lambda \in \sigma_\epsilon(a)$ then there exists a $b \in Sing(A)$ such that

$$\|b\| \leq \epsilon \|\lambda - a\|, \quad \lambda \in \sigma(a+b).$$

Theorem

Let A be complex commutative Banach algebra with unit 1 and let ϕ be a δ-amf on A and $\phi(1) = 1$. Then

$$\phi(a) \in \sigma_\delta(a) \quad \forall a \in A.$$

Assumption: The class of complex commutative Banach algebras with this property

$$(*) \qquad \forall a \in Inv(A), \exists b \in Sing(A) \text{ such that } \|a-b\| = \frac{1}{\|a^{-1}\|}.$$

Example: Function algebras

Lemma

Let A be a complex commutative Banach algebra satisfying $(*)$ *and let* $\lambda \in \sigma_\epsilon(a)$. *Then,*

$$d\Big(\lambda, \sigma(a)\Big) \leq \frac{2\epsilon}{1-\epsilon}\,\|a\|\,.$$

Theorem

Let A be a complex commutative unital Banach algebra with the property given in (∗). Let $a \in A$ *and* $\lambda \in \sigma_\epsilon(a)$*. Then, there exists an almost* δ*-amf* ψ *such that* $\psi(1) = 1$ *and* $\lambda = \psi(a)$*, where*

$$\delta = \alpha(3+\alpha), \quad \alpha = \frac{2\epsilon^2 \|a\|}{(1-\epsilon)m}, \quad m = \inf\{\|z-a\| : z \in \mathbb{C}\}.$$

Theorem

Let A be a function algebra and $\phi : A \to \mathbb{C}$ be a linear function. If $\phi(a) \in \sigma_\epsilon(a)$ for every a in A. Then ϕ is δ-amf, where

$$\delta = \log\left(\kappa^{-1}\right)^{-1} 2(2\kappa + 1) \quad \textit{with} \quad \kappa = \frac{2\epsilon}{1-\epsilon}.$$

Theorem (GKŻ Theorem)

Let A be complex Banach algebra and $\phi : A \to \mathbb{C}$ be a linear map with $\phi(1) = 1$. If, for every $a \in A$,

$$\phi(a) \in \sigma(a)$$

then ϕ is multiplicative.

Conclusion

1. If ϕ is δ-amf, then $\phi(a) \in \sigma_\delta(a)$ for all a in A.
2. If $\lambda \in \sigma_\epsilon(a)$, then $\lambda = \phi(a)$ for some $\delta(\epsilon)$-amf ϕ.
3. If ϕ is linear and
$$\phi(a) \in \sigma_\epsilon(a) \quad \forall a \in A,$$
then ϕ is δ-amf for some $\delta(\epsilon)$.

References

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