Eigenvalues, eigenvectors and applications

Dr. D. Sukumar

Department of Mathematics
Indian Institute of Technology Hyderabad

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Department of Applied Science
Government Engineering College, Kozhikode, Kerala
Maps which preserve

- Origin
- lines passing through origin
- parallelograms with one corner as origin
Outline

1. Linear transformations on plane
   - Typical Examples
   - Properties

2. Eigen values
   - Eigen value and eigen vector

3. Markov Matrices
   - Formation
   - Interpretation
   - Properties
Rotation \[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]
Rotation \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)
Reflection \[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]
Reflection \[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]
Linear transformations on plane
Eigen values
Markov Matrices

Typical Examples
Properties

Expansion \( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \)  
Compression \( \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \)
Linear transformations on plane

Eigenvalues

Markov Matrices

Typical Examples

Properties

Expansion \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}  

Compression \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}

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Eigenvalues
Linear transformations on plane

Eigenvalues

Markov Matrices

Typical Examples

Properties

Expansion \( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \) Compression \( \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \)
Multi-scaling or Stretching \( \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \)
Multi-scaling or Stretching \[
\begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}
\]
Linear transformations on plane
Eigen values
Markov Matrices

Projection \[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]
Linear transformations on plane
Eigen values
Markov Matrices

Typical Examples
Properties

Projection \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \)

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Eigenvalues
Shear transformation \[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]
Shear transformation \[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\]
Outline

1. Linear transformations on plane
   - Typical Examples
   - Properties

2. Eigen values
   - Eigen value and eigen vector

3. Markov Matrices
   - Formation
   - Interpretation
   - Properties
Properties

- Area
Properties

- Area
- Eigen vectors
Properties

- Area
- Eigen vectors
- Eigen values
Properties

- Area
- Eigen vectors
- Eigen values
- Determinant
Properties

- Area
- Eigen vectors
- Eigen values
- Determinant
- Diagonalizable
## Table of properties

<table>
<thead>
<tr>
<th>Map</th>
<th>Area</th>
<th>Fixed Dir</th>
<th>Scale in FD Eigenvector</th>
<th>Det</th>
<th>Diagonable</th>
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<tbody>
<tr>
<td>Rotation</td>
<td>1</td>
<td>NO</td>
<td>NO</td>
<td>1</td>
<td>NO</td>
</tr>
<tr>
<td>Reflection</td>
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<td>x-axis, y-axis</td>
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<td>-1</td>
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<td>Expansion</td>
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<td>Compression</td>
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<td>x-axis, y-axis</td>
<td>1/2,1/2</td>
<td>1/4</td>
<td>Yes</td>
</tr>
<tr>
<td>Multi-scaling</td>
<td>6</td>
<td>x-axis, y-axis</td>
<td>2,3</td>
<td>6</td>
<td>Yes</td>
</tr>
<tr>
<td>Projection</td>
<td>0</td>
<td>x-axis, y-axis</td>
<td>1,0</td>
<td>0</td>
<td>Yes</td>
</tr>
<tr>
<td>Shear</td>
<td>1</td>
<td>x-axis</td>
<td>1</td>
<td>1</td>
<td>NO</td>
</tr>
</tbody>
</table>

**Table:** Properties

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Problem

Big Problem

• Getting a common opinion from individual opinion

Purpose
Big Problem

- Getting a common opinion from individual opinion
- From individual preference to common preference

Purpose
Problem

Big Problem
- Getting a common opinion from individual opinion
- From individual preference to common preference

Purpose
- Showing all steps of this process using linear algebra
Problem

Big Problem

- Getting a common opinion from individual opinion
- From individual preference to common preference

Purpose

- Showing all steps of this process using linear algebra
- Mainly using eigenvalues and eigenvectors
Outline

1. Linear transformations on plane
   - Typical Examples
   - Properties

2. Eigen values
   - Eigen value and eigen vector

3. Markov Matrices
   - Formation
   - Interpretation
   - Properties
Let $A$ be a square matrix. 

- **Eigen values** of $A$ are solutions or roots of 

$$ \text{det}(A - \lambda I) = 0. $$
Let \( A \) be a square matrix.

- **Eigenvalues** of \( A \) are solutions or roots of
  \[
  \det(A - \lambda I) = 0.
  \]

- If
  \[
  Ax = \lambda x \quad \text{or} \quad (A - \lambda I)x = 0,
  \]
  for a non-zero vector \( x \) then
Let $A$ be a square matrix.

- **Eigen values** of $A$ are solutions or roots of

  $$\det(A - \lambda I) = 0.$$ 

- If

  $$Ax = \lambda x \quad \text{or} \quad (A - \lambda I)x = 0,$$

  for a non-zero vector $x$ then

  - $\lambda$ is an eigenvalue of $A$ and
Let $A$ be a square matrix.

- **Eigen values** of $A$ are solutions or roots of

$$\det(A - \lambda I) = 0.$$

- If

$$Ax = \lambda x \quad \text{or} \quad (A - \lambda I)x = 0,$$

for a non-zero vector $x$ then

- $\lambda$ is an eigenvalue of $A$ and
- $x$ is an eigenvector corresponding to the eigenvalue $\lambda$. 
Example

Consider the matrix \( A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}. \)

Example (Eigen value)

\[
A - \lambda I = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 - \lambda & 5 \\ 3 & -\lambda \end{bmatrix}
\]

\[
\det(A - \lambda I) = (2 - \lambda)(-\lambda) - (3 \times 5) = \lambda^2 - 2\lambda - 15
\]

The roots of the polynomial are the eigen values: -3 and 5.

Example (When \( \lambda = -3 \))

\[
\begin{bmatrix} 2 - (-3) & 5 \\ 3 & -(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Eigen vector \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \)

Example (When \( \lambda = 5 \))

\[
\begin{bmatrix} 2 - 5 & 5 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Eigen vector \( \begin{bmatrix} 5 \\ 3 \end{bmatrix} \)
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\[
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\[
\det(A - \lambda I) = (2 - \lambda)(-\lambda) - (3 \times 5) = \lambda^2 - 2\lambda - 15
\]

The roots of the polynomial are the eigen values: -3 and 5.
Example

Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$.

Example (Eigen value)

$A - \lambda I = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 - \lambda & 5 \\ 3 & -\lambda \end{bmatrix}$

det($A - \lambda I$) = $(2 - \lambda)(-\lambda) - (3 \times 5) = \lambda^2 - 2\lambda - 15$

The roots of the polynomial are the eigen values: -3 and 5.

Example (When $\lambda = -3$)

$\begin{bmatrix} 2 - (-3) & 5 \\ 3 & -(3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigen vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
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Consider the matrix \( A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \).

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Eigen vector \( \begin{pmatrix} 5 \\ 3 \end{pmatrix} \)
For the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$

- the eigenvalues are -3 and 5
For the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$

- the eigenvalues are -3 and 5
- the eigenvector corresponding to -3 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
For the matrix \( A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \)

- the eigenvalues are -3 and 5
- the eigenvector corresponding to -3 is \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \)
  
\[
\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

- the eigenvector corresponding to 5 is \( \begin{bmatrix} 5 \\ 3 \end{bmatrix} \)
  
\[
\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} =
\]
For the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$

- the eigenvalues are -3 and 5
- the eigenvector corresponding to -3 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- the eigenvector corresponding to 5 is $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} =$$
For the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$

- the eigenvalues are -3 and 5
- the eigenvector corresponding to -3 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- the eigenvector corresponding to 5 is $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$
Outline

1. Linear transformations on plane
   - Typical Examples
   - Properties

2. Eigen values
   - Eigen value and eigen vector

3. Markov Matrices
   - Formation
   - Interpretation
   - Properties

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5 Volunteers

1. **Data:** Each one should give your marking for each one of you.

\[
\begin{bmatrix}
0 & 0.1 & 0.2 & 0.1 & 0.1 \\
0.5 & 0 & 0.8 & 0 & 0.1 \\
0 & 0.2 & 0 & 0 & 0.3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

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Eigenvalues
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Data: Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0

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1. **Data:** Each one should give your marking for each one of you. For example: Isha: (I) 100, (J) 20, (K) 50, (L) 30, (M) 0

2. **Normalizing** or averaging:
5 Volunteers

1. **Data:** Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0

2. **Normalizing** or averaging: How much you have given from the total marking

   Total mark: $100 + 20 + 50 + 30 + 0 = 200$
5 Volunteers

1. **Data:** Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0

2. **Normalizing** or averaging: How much you have given from the total marking
   - **Total mark:** $100 + 20 + 50 + 30 + 0 = 200$
   - **Normal mark:** $100/200, 20/200, 50/200, 30/200, 0/200$
5 Volunteers

1. **Data:** Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0

2. **Normalizing or averaging:** How much you have given from the total marking
   - Total mark: $100 + 20 + 50 + 30 + 0 = 200$
   - Normal mark: 0.50, 0.1, 0.25, 0.15, 0.0

3. **Markov Matrices:**
   - Formation
   - Interpretation
   - Properties

4. **Eigenvalues**

---

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5 Volunteers

1. **Data**: Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0

2. **Normalizing** or averaging: How much you have given from the total marking

   - **Total mark**: 100+20+50+30+0=200
   - **Normal mark**: 0.50, 0.1, 0.25, 0.15, 0.0

3. **Arranging**: Writing them in a matrix form

   $$
   \begin{pmatrix}
   0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\
   0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\
   0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\
   0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\
   0.0 & 0.2 & 0.0 & 0.2 & 0.2
   \end{pmatrix}
   $$

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<table>
<thead>
<tr>
<th></th>
<th>Isha</th>
<th>Jose</th>
<th>Kumar</th>
<th>Latha</th>
<th>Mani</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isha</td>
<td>0.5</td>
<td>0.8</td>
<td>0.0</td>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>Jose</td>
<td>0.1</td>
<td>0.32</td>
<td>0.0</td>
<td>0.0</td>
<td>0.2</td>
</tr>
<tr>
<td>Kumar</td>
<td>0.25</td>
<td>0.24</td>
<td>1.0</td>
<td>0.0</td>
<td>0.2</td>
</tr>
<tr>
<td>Latha</td>
<td>0.15</td>
<td>0.16</td>
<td>0.0</td>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>Mani</td>
<td>0.0</td>
<td>0.2</td>
<td>0.0</td>
<td>0.2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

∑ = 1  1  1  1  1
**Markov Matrix**

A real $n \times n$ matrix is called a Markov matrix if:

1. Each entry is non-negative: $m_{ij} \geq 0$ for all $1 \leq i \leq n, 1 \leq j \leq n$. 

\[
M = \begin{bmatrix}
m_{11} & m_{12} & \cdots & m_{1n} \\
m_{21} & m_{22} & & m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{n1} & m_{n2} & & m_{nn}
\end{bmatrix}_{n \times n}
\]
**Markov Matrix**

A real $n \times n$ matrix is called a Markov matrix if

1. Each entry is non-negative: $m_{ij} \geq 0$ for all $1 \leq i \leq n, 1 \leq j \leq n$.
2. Each column sum is 1: $\sum_{i=1}^{n} m_{ij} = 1$ for each $1 \leq j \leq n$. 

**Definition (Markov Matrix)**

$$M = \begin{bmatrix}
m_{11} & m_{12} & \cdots & m_{1n} \\
m_{21} & m_{22} & \cdots & m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{n1} & m_{n2} & \cdots & m_{nn}
\end{bmatrix}_{n \times n}$$
Outline

1. Linear transformations on plane
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   - Eigen value and eigen vector

3. Markov Matrices
   - Formation
   - Interpretation
   - Properties
We have formed a Markov matrix from individual opinions.

\[
\begin{pmatrix}
0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\
0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\
0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\
0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\
0.0 & 0.2 & 0.0 & 0.2 & 0.2
\end{pmatrix}
\]
We have formed a Markov matrix from individual opinions.
If Isha's opinion is full value then Isha is better than others.

\[
\begin{pmatrix}
0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\
0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\
0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\
0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\
0.0 & 0.2 & 0.0 & 0.2 & 0.2 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
= 
\begin{pmatrix}
0.5 \\
0.1 \\
0.25 \\
0.15 \\
0 \\
\end{pmatrix}
\]
Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha’s opinion is full value then Isha is better than others.
- If Jose’s opinion is full value then Isha is better than others.

\[
\begin{bmatrix}
0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\
0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\
0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\
0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\
0.0 & 0.2 & 0.0 & 0.2 & 0.2
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
0.8 \\
0.32 \\
0.24 \\
0.16 \\
0.2
\end{bmatrix}
\]
Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha’s opinion is full value then Isha is better than others.
- If Jose’s opinion is full value then Isha is better than others.
- If Mani’s opinion is full value then all are equal.

\[
\begin{pmatrix}
0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\
0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\
0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\
0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\
0.0 & 0.2 & 0.0 & 0.2 & 0.2
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
0.2 \\
0.2 \\
0.2 \\
0.2 \\
0.2
\end{pmatrix}
\]
Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha’s opinion is full value then Isha is better than others.
- If Jose’s opinion is full value then Isha is better than others.
- If Mani’s opinion is full value then all are equal.
- If all opinions are equal value then Isha is better than others.

\[
\begin{pmatrix}
0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\
0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\
0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\
0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\
0.0 & 0.2 & 0.0 & 0.2 & 0.2
\end{pmatrix}
\begin{pmatrix}
0.2 \\
0.2 \\
0.2 \\
0.2 \\
0.2
\end{pmatrix}
= 
\begin{pmatrix}
0.380 \\
0.124 \\
0.338 \\
0.182 \\
0.120
\end{pmatrix}

When the opinion values matches with the conclusion value?
For which opinion value we will get the same conclusion value.
Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha’s opinion is full value then Isha is better than others.
- If Jose’s opinion is full value then Isha is better than others.
- If Mani’s opinion is full value then all are equal.
- If all opinions are equal value then Isha is better than others.
- If opinion has different value then Kumar is better than others.

\[
\begin{pmatrix}
0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\
0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\
0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\
0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\
0.0 & 0.2 & 0.0 & 0.2 & 0.2 \\
\end{pmatrix}
\begin{pmatrix}
0.2 \\
0.4 \\
0.1 \\
0.2 \\
0.1 \\
\end{pmatrix}
=
\begin{pmatrix}
0.320 \\
0.072 \\
0.394 \\
0.186 \\
0.100 \\
\end{pmatrix}
\]
Interpretation

- We have formed a **Markov** matrix from individual opinions.
- If Isha’s opinion is full value then Isha is better than others.
- If Jose’s opinion is full value then Isha is better than others.
- If Mani’s opinion is full value then all are equal.
- If all opinions are equal value then Isha is better than others.
- If opinion has different value then Kumar is better than others.

\[
\begin{pmatrix}
0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\
0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\
0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\
0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\
0.0 & 0.2 & 0.0 & 0.2 & 0.2 \\
\end{pmatrix}
\begin{pmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
\end{pmatrix}
= 
\begin{pmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
\end{pmatrix}
\]

- When the opinion values matches with the conclusion value?
  For which opinion value we will get the same conclusion value.
Big Problem

• Getting a common opinion from individual opinion

For a Markov matrix $M$, we need a stable opinion value $x$. Find a vector $x$ such that $Mx = x$.

For $M$, find an eigenvector $x$ corresponding to eigenvalue 1.
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- From individual preference to common preference

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- For a Markov matrix $M$, we need a stable opinion value $x$.
- Find a vector $x$ such that
  \[ Mx = x \]
- For $M$, find an eigenvector $x$ corresponding to eigenvalue 1.
1. Linear transformations on plane
   - Typical Examples
   - Properties

2. Eigen values
   - Eigen value and eigen vector

3. Markov Matrices
   - Formation
   - Interpretation
   - Properties
Theorem

*For every Markov matrix 1 is surely an eigenvalue.*
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Proof.

For a Markov matrix $M$ every column sum is 1. Consider the transpose matrix $M^T$. Now its row sum is 1. Let $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

- $M^T e = e$
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- 1 is an eigenvalue of $M^T$ with $e$ as its eigenvector.
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- We know $\det(A) = \det(A^T)$, so $\det(M - \lambda I) = \det(M^T - \lambda I)$
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- We know $\det(A) = \det(A^T)$, so $\det(M - \lambda I) = \det(M^T - \lambda I)$
- Hence 1 is an eigenvalue of $M$
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- $M^T e = e$
- 1 is an eigenvalue of $M^T$ with $e$ as its eigenvector.
- We know $\det(A) = \det(A^T)$, so $\det(M - \lambda I) = \det(M^T - \lambda I)$
- Hence 1 is an eigenvalue of $M$
- But eigen vectors of $A$ and $A^T$ need not be same.
The previous result guaranteed the eigenvalue 1, but there is a problem if it has more eigenvectors corresponding to eigenvalue 1.
Further Properties

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Theorem

For a Markov matrix $M$ with all entries $m_{ij} > 0$,

1. 1 is always an eigenvalue.
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**Theorem**

*For a Markov matrix $M$ with all entries $m_{ij} > 0$,*

1. *1 is always an eigenvalue.*
2. *Other eigenvalues have magnitude less than 1 ($|\lambda| < 1$).*
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Theorem

For a Markov matrix $M$ with all entries $m_{ij} > 0$,

1. 1 is always an eigenvalue.
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**Theorem**

For a Markov matrix $M$ with all entries $m_{ij} > 0$,

1. 1 is always an eigenvalue.
2. Other eigenvalues have magnitude less than 1 ($|\lambda| < 1$).
3. There is an eigenvector corresponding to the eigenvalue 1.
4. This eigenvector has all positive entries with sum 1.
Further Properties

The previous result guaranteed the eigenvalue 1, but there is a problem if it has more eigenvectors corresponding to eigenvalue 1.

**Theorem**

For a Markov matrix $M$ with all entries $m_{ij} > 0$,

1. **1 is always an eigenvalue.**
2. **Other eigenvalues have magnitude less than 1 ($|\lambda| < 1$).**
3. **There is an eigenvector corresponding to the eigenvalue 1.**
4. **This eigenvector has all positive entries with sum 1.**
5. **Such an eigenvector is unique.**
Finding the stable opinion

Now we know that every Markov matrix with positive entries has a unique eigenvector corresponding to the eigenvalue $\lambda = 1$. 
Finding the stable opinion

Now we know that every Markov matrix with positive entries has a unique eigenvector corresponding to the eigenvalue $\lambda = 1$.

$$M - \lambda I = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}_{n \times n} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n \times n}$$
Finding the stable opinion

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$$(M - \lambda I)x = \begin{bmatrix} m_{11} - \lambda & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} - \lambda & & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} - \lambda \end{bmatrix}_{n \times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
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Now we know that every Markov matrix with positive entries has a unique eigenvector corresponding to the eigenvalue \( \lambda = 1 \).

\[
M - \lambda I = \begin{bmatrix}
m_{11} & m_{12} & \cdots & m_{1n} \\
m_{21} & m_{22} & \cdots & m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{n1} & m_{n2} & \cdots & m_{nn}
\end{bmatrix}_{n \times n} - \lambda
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1
\end{bmatrix}_{n \times n}
\]

\[
(M - \lambda I)x = \begin{bmatrix}
m_{11} - \lambda & m_{12} & \cdots & m_{1n} \\
m_{21} & m_{22} - \lambda & \cdots & m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{n1} & m_{n2} & \cdots & m_{nn} - \lambda
\end{bmatrix}_{n \times n} \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
v_n
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Solve \((M - \lambda I)x = 0\) by Gauss Elimination Process
Other Views

- It is a simpler version of page-rank.
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- It represents transition matrix and steady state.
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- Markov process in input-output business models.
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- ...
Questions
Thank you
In aeronautical engineering, eigenvalues may determine whether the flow over a wing is laminar or turbulent.

In electrical engineering, they may determine the frequency response of an amplifier or the reliability of a national power system.

In structural mechanics, eigenvalues may determine whether an automobile is too noisy or whether a building will collapse in an earthquake.

In probability, they may determine the rate of convergence of a Markov process.

In ecology, they may determine whether a food web will settle into a steady equilibrium.

In numerical analysis, they may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge.