Biggest open ball in invertible elements of a Banach algebra

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Group of invertible elements in a Banach Algebra

Let A be a complex unital Banach Algebra and $G(A)$ denote the group of invertible elements of A.

Theorem

$G(A)$ is an open set in A.

$$B\left(a, \frac{1}{\|a^{-1}\|}\right) \subseteq G(A)$$

Is this the biggest ball?

Does there exists a non-invertible $b \in A$ such that $\|b - a\| = \frac{1}{\|a^{-1}\|}$

Biggest Open Ball Property (BOBP)

Definition (An element has BOBP)

An element $a \in G(A)$ has BOBP if the boundary of the ball $B\left(a, \frac{1}{\|a^{-1}\|}\right)$ intersects $Sing(A)$.

Definition (BOBP)

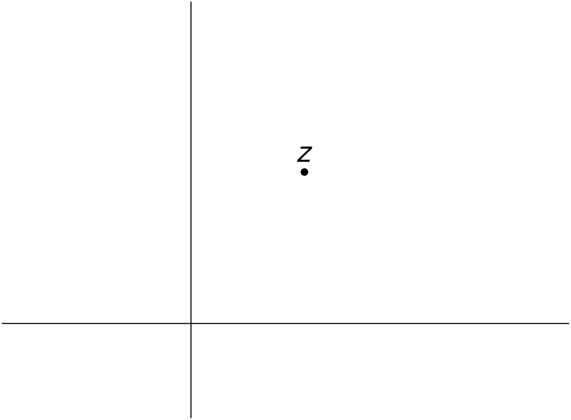
A Banach algebra has BOBP if every element of $G(A)$ has BOBP.

Can we characterise all Banach Algebras which satisfy BOBP?
Can we characterise all the elements of a Banach Algebra, which *do not* satisfy BOBP?

Positive observations for BOBP

The Banach algebra of complex numbers $\mathbb{C}$

Let $z \in \mathbb{C}$ be invertible. $|z^{-1}| = \frac{1}{|z|}$. Then $B\left(z, \frac{1}{|z^{-1}|}\right) = B(z, |z|)$.
Here 0 is the singular element on the boundary.



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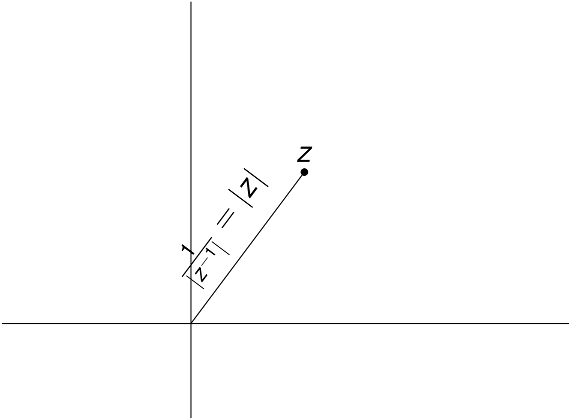

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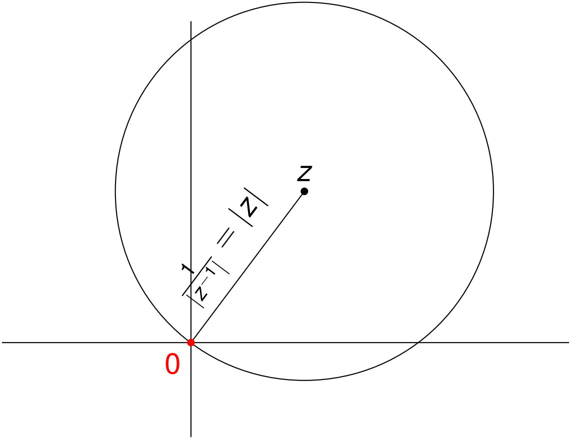

Positive observations for BOBP

The uniform algebra $C(X)$, X compact Hausdorff Space

Let $f \in C(X)$ be invertible.

$$\left\|f^{-1}\right\|_{\infty} = \sup_{x \in X} \left\{ \frac{1}{|f(x)|} \right\} = \frac{1}{\inf_{x \in X} \{|f(x)|\}} = \frac{1}{|f(x_0)|} \text{ for some } x_0 \in X$$

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Consider $g(x) = f(x) - f(x_0)$. Then g is singular and

$$\|f - g\|_{\infty} = |f(x_0)| = \frac{1}{\|f^{-1}\|}$$

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Commutative unital C^* algebra

Since it is true for $C(X)$, X compact and Hausdorff, by representation theorem, any commutative unital C^* algebras has BOBP.

For general C^* algebra. If $x \in G(A)$ is normal, the x satisfies BOBP.

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Operator algebra $B(H)$

Condition number 1

Let $T \in B(H)$ be invertible such that $\|T\|\|T^{-1}\| = 1$, then T has BOBP.

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Example

The matrix algebra $\mathcal{M}_{n\times n}(\mathbb{C})$ has BOBP.

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Proof.

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Let T be invertible in $B(H)$. Let $T = V|T|$ be the polar decomposition. As T is invertible, V is unitary and $|T|$ also invertible. As |T| is self-adjoint

$$\left\| T^{-1} \right\| = \left\| |T|^{-1} \right\| = \sup_{\lambda \in \sigma(|T|)} \left\{ \frac{1}{|\lambda|} \right\} = \frac{1}{|\lambda_0|} \text{ for some } \lambda_0 \in \sigma(|T|)$$

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$$S = V(|T| - \lambda_0 I) = V|T| - \lambda_0 V = T - \lambda_0 V.$$

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Then S is not invertible and $\|S - T\| = |\lambda_0| = \frac{1}{\|T^{-1}\|}$ □

Negative observations for BOBP

$B\left(a, \frac{1}{\|a^{-1}\|}\right)$ is not the biggest open ball around a contained in $G(A)$
There exists $a \in G(A)$ such that for every b singular

$$\|a - b\| > \frac{1}{\|a^{-1}\|}.$$

Negative observations for BOBP

Banach function algebra $C^1[0,1]$

$C^1[0,1]$ equipped with the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty \ \forall\ f \in C^1[0,1].$$

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Let $f(x) = e^x$ be an invertible element. Here

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Let g be singular. That is $g(x_0) = 0$ for some $x_0 \in [0,1]$. Then

$$\|f-g\| = \|f-g\|_\infty + \left\|f' - g'\right\|_\infty \geq |f(x_0) - g(x_0)| = e^{x_0} > \frac{1}{2} = \frac{1}{\|f^{-1}\|}$$

Negative observations for BOBP

The group algebra $\ell^1(\mathbb{Z}_2)$

Wiener algebra $A(\mathbb{T})$

The set of all complex valued functions on $[-\pi, \pi]$ with absolutely convergent Fourier series, that is, functions of the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, t \in [-\pi, \pi]$$

with $\|f\| = \sum_{n=-\infty}^{\infty} |c_n| < \infty$.

Summary

BOBP

- $C(X)$, X compact T_2
- Commutative C^* algebra
- $M_n(\mathbb{C})$
- $B(H)$

Does not have BOBP

- $C^1[0, 1]$
- $\ell^1(\mathbb{Z}_2)$
- $A(\mathbb{T})$ - Wiener algebra

Thank you