Conflict-Free Coloring: Graphs of Bounded Clique Width and Intersection Graphs

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Outline

- Introduce CONFLICT-FREE COLORING PROBLEM
- Our Results
- Discuss a couple of our results
- Open Questions/Present Status of our results

Definition (Conflict-free Coloring)

Given a graph $G = (V, E)$, a conflict-free coloring is an assignment of colors to a subset of V such that

- Every vertex in G has a uniquely colored vertex in its neighborhood.

The minimum number of colors required for such a coloring is called the conflict-free chromatic number.

Uniquely colored vertex in the neighborhood of a vertex v is the vertex which is distinctly colored among all neighbors of v.

Definition (Conflict-free Coloring on Open Neighborhoods)

Given a graph $G = (V, E)$, a conflict-free coloring with respect to open neighborhoods is an assignment of colors to a subset of V such that

- Every vertex has a uniquely colored vertex in its open neighborhood.

The minimum number of colors required for such a coloring is called the conflict-free chromatic number denoted by $\chi^*_{ON}(G)$.

- Open Neighborhood of a vertex v is $N(v) = \{w \mid \{v, w\} \in E(G))\}$.
- CFON* COLORING PROBLEM.

Definition (Conflict-free Coloring on Closed Neighborhoods)

Given a graph $G = (V, E)$, a conflict-free coloring with respect to closed neighborhoods is an assignment of colors to a subset of V such that

- Every vertex has a uniquely colored vertex in its closed neighborhood.

The minimum number of colors required for such a coloring is called the conflict-free chromatic number denoted by $\chi^*_{CN}(G)$.

- Closed Neighborhood of a vertex v is $N[v] = N(v) \cup \{v\}$.
- CFCN* COLORING PROBLEM.

CFON* vs CFCN*

Figure 1: CFON* Coloring

Figure 2: CFCN* Coloring

CFON* vs CFCN*

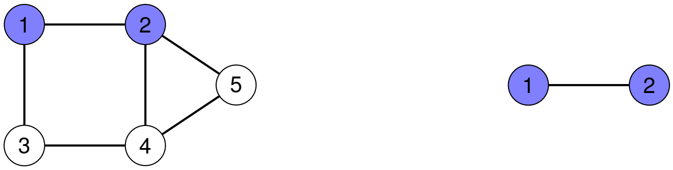

Figure 1: CFON* Coloring

Figure 2: CFCN* Coloring

Figure 3: CFON* Coloring

Figure 4: CFCN* Coloring

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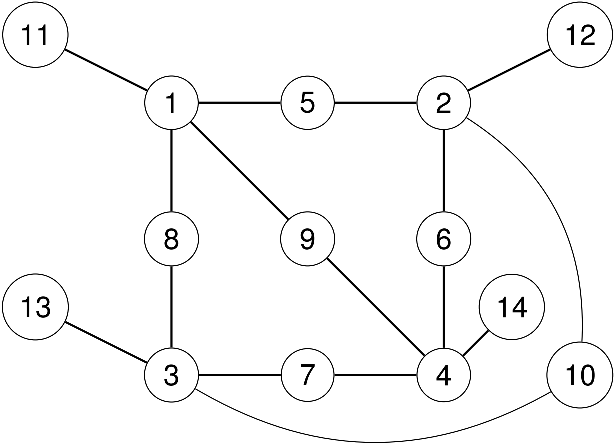
K_n^* : Subdivision graph of the Clique



- $\chi_{ON}^*(K_n^*) = n$.
- K_n^* is bipartite and hence $\chi_{CN}^*(K_n^*) = 2$.

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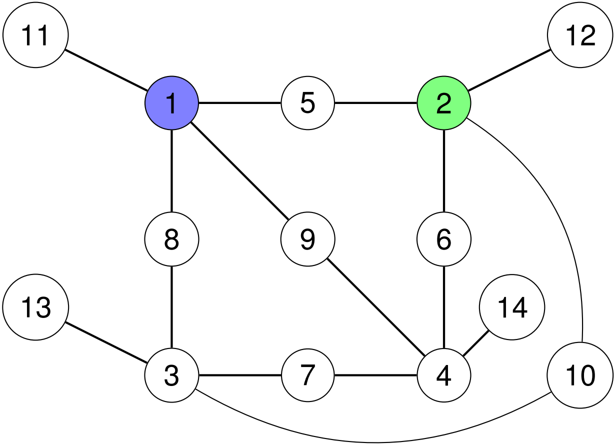

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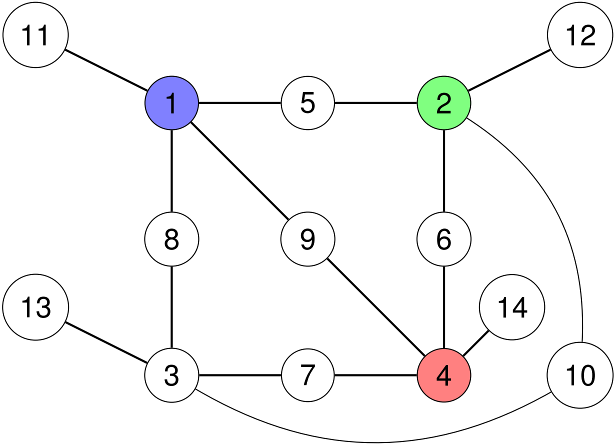

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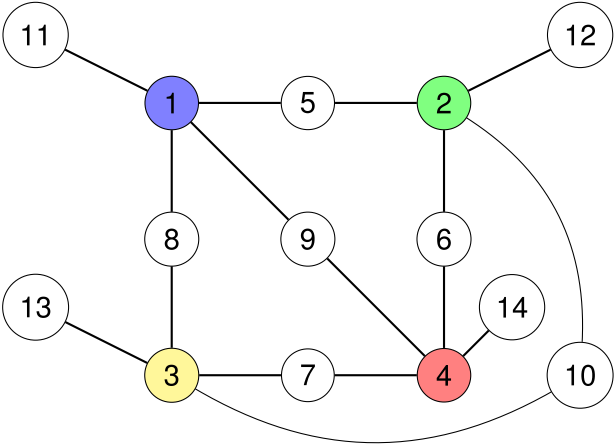

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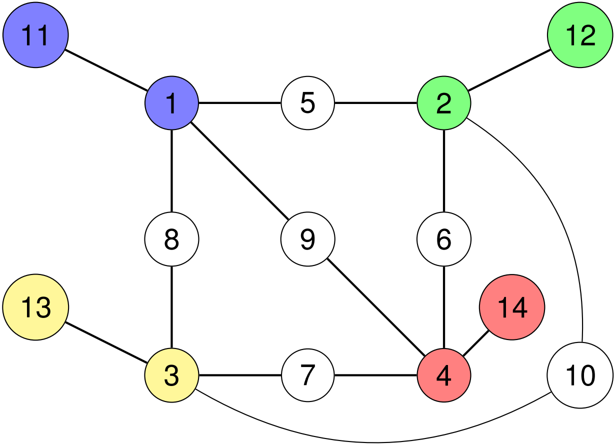

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Motivation & History

- Introduced by Even, Lotker, Ron and Smorodinsky in 2004, motivated by the Frequency Assignment Problem.
- The problem has been studied with respect to both the open neighborhoods and the closed neighborhoods.
- $\chi^*_{ON}(G) = \Theta(\sqrt{n})$ and $\chi^*_{CN}(G) = \Theta(\log^2 n)$.
- Geometric intersection graphs like disk, square, rectangle, interval graphs, etc have attracted special interest.
- Most of the variants are NP-complete.

Our Results (CFON*)

Graph Class	Upper Bound	Tight?	Complexity
(G, cw, k)	-	-	FPT
Block graphs	3	3	P
Cographs	2	2	P
Interval graphs	3	3	-
Proper Interval graphs	2	2	-
Unit square	27	3	-
Unit disk	51	3	-
Kneser graphs $K(n, k)$	$k + 1$	$k + 1$	-
Split graphs	-	-	NP-complete

Our results are marked in red color.

Our Results (Interval graphs)

Interval Graph: A graph $G = (V, E)$ is an *interval graph* if there exists a set $\mathcal{I}$ of intervals on the real line such that there is a bijection $f : V \to \mathcal{I}$ satisfying the following: $\{v_1, v_2\} \in E$ if and only if $f(v_1) \cap f(v_2) \neq \emptyset$.

- Reddy [2018] studied the full coloring variant of the problem.
- Fekete and Keldenich [2017] showed that $\chi^*_{CN}(G) \leq 2$.
- We show that $\chi^*_{ON}(G) \leq 3$. We also show existence of interval graph G' for which $\chi^*_{ON}(G') = 3$, making the bound tight.

  Moreover the graph G' is a tight example for the full coloring variant of the problem.

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Interval graphs

Lemma

If G is an interval graph, then $\chi^*_{ON}(G) \leq 3$.

Proof: Let $\mathcal{I}$ be the set of intervals. For each interval $I \in \mathcal{I}$, its right end point is denoted by $R(I)$. $C : \mathcal{I} \rightarrow \{0\} \cup \{1, 2, 3\}$.

- Assign the colors 1, 2 and 3 alternately, one in each iteration $1 \leq j \leq \ell$.
- Start with the interval I_1 for which $R(I_1)$ is the least and assign $C(I_1) = 1$.
- Choose an interval $I_2 \in N(I_1)$ s.t. $R(I_2) \geq R(I), \forall I \in N(I_1)$ and assign $C(I_2) = 2$.

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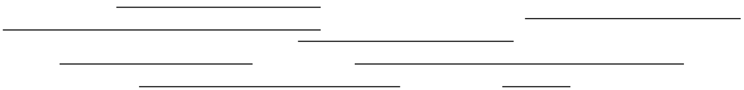

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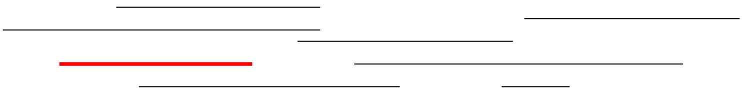

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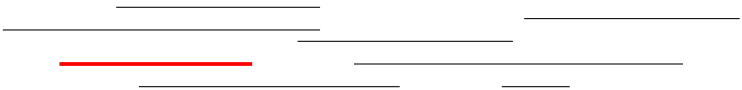

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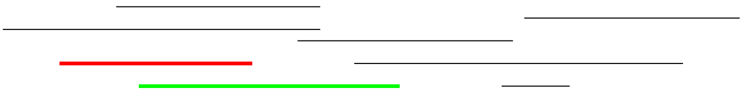
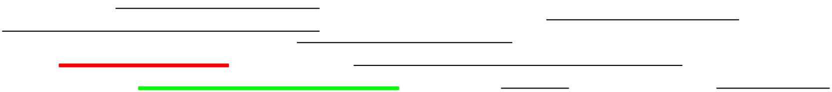


Interval graphs

- For $j \geq 3$, we do the following.
  - Choose the interval $I_j \in N(I_{j-1})$ s.t. $R(I_j) \geq R(I)$, for all $I \in N(I_{j-1})$.
  - Assign color $\{1, 2, 3\} \setminus \{C(I_{j-1}), C(I_{j-2})\}$ to the interval I_j.
- Note that the interval I_ℓ chosen in the last iteration ℓ, is such that $R(I_\ell)$ maximizes $R(I)$ amongst all $I \in \mathcal{I}$.
- All the uncolored intervals are assigned the color 0.



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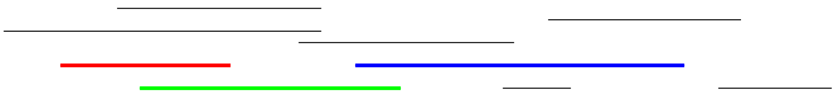

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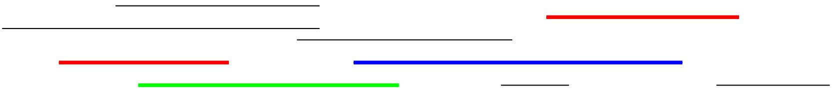

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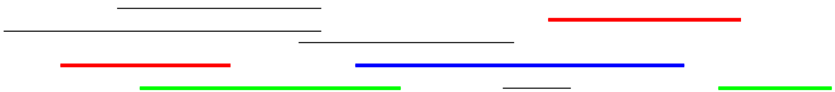

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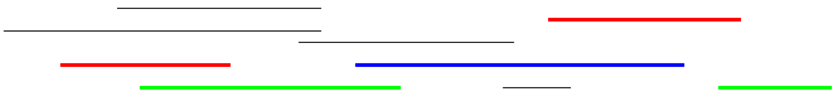
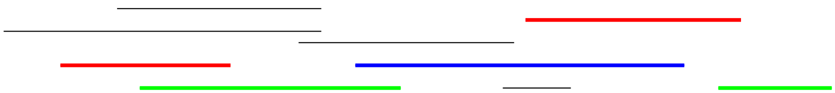


Correctness



- All vertices assigned the color 0 have a uniquely colored neighbor.
- All vertices assigned a non-zero color also have a uniquely colored neighbor.

Unit Interval graphs

Proper Interval Graph: An interval graph is a *proper interval graph* if it has an interval representation $\mathcal{I}$ such that no interval in $\mathcal{I}$ is properly contained in any other interval of $\mathcal{I}$.

Unit Interval Graph: An interval graph G is a *unit interval graph* if it has an interval representation $\mathcal{I}$ where all the intervals are of unit length.

- We show that $\chi^*_{ON}(G) \leq 2$.
- There is a unit interval graph K_3 such that $\chi^*_{ON}(K_3) = 2$, making the above bound tight.

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Lemma

If G is a proper interval graph, then $\chi^*_{ON}(G) \leq 2$.

Proof: We denote the left endpoint of an interval $I \in \mathcal{I}$ by $L(I)$.

- We assign $C : \mathcal{I} \rightarrow \{0\} \cup \{1, 2\}$
- At each iteration i, we pick two intervals $I_1^i, I_2^i \in \mathcal{I}$.
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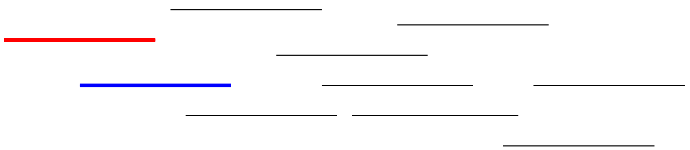

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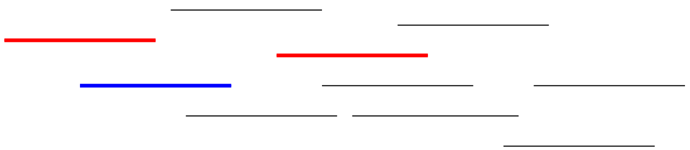

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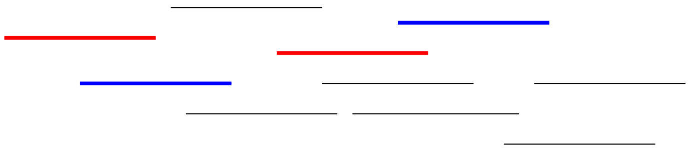
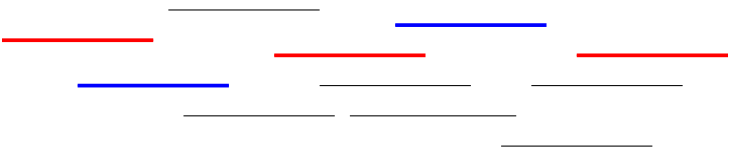


Unit Interval graphs

- I_2^i does not exist ? All neighbors of I_1^i are already colored. This can happen only in the very last iteration ℓ of the algorithm.



- Correctness when I_2^ℓ exists for the last iteration ℓ
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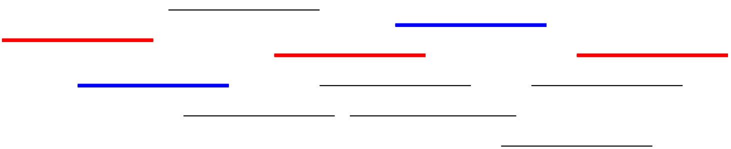

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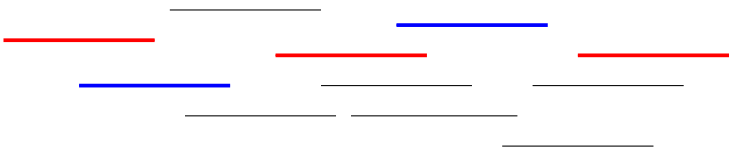

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Unit Interval graphs

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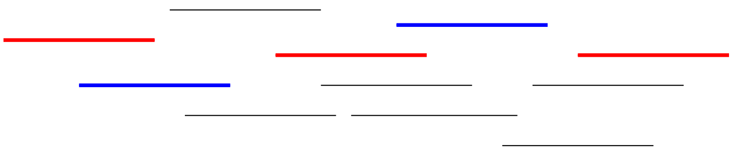


- Correctness when I_2^ℓ exists for the last iteration ℓ
  - I_1^i and I_2^i act as the uniquely colored neighbors for each other in each iteration i.
  - All intervals that are assigned color 0 are adjacent to either I_1^i or I_2^i, and thus will have a uniquely colored neighbor.
  - The vertices I_1^i (or I_2^i) and I_1^{i+1} (or I_2^{i+1}) are assigned the same color. This is fine as there is no interval that intersects both I_1^i and I_1^{i+1}.

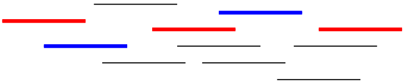
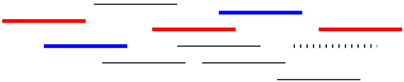


Figure 5: Before

Figure 6: After

- I_2^i does not exist
  - This can happen only during the last iteration $i = \ell$.
  - I_1^ℓ is the only interval for which C is yet to be assigned.
  - Choose an interval $I_m \in N(I_2^{\ell-1}) \cap N(I_1^\ell)$. Such an I_m exists ?
  - We reassign $C(I_1^{\ell-1}) = 0$, $C(I_2^{\ell-1}) = 1$, $C(I_m) = 2$ and assign $C(I_1^\ell) = 0$.
  - Any effect for reassigning $C(I_1^{\ell-1}) = 0$?



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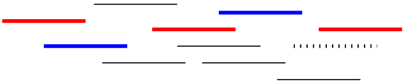

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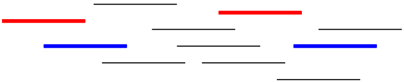

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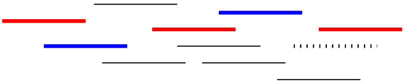
Unit Interval graphs



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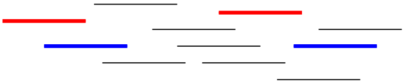

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Present Status of Our Results

Graph Class	Upper Bound	Tight?	Complexity
(G, cw, k)	-	-	FPT
Block graphs	3	3	P
Cographs	2	2	P
Interval graphs	3	3	-
Proper Interval graphs	2	2	-
Unit square	27	3	NP-complete
Unit disk	51	3	NP-complete
Kneser graphs $K(n, k)$	$k + 1$	$k + 1$	-
Split graphs	-	-	NP-complete

THANK YOU

Questions?