# Conflict-free Coloring on Claw-free graphs and Interval graphs

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# Agenda

- Conflict-free Coloring
- Known Results
- Our Results

# Conflict-free Coloring

**Definition (Conflict-free Coloring)**

Given a graph $G = (V, E)$, a conflict-free coloring is an assignment of colors to a subset of $V$ such that

- Every vertex in $G$ has a uniquely colored vertex in its neighborhood.

The minimum number of colors required for such a coloring is called the conflict-free chromatic number.

# Conflict-free Coloring

**Definition (Conflict-free Coloring on Open Neighborhoods)**

Given a graph $G = (V, E)$, a conflict-free coloring is an assignment of colors to a subset of $V$ such that

- Every vertex in $G$ has a uniquely colored vertex in its open neighborhood.

The minimum number of colors required for such a coloring is called the conflict-free chromatic number denoted by $\chi^*_{ON}(G)$.

- Open Neighborhood of a vertex $v$ is $N(v) = \{w \mid \{v, w\} \in E(G))\}$.
- This problem is abbreviated as CFON* Coloring Problem.

# Conflict-free Coloring

**Definition (Conflict-free Coloring on Closed Neighborhoods)**

Given a graph $G = (V, E)$, a conflict-free coloring with respect to closed neighborhoods is an assignment of colors to a subset of $V$ such that

- Every vertex has a uniquely colored vertex in its closed neighborhood.

The minimum number of colors required for such a coloring is called the conflict-free chromatic number denoted by $\chi^*_{CN}(G)$.

- Closed Neighborhood of a vertex $v$ is $N[v] = N(v) \cup \{v\}$.
- This problem is abbreviated as CFCN* Coloring Problem.

# CFON* vs CFCN*

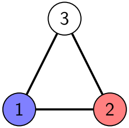

Figure 1: CFON* Coloring

# CFON* vs CFCN*


Figure 1: CFON* Coloring


Figure 2: CFCN* Coloring

# CFON* vs CFCN*



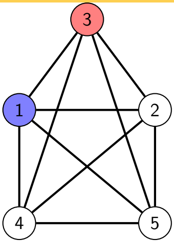



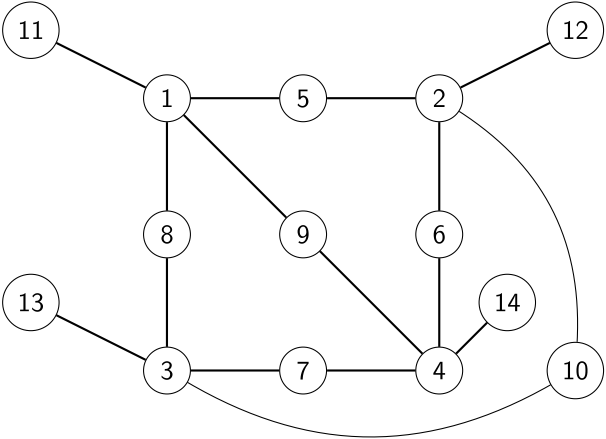

Figure 3: CFON* Coloring

# CFON* vs CFCN*

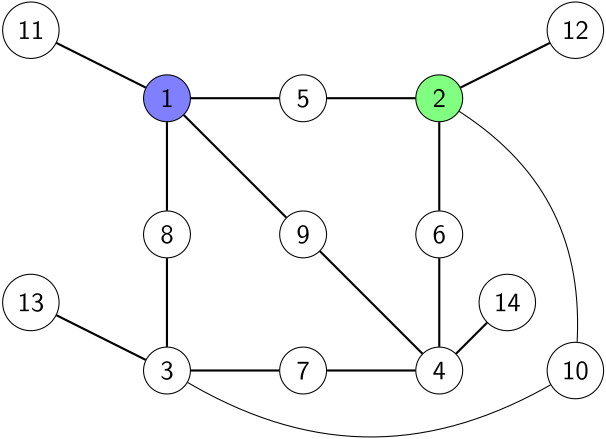

Figure 3: CFON* Coloring


Figure 4: CFCN* Coloring

# $K_n^*$ : Subdivision graph of the Clique



# $K_n^*$ : Subdivision graph of the Clique



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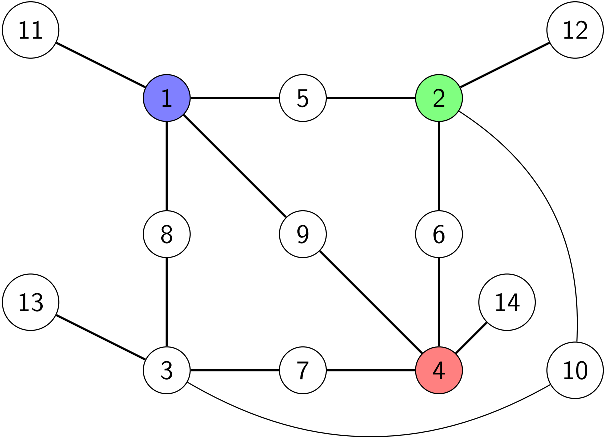

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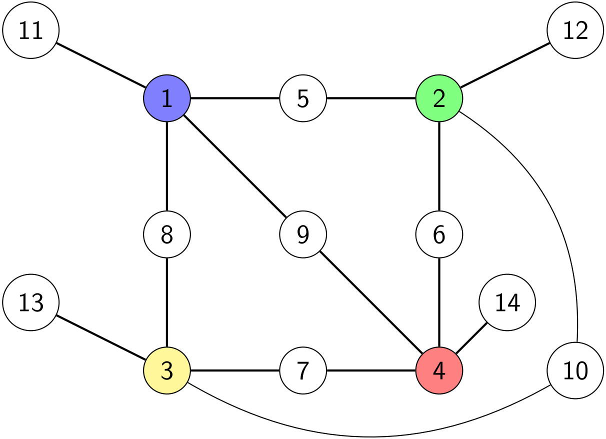
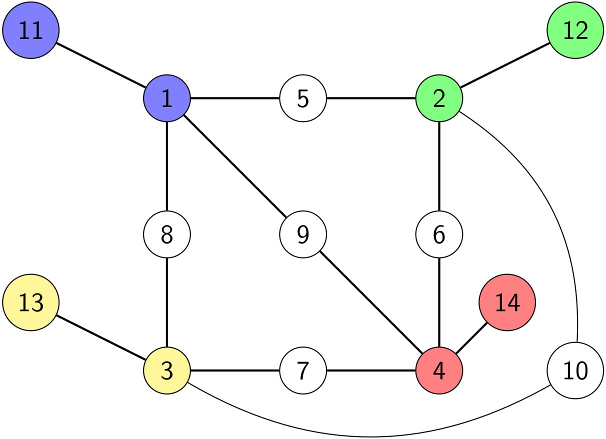


# $K_n^*$ : Subdivision graph of the Clique




- $\chi^*_{ON}(K_n^*) = n$.
- $K_n^*$ is bipartite and hence $\chi^*_{CN}(K_n^*) = 2$.

# Motivation & History

- Introduced by Even, Lotker, Ron and Smorodinsky in 2004, motivated by the Frequency Assignment Problem.
- The problem has been studied with respect to both the open neighborhoods and the closed neighborhoods.
- $\chi^*_{ON}(G) = O(\sqrt{n})$ and $\chi^*_{CN}(G) = O(\log^2 n)$.
- Geometric intersection graphs like disk, square, rectangle, interval graphs, etc have attracted special interest.
- Most of the variants are NP-complete.

# Our Results 1

- Dębski and Przybyło in [J. Graph Theory, 2021] had shown that if $G$ is a line graph, then $\chi^*_{CN}(G) = O(\log \Delta)$.
  - **Open Question:** Can it be extended to claw-free ($K_{1,3}$-free) graphs, which are a superclass of line graphs ?
- In the same paper, they showed that if the minimum degree of any vertex in $G$ is $\Omega(\Delta)$, then $\chi^*_{ON}(G) = O(\log \Delta)$.

## Our Results

- For $k \geq 3$, we show that if $G$ is a $K_{1,k}$-free graph then $\chi^*_{ON}(G) = O(k^2 \log \Delta)$, where $\Delta$ denotes the maximum degree of $G$. Since $\chi^*_{CN}(G) \leq 2\chi^*_{ON}(G)$, we have $\chi^*_{CN}(G) = O(k^2 \log \Delta)$ as well
- If the minimum degree of any vertex in $G$ is $\Omega(\frac{\Delta}{\log^\epsilon \Delta})$ for some $\epsilon \geq 0$, then $\chi^*_{ON}(G) = O(\log^{1+\epsilon} \Delta)$.

# Our Results 2

- Reddy in [Theo. Comp. Sci., 2018] showed that, for an interval graph $G$, $\chi^*_{ON}(G) \leq 3$. Bhyravarapu et. al. [IWOCA 2021] showed that there exists an interval graph that requires three colors.
- It was asked by Reddy if there is a polynomial time algorithm to compute $\chi^*_{ON}(G)$ for interval graphs.

## Our Results

- We show that CFON$^*$ Coloring Problem is polynomial time solvable on interval graphs.

# Our Results 3

- Abel et. al. [SIDMA 2018] showed that it is NP-complete to decide if $k$ colors are sufficient to CFON$^*$ color a planar bipartite graph, even when $k \in \{1, 2, 3\}$.

## Our Results

- We explore sub-classes of bipartite graphs that includes biconvex graphs, biconvex permutation graphs, etc, and show polynomial time algorithms for CFON$^*$ Coloring Problem.

# CFON* Coloring for claw-free graphs

**Claw:** The complete bipartite graph $K_{1,3}$ is called a *claw*. A graph is called a *claw-free graph* if it does not contain a claw as an induced subgraph.

**Claw number:** The *claw number* of a graph $G$ is the smallest $k$ such that $G$ is $K_{1,k+1}$-free. In other words, it is the largest $k$ such that $G$ contains an induced $K_{1,k}$.

# Bounded Claw Number

**Theorem**

*Let $G$ be a $K_{1,k}$-free graph with no isolated vertices. Then, $\chi^*_{ON}(G) = O(k^2 \log \Delta)$, where $\Delta$ is the maximum degree of $G$.*

**Proof:** We start with a proper coloring $h : V(G) \to [\Delta + 1]$ of $G$.

- Let $C_1, C_2, \dots, C_{\Delta+1}$ be the color classes w.r.t. the coloring $h$.
- WLOG we assume that every vertex in $C_i$ has a neighbor in each $C_j$, $1 \leq j < i$.
- Observe that, any vertex in $G$ has at most $k - 1$ neighbors in $C_i$, for every $i \in [\Delta + 1]$.

# Two ingredients

**Theorem (Pach and Tardos, 2009)**

*Let $\mathcal{H}$ be a hypergraph and let $\Delta$ be the maximum degree of any vertex in $\mathcal{H}$. Then, $\chi_{CF}(\mathcal{H}) \leq \Delta + 1$.*

**Lemma**

*Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph where (i) every hyperedge intersects with at most $\Gamma$ other hyperedges, and (ii) for every hyperedge $E \in \mathcal{E}$, $r \leq |E| \leq \ell r$, where $\ell \geq 1$ is some integer and $r \geq 2\log(4\Gamma)$. Then, $\chi_{CF}(\mathcal{H}) \leq e\ell r$, where $e$ is the base of natural logarithm.*

# Proofs

**Theorem (Pach and Tardos, 2009)**

*Let $\mathcal{H}$ be a hypergraph and let $\Delta$ be the maximum degree of any vertex in $\mathcal{H}$. Then, $\chi_{CF}(\mathcal{H}) \leq \Delta + 1$.*

**Proof.**

- Consider the vertices in arbitrary order.
- We want the first vertex in any hyperedge to be the uniquely colored vertex.
- A vertex appears in at most $\Delta$ hyperedges and hence needs to avoid at most $\Delta$ other colors.

$\square$

# Proofs

## Lemma

*Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph where (i) every hyperedge intersects with at most $\Gamma$ other hyperedges, and (ii) for every hyperedge $E \in \mathcal{E}$, $r \leq |E| \leq \ell r$, where $\ell \geq 1$ is some integer and $r \geq 2\log(4\Gamma)$. Then, $\chi_{CF}(\mathcal{H}) \leq e\ell r$, where $e$ is the base of natural logarithm.*

## Proof.

- Each vertex is assigned a color, that is chosen independently and uniformly at random from a set of $e\ell r$ colors.
- For a hyperedge $E$, let $A_E$ be the event that $E$ is colored with $\leq |E|/2$ colors.
- $A_E$ contains the event that $E$ is not conflict-free colored.
- We show that $P(A_E) \leq 1/4\Gamma$ and hence Local Lemma implies that $P[\cap_{E \in \mathcal{E}}(\overline{A}_E)] > 0$.

□

# Calculations

$$
\begin{aligned}
Pr[A_E] &\leq \binom{e\ell r}{m/2}\left(\frac{m/2}{e\ell r}\right)^m \\
&\leq \left(\frac{e^2\ell r}{m/2}\right)^{m/2}\left(\frac{m/2}{e\ell r}\right)^m \qquad \left(\text{since } \binom{n}{k} \leq \left(\frac{en}{k}\right)^k\right) \\
&= \frac{(m/2)^{m/2}}{(\ell r)^{m/2}} \\
&= \left(\frac{m}{2\ell r}\right)^{m/2} \\
&\leq (1/2)^{m/2} \quad \leq \quad \frac{1}{4\Gamma}.
\end{aligned}
$$

Here the penultimate inequality follows since $m \leq \ell r$, and the last inequality follows since $m \geq r \geq 2\log(4\Gamma)$.

# Bounded Claw Number

**Theorem**

*Let $G$ be a $K_{1,k}$-free graph with no isolated vertices. Then, $\chi^*_{ON}(G) = O(k^2 \log \Delta)$, where $\Delta$ is the maximum degree $\Delta$ of $G$.*

**Proof:** We start with a proper coloring $h : V(G) \to [\Delta + 1]$ of $G$.

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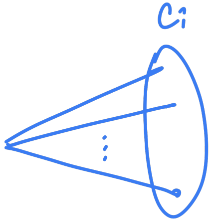
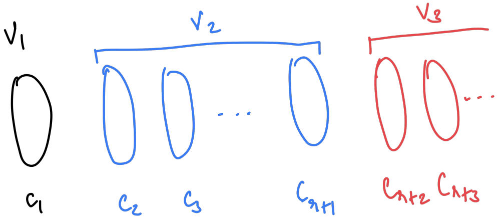
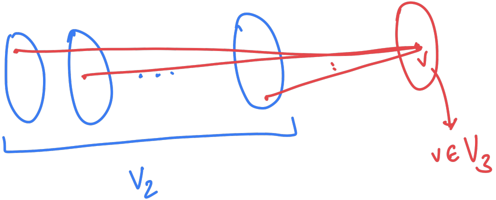


# Bounded Claw Number

- Let $r = 2\log(4\Delta^2)$.
- Partition the vertices of $G$ into three sets $V_1, V_2$ and $V_3$ as follows:
  - $V_1 = C_1$.
  - $V_2 = C_2 \cup C_3 \cup \cdots \cup C_{r+1}$.
  - $V_3 = C_{r+2} \cup C_{r+3} \cup \cdots \cup C_{\Delta+1}$.
- Color $V_1, V_2$ and $V_3$ such that every vertex has a uniquely colored neighbor.



# Uniquely colored neighbors for $V_3$

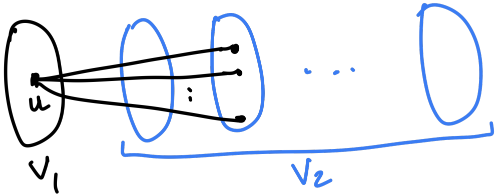



- Let $\mathcal{H}_3 = (V_2, \mathcal{E}_3)$ where $E_v \in \mathcal{E}_3$ if $E_v = N(v) \cap V_2$, for $v \in V_3$.
- Note that $r \leq |E_v| \leq r(k-1)$, for all $E_v \in \mathcal{E}_3$
- The below lemma implies $\chi_{CF}(\mathcal{H}_3) \leq e(k-1)r$

**Lemma**

*Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph where (i) every hyperedge intersects with at most $\Gamma$ other hyperedges, and (ii) for every hyperedge $E \in \mathcal{E}$, $r \leq |E| \leq \ell r$, where $\ell \geq 1$ is some integer and $r \geq 2\log(4\Gamma)$. Then, $\chi_{CF}(\mathcal{H}) \leq e\ell r$, where e is the base of natural logarithm.*

# Uniquely colored neighbors for $V_2$

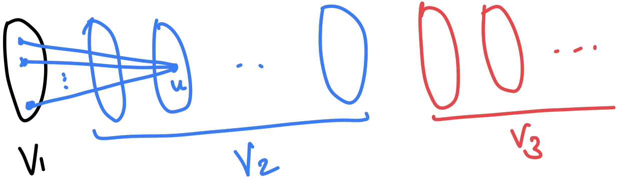



- Let $\mathcal{H}_2 = (V_1, \mathcal{E}_2)$ where $E_v \in \mathcal{E}_2$ if $E_v = N(v) \cap V_1$, for $v \in V_2$.
- Note that any $u \in V_1$ appears in at most $r(k-1)$ hyperedges of $\mathcal{H}_2$.
- The below theorem implies $\chi_{CF}(\mathcal{H}_1) \leq r(k-1) + 1$

**Theorem (Pach and Tardos, 2009)**

*Let $\mathcal{H}$ be a hypergraph and let $\Delta$ be the maximum degree of any vertex in $\mathcal{H}$. Then, $\chi_{CF}(\mathcal{H}) \leq \Delta + 1$.*

# Uniquely colored neighbors for $V_1$




- Let $\mathcal{H}_1 = (V_2 \cup V_3, \mathcal{E}_1)$ where $E_v \in \mathcal{E}_2$ if $E_v = N(v)$, for $v \in V_1$.
- Note that any $u \in V_2 \cup V_3$ appears in at most $(k-1)$ hyperedges of $\mathcal{H}_1$.
- The below theorem implies $\chi_{CF}(\mathcal{H}_1) \leq k$

**Theorem (Pach and Tardos, 2009)**

*Let $\mathcal{H}$ be a hypergraph and let $\Delta$ be the maximum degree of any vertex in $\mathcal{H}$. Then, $\chi_{CF}(\mathcal{H}) \leq \Delta + 1$.*

# Summarizing

- Vertices in $V_1$ are taken care by coloring $\mathcal{H}_1$, i.e., $V_2 \cup V_3$ using $k$ colors.
- Vertices in $V_2$ are taken care by coloring $\mathcal{H}_2$, i.e., $V_1$ using $r(k-1)+1$ colors.
- Vertices in $V_3$ are taken care by coloring $\mathcal{H}_3$, i.e., $V_2$ using $r(k-1)e$ colors.

# Summarizing

- Vertices in $V_1$ are taken care by coloring $\mathcal{H}_1$, i.e., $V_2 \cup V_3$ using $k$ colors.
- Vertices in $V_2$ are taken care by coloring $\mathcal{H}_2$, i.e., $V_1$ using $r(k-1)+1$ colors.
- Vertices in $V_3$ are taken care by coloring $\mathcal{H}_3$, i.e., $V_2$ using $r(k-1)e$ colors.

- Vertices in $V_2$ can be colored by using a Cartesian product, needing $r(k-1)ke \approx O(rk^2)$ colors. This turns out to be the dominating quantity.
- Noting that $r = O(\log \Delta)$, we have a CFON* coloring of $G$ with $O(k^2 \log \Delta)$ colors.

# Summarizing

- Vertices in $V_2$ can be colored by using a Cartesian product, needing $r(k-1)ke = O(rk^2)$ colors. This turns out to be the dominating quantity.
- Noting that $r = O(\log \Delta)$, we have a CFON* coloring of $G$ with $O(k^2 \log \Delta)$ colors.

**Theorem**

*Let $G$ be a $K_{1,k}$-free graph with no isolated vertices. Then, $\chi^*_{ON}(G) = O(k^2 \log \Delta)$, where $\Delta$ is the maximum degree of $G$.*

# CFON* Coloring on Interval Graphs

**Theorem**

*The CFON* Coloring Problem is polynomial time solvable on interval graphs.*

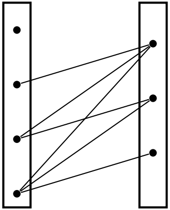
- If $G$ is an interval graph, it is known that $\chi^*_{ON}(G) \leq 3$.
- Characterization algorithms for interval graphs $G$ that decide if $\chi^*_{ON}(G) \in \{1, 2, 3\}$.
- The main tool that we use is the **multi-chain ordering** of interval graphs.
- It was shown by Enright, Stewart and Tardos [SIDMA 2014] that connected interval graphs admit multi chain orderings.

# Multi-chain ordering

**Definition (Chain Graph)**

A bipartite graph $G = (A, B)$ is a *chain graph* if and only if for any two vertices $u, v \in A$, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. If $G$ is a chain graph, it follows that for any two vertices $u, v \in B$, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.

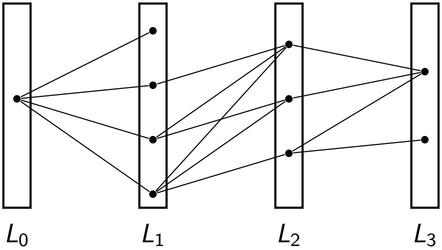
As a consequence, we can order the vertices in $B$ in the decreasing order of the degrees. We can break ties arbitrarily. If $b_1 \in B$ appears before $b_2 \in B$ in the ordering, then it follows that $N(b_2) \subseteq N(b_1)$.



# Multi-chain ordering

**Definition (Multi-chain Ordering)**

We say that distance layers form a *multi-chain ordering* of $G$ if for every two consecutive layers $L_i$ and $L_{i+1}$, where $i \in \{0, 1, \ldots, p-1\}$, we have that the vertices in $L_i$ and $L_{i+1}$, and the edges connecting these layers form a chain graph.



# Interval Graphs

**Theorem (Enright, Stewart and Tardos (SIDMA 2014))**

*All connected interval graphs admit multi-chain orderings.*

**Theorem (Our Result)**

*Given an interval graph $G$, there is a polynomial time algorithm that determines $\chi^*_{ON}(G)$.*

**Overall Idea of the Proof.**

- We give a characterization of interval graphs that require one color and two colors in polynomial time.
- If $G$ is not CFON$^*$ colorable using one color or two colors, we conclude that $G$ is CFON$^*$ colorable using three colors (since it is known that for an interval graph $G$, $\chi^*_{ON}(G) \leq 3$).
- One of the key ideas used to decide if $G$ can be CFON$^*$ 2-colorable is sort of a bootstrapping idea.

# 1-Colorable?

**Observation**

If $G$ admits a multi-chain ordering, then every distance layer $L_i$, for $0 \leq i < p$ contains a vertex $v$ such that $N(v) \supseteq L_{i+1}$.

- This means that if $G$ is CFON* colorable with 1 color, then, $L_{i+1}$ has at most one vertex that is colored.
- There are $|L_{i+1}|$ possible colorings to check for $L_{i+1}$.
- We also need to check if the colorings are consistent across neighboring layers.
- This leads to a dynamic programming algorithm.

**Theorem**

*Given an interval graph $G = (V, E)$, we can decide in $O(n^5)$ time if $\chi^*_{ON}(G) = 1$.*

# 2-Colorable?

- The idea is similar to checking 1-colorability, but there are more cases to deal with.
- One of the cases require us to verify that a subgraph is 1-CFON$^*$ colorable.
- We use the algorithm for 1-colorability since subgraphs of interval graphs are interval graphs.

**Theorem**

*Given an interval graph $G = (V, E)$, we can decide in $O(n^{20})$ time if $\chi^*_{ON}(G) = 2$.*

# Interval Graphs

**Remark**

Recently, the work of Gonzalez and Mann [**Gonzalez-Mann**] (done simultaneously and independently from ours) on mim-width showed that the CFON* coloring problem is polynomial-time solvable on graph classes for which a branch decomposition of constant mim-width can be computed in polynomial time.
This includes the class of interval graphs. We note that our work gives a more explicit algorithm without having to go through the machinery of mim-width.
We also note that the mim-width algorithm, as presented in [**Gonzalez-Mann**], requires a running time in excess of $\Omega(n^{300})$. Hence our algorithm is better in this regard as well.

[**Gonzalez-Mann**] Carolina Lucía Gonzalez and Felix Mann, "On d-stable locally checkable problems on bounded mim-width graphs", CoRR, abs/2203.15724, 2022.

# Conclusion

In this paper, we study CFON* coloring on claw-free graphs, interval graphs and biconvex graphs.

- We first show that if $G$ is a $K_{1,k}$-free graph with maximum degree $\Delta$, then $\chi^*_{ON}(G) = O(k^2 \log \Delta)$.
- We then show that if the minimum degree of $G$ is $\Omega(\frac{\Delta}{\log^{\epsilon} \Delta})$ for some $\epsilon \geq 0$, then $\chi^*_{ON}(G) = O(\log^{1+\epsilon} \Delta)$.

Question 1: The tightness of these bounds is a natural open question.

- We show polynomial time algorithms for the CFON* coloring problem on interval graphs and biconvex graphs. (can be extended to CFON coloring also)

Question 2: It may be of interest to study the problem on other subclasses of bipartite graphs, such as convex bipartite graphs, chordal bipartite graphs and tree-convex bipartite graphs.

Thank You!