Conflict-free Coloring on Claw-free graphs and Interval graphs

Sriram Bhyravarapu ✉
The Institute of Mathematical Sciences, HBNI, Chennai, India

Subrahmanyam Kalyanasundaram ✉
Department of Computer Science and Engineering, Indian Institute of Technology Hyderabad, India

Rogers Mathew ✉
Department of Computer Science and Engineering, Indian Institute of Technology Hyderabad, India

Abstract

A *Conflict-Free Open Neighborhood coloring*, abbreviated CFON* coloring, of a graph $G = (V, E)$ using k colors is an assignment of colors from a set of k colors to a subset of vertices of $V(G)$ such that every vertex sees some color exactly once in its open neighborhood. The minimum k for which G has a CFON* coloring using k colors is called the *CFON* chromatic number* of G, denoted by $\chi^*_{ON}(G)$. The analogous notion for closed neighborhood is called CFCN* coloring and the analogous parameter is denoted by $\chi^*_{CN}(G)$. The problem of deciding whether a given graph admits a CFON* (or CFCN*) coloring that uses k colors is NP-complete. Below, we describe briefly the main results of this paper.

- For $k \geq 3$, we show that if G is a $K_{1,k}$-free graph then $\chi^*_{ON}(G)= O(k^2 \log \Delta)$, where Δ denotes the maximum degree of G. Dębski and Przybyło in [J. Graph Theory, 2021] had shown that if G is a line graph, then $\chi^*_{CN}(G)= O(\log \Delta)$. As an open question, they had asked if their result could be extended to claw-free ($K_{1,3}$-free) graphs, which are a superclass of line graphs. Since it is known that the CFCN* chromatic number of a graph is at most twice its CFON* chromatic number, our result positively answers the open question posed by Dębski and Przybyło.
- We show that if the minimum degree of any vertex in G is $\Omega(\frac{\Delta}{\log^\epsilon \Delta})$ for some $\epsilon \geq 0$, then $\chi^*_{ON}(G)= O(\log^{1+\epsilon} \Delta)$. This is a generalization of the result given by Dębski and Przybyło in the same paper where they showed that if the minimum degree of any vertex in G is $\Omega(\Delta)$, then $\chi^*_{ON}(G)= O(\log \Delta)$.
- We give a polynomial time algorithm to compute $\chi^*_{ON}(G)$ for interval graphs G. This answers in positive the open question posed by Reddy [Theoretical Comp. Science, 2018] to determine whether the CFON* chromatic number can be computed in polynomial time on interval graphs.
- We explore biconvex graphs, a subclass of bipartite graphs and give a polynomial time algorithm to compute their CFON* chromatic number. This is interesting as Abel et al. [SIDMA, 2018] had shown that it is NP-complete to decide whether a planar bipartite graph G has $\chi^*_{ON}(G) = k$ where $k \in \{1, 2, 3\}$.

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1 Introduction

A *Conflict-Free Open Neighborhood coloring*, abbreviated CFON* coloring, of a graph $G = (V, E)$ using k colors is an assignment of colors from a set of k colors to a subset of vertices of $V(G)$ such that every vertex sees some color exactly once in its open neighborhood. The minimum k for which G has a CFON* coloring using k colors is called the *CFON* chromatic number* of G, denoted by $\chi^*_{ON}(G)$.[1] The analogous notion for closed neighborhood is called CFCN* coloring and the analogous parameter is denoted by $\chi^*_{CN}(G)$. It is known (see for instance, Equation 1.3 from [26]) that if G has no isolated vertices, then $\chi^*_{CN}(G)$ is at most twice $\chi^*_{ON}(G)$. Given a graph G and integer $k > 0$, the *CFON* coloring problem* is the problem of determining if $\chi^*_{ON}(G) \leq k$. The CFON* variant is considered to be harder than the CFCN* variant, see for instance, remarks in [22, 26].

The notion of conflict-free coloring was introduced by Even, Lotker, Ron and Smorodinsky in 2004, motivated by the frequency assignment problem in wireless communication [14]. The conflict-free coloring problem on graphs was introduced and first studied by Cheilaris [8] and Pach and Tardos [26]. Conflict-free coloring has found applications in the area of sensor networks [17, 25] and coding theory [23]. Since its introduction, the problem has been extensively studied, see for instance [1, 3, 5, 6, 8, 18, 19, 26, 28]. The decision version of the CFON* coloring problem and many of its variants are known to be NP-complete [1,18]. In [18], Gargano and Rescigno showed that the optimization version of the CFON* coloring problem is hard to approximate within a factor of $n^{1/2-\epsilon}$, unless P = NP. Fekete and Keldenich [15] and Hoffmann et al. [21] studied a conflict-free variant of the chromatic Art Gallery Problem, which is about guarding a simple polygon P using a finite set of colored point guards such that each point $p \in P$ sees at least one guard whose color is distinct from all the other guards visible from p.

The conflict-free coloring problem has been studied on several graph classes like planar graphs, split graphs, geometric intersection graphs like interval graphs, unit disk intersection graphs and unit square intersection graphs, graphs of bounded degree, block graphs, etc. [1, 4, 6, 9, 16, 22, 26, 27]. The problem has been studied from parameterized complexity perspective. The problem is fixed-parameter tractable when parameterized by tree-width, neighborhood diversity, distance to cluster, or the combined parameters clique-width and the number of colors [2, 4, 6, 18, 27].

1.1 Our Contribution and Discussion

Below, we discuss the main results of this paper.

The complete bipartite graph $K_{1,3}$ is known as a *claw*. If a graph does not contain a claw as an induced subgraph, then it is called a *claw-free graph*. The *claw number* of a graph G is the largest integer k such that G contains an induced $K_{1,k}$. Dębski and Przybyło [10] showed that if G is a line graph with maximum degree Δ, then $\chi^*_{CN}(G) = O(\log \Delta)$. This bound is tight up to constants. Line graphs are a subclass of claw-free graphs. In [10], it was asked whether the above result can be extended to claw-free graphs. We do this by proving a more general result. We show that if G is $K_{1,k}$-free with maximum degree Δ, then

[1] It is also known by the name 'partial conflict-free chromatic number' as only a subset of vertices are assigned colors. The '(full) conflict-free chromatic number' of a graph, which requires assigning colors to all the vertices, is at most one more than its partial conflict-free chromatic number. We use the notations $\chi^*_{ON}(G)$ and $\chi^*_{CN}(G)$ to be consistent with our other papers on related topics. In our other papers, we use $\chi_{ON}(G)$ and $\chi_{CN}(G)$ to refer to the versions of the problem that require all the vertices to be assigned a color.

$\chi^*_{ON}(G)= O(k^2 \log \Delta)$. Since $\chi^*_{CN}(G) \leq 2\chi^*_{ON}(G)$, we have $\chi^*_{CN}(G)= O(k^2 \log \Delta)$ as well. This result is presented in Section 3.2.

What is the maximum number of colors required to CFON* color a graph whose maximum degree is Δ? It can be seen that the graph obtained by subdividing every edge of a complete graph requires $\Delta + 1$ colors. It is known that for a graph G with maximum degree Δ, $\chi^*_{ON}(G)$ is at most $\Delta + 1$ [26]. Pach and Tardos [26] showed that if the minimum degree of any vertex in G is $\Omega(\log \Delta)$, then $\chi^*_{ON}(G)= O(\log^2 \Delta)$. In this direction, Dębski and Przybyło [10] showed that if the minimum degree of any vertex in G is $\Omega(\Delta)$, then the previous upper bound can be improved to show $\chi^*_{ON}(G)= O(\log \Delta)$. We extend the proof idea of [10] to generalize their result. We show that if the minimum degree of any vertex in G is $\Omega(\frac{\Delta}{\log^{\epsilon} \Delta})$ for some $\epsilon \geq 0$, then $\chi^*_{ON}(G)= O(\log^{1+\epsilon} \Delta)$. This result is presented in Section 3.3. A natural open question we have here is, can we get a stronger upper bound for the CFON* chromatic number of a graph with minimum degree $\omega(1)$? When the minimum degree is $o(\log \Delta)$, the only upper bound known is $O(\Delta)$ mentioned above due to [26]. In this situation our first result does give a better (than $O(\Delta)$) upper bound for CFON* chromatic number, if the claw number of the graph under consideration is $o\left(\sqrt{\frac{\Delta}{\log \Delta}}\right)$.

For an interval graph G, it has been shown that [4, 27] $\chi^*_{ON}(G) \leq 3$. It was shown in [4] that there exists an interval graph that requires 3 colors, making the above bound tight. It was asked in [27] if there is a polynomial time algorithm that given an interval graph G, computes $\chi^*_{ON}(G)$. We answer this in the affirmative and give polynomial time characterization algorithms for interval graphs G that decide if $\chi^*_{ON}(G) \in \{1, 2, 3\}$. These results are presented in Section 4.

For a bipartite graph G, it is easy to see that $\chi^*_{CN}(G) \leq 2$. On the contrary, there exist bipartite graphs G, for which $\chi^*_{ON}(G) = \Theta(\sqrt{n})$. It is NP-complete [1] to decide if a planar bipartite graph is CFON* colorable using k colors, where $k \in \{1, 2, 3\}$. We study the problem on some subclasses of bipartite graphs that include chain graphs, biconvex bipartite graphs, and bipartite permutation graphs. We show that three colors are sufficient to CFON* color a biconvex bipartite graph and give characterization algorithms to decide the CFON* chromatic number. The results are presented in Section 5.

2 Preliminaries

Throughout the paper, we consider simple undirected graphs. We denote the vertex set and the edge set of a graph $G = (V, E)$, by $V(G)$ and $E(G)$. For standard graph notations, we refer to the graph theory book by R. Diestel [11]. For a vertex $v \in G$, its *open neighborhood*, denoted by $N_G(v)$, is the set of neighbors of v in G. The *closed neighborhood* of v, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. We use log to denote the logarithm to the base 2, and ln to denote the natural logarithm. Proofs of the results marked with ($\star$) are omitted due to space constraints.

3 Improved bounds for $\chi^*_{ON}(G)$ for graphs with bounded claw number

The graph $K_{1,k}$ is the complete bipartite graph on $k + 1$ vertices with one vertex in one part and the remaining k vertices in the other part.

▶ **Definition 1** (Claw number). *The* claw number *of a graph G is the smallest k such that G is $K_{1,k+1}$-free. In other words, it is the largest k such that G contains an induced $K_{1,k}$.*

The complete bipartite graph $K_{1,3}$ is called a *claw*. A graph is called a *claw-free graph* if it does not contain a claw as an induced subgraph.

In this section, we prove two results: (i) an improved bound for $\chi^*_{ON}(G)$ in terms of the claw number and maximum degree of G, and (ii) an improved bound for $\chi^*_{ON}(G)$ for graphs with high minimum degree. We begin by stating a couple of results from probability theory which will be useful.

▶ **Lemma 2** (*The Local Lemma*, [13]). *Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that $Pr[A_i] \leq p$ for all $i \in [n]$. If $4pd \leq 1$, then $Pr[\cap_{i=1}^{n} \overline{A_i}] > 0$.*

▶ **Theorem 3** (Chernoff Bound, Corollary 4.6 in [24]). *Let $X_1, \ldots, X_n$ be independent Poisson trials such that $Pr[X_i] = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. For $0 < \delta < 1$, $Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3}$.*

3.1 Auxiliary lemmas

In this subsection, we state some auxiliary lemmas on conflict-free chromatic number of graphs and hypergraphs having certain structural characteristics that will be used to prove the main theorems in Sections 3.2 and 3.3. Before we begin, let us define the conflict-free chromatic number of a hypergraph.

▶ **Definition 4.** *Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, a coloring $c : V \to [r]$ is a* conflict-free coloring *of $\mathcal{H}$ if for every hyperedge $E \in \mathcal{E}$, there is a vertex in E that receives a color under c that is distinct from the colors received by all the other vertices in E. The minimum r such that $c : V \to [r]$ is a conflict-free coloring of $\mathcal{H}$ is called the* conflict-free chromatic number *of $\mathcal{H}$. This is denoted by $\chi_{CF}(\mathcal{H})$.*

The following theorem on conflict-free coloring of hypergraphs is from [26]. The degree of a vertex in a hypergraph is the number of hyperedges it is part of.

▶ **Theorem 5** (Theorem 1.1(b) in [26]). *Let $\mathcal{H}$ be a hypergraph and let Δ be the maximum degree of any vertex in $\mathcal{H}$. Then, $\chi_{CF}(\mathcal{H}) \leq \Delta + 1$.*

We prove an upper bound for the conflict-free chromatic number of a 'near uniform hypergraph' in Lemma 6 below.

▶ **Lemma 6.** *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph where (i) every hyperedge intersects with at most Γ other hyperedges, and (ii) for every hyperedge $E \in \mathcal{E}$, $r \leq |E| \leq \ell r$, where $\ell \geq 1$ is some integer and $r \geq 2\log(4\Gamma)$. Then, $\chi_{CF}(\mathcal{H}) \leq e\ell r$, where e is the base of natural logarithm.*

Proof. For each vertex in V, assign a color that is chosen independently and uniformly at random from a set of $e\ell r$ colors. We will first show that the probability of this coloring being bad for an edge is small, and then use Local Lemma to show the existence of conflict-free coloring for $\mathcal{H}$ using at most $e\ell r$ colors.

Consider a hyperedge $E \in \mathcal{E}$ with $m := |E|$. By assumption, we have $r \leq m \leq \ell r$. Let A_E denote the bad event that E is colored with $\leq |E|/2$ colors. Note that if A_E does not occur, then E is colored with $> |E|/2$ colors, hence there is at least one color that appears exactly once in E.

$$
\begin{aligned}
Pr[A_E] &\leq \binom{e\ell r}{m/2}\left(\frac{m/2}{e\ell r}\right)^m \\
&\leq \left(\frac{e^2\ell r}{m/2}\right)^{m/2}\left(\frac{m/2}{e\ell r}\right)^m \qquad (\text{since } \binom{n}{k} \leq \left(\frac{en}{k}\right)^k) \\
&= \frac{(m/2)^{m/2}}{(\ell r)^{m/2}} \quad = \quad \left(\frac{m}{2\ell r}\right)^{m/2} \\
&\leq (1/2)^{m/2} \quad \leq \quad \frac{1}{4\Gamma}.
\end{aligned}
$$

Here the penultimate inequality follows since $m \leq \ell r$, and the last inequality follows since $m \geq 2\log(4\Gamma)$.

We apply the Local Lemma (Lemma 2) on the events A_E for all hyperedges $E \in \mathcal{E}$. Since each hyperedge intersects with at most Γ other hyperedges, and $4 \cdot \frac{1}{4\Gamma} \cdot \Gamma \leq 1$, we get $Pr[\cap_{E\in\mathcal{E}}(\overline{A}_E)] > 0$. That is, there is a conflict free coloring of $\mathcal{H}$ that uses at most $e\ell r$ colors. This completes the proof of the lemma. ◀

Lemmas 7 and 8 prove upper bounds for $\chi^*_{ON}(G)$ when G satisfies certain degree restrictions.

▶ **Lemma 7.** *Let G be a graph with (i) $V(G) = X \uplus Y$, $X, Y \neq \emptyset$, (ii) every vertex in G has at most d_X neighbors in X, (iii) every vertex in Y has at least one neighbor in X, and (iv) every vertex in X has at most d_Y neighbors in Y. Then, there is a coloring of vertices of X with $d_X d_Y + d_X - d_Y + 1$ colors such that every vertex in Y sees some color exactly once among its neighbors in X.*

Proof. For each vertex $y \in Y$, we arbitrarily choose one of its neighbors in X. Let us call this neighbor $f(y)$. For each $y \in Y$, contract the edges $\{y, f(y)\}$ to obtain a resulting graph G_X. Note that the vertex set of G_X is $V(G_X) = X$. The maximum degree of a vertex in the new graph G_X is at most $(d_X - 1)d_Y + d_X$. Thus, we can do a proper coloring (such that no pair of adjacent vertices receive the same color) of G_X using $d_X d_Y + d_X - d_Y + 1$ colors. We note that this coloring of the vertices of X satisfies our requirement: in the original graph G, for each $y \in Y$, the neighbor $f(y)$ is colored distinctly from all the other neighbors of y in X. ◀

▶ **Lemma 8.** *Let G be a graph with (i) $V(G) = X \uplus Y$, $X, Y \neq \emptyset$, (ii) every vertex in Y has at most t_X neighbors in X, and (iii) every vertex in X has at least one neighbor in Y. Then, there is a coloring of the vertices of Y using at most $(t_X + 1)$ colors such that every vertex in X sees some color exactly once among its neighbors in Y.*

Proof. For every vertex $v \in X$, let $N^Y_G(v)$ denote the set $N_G(v) \cap Y$, i.e., the neighbors of v in Y in the graph G. Since every vertex in X has at least one neighbor in Y, we have, $|N^Y_G(v)| \geq 1$. We construct a hypergraph $\mathcal{H} = (V, \mathcal{E})$ from G as described below. We have (i) $V = Y$, and (ii) $\mathcal{E} = \{N^Y_G(v) \ : \ v \in X\}$. Since every vertex in Y has at most t_X neighbors in X in the graph G, the maximum degree of a vertex in the hypergraph $\mathcal{H}$ (that is, the maximum number of hyperedges a vertex in $\mathcal{H}$ is part of) is at most t_X. From Theorem 5, we have $\chi_{CF}(\mathcal{H}) \leq t_X + 1$. Observe that in this coloring of the vertices of Y using at most $(t_X + 1)$ colors, every vertex in X sees some color exactly once among its neighbors in Y. ◀

The following lemma, which will be used in the proof of Theorem 12, shows that given a graph with high minimum degree there exists a subset of vertices that, for every vertex, intersects its neighborhood at a small number of vertices.

▶ **Lemma 9.** *Let Δ denote the maximum degree of a graph G. It is given that every vertex in G has degree at least $\frac{c\Delta}{\log^{\epsilon}\Delta}$ for some $\epsilon \geq 0$ and c is a constant. Then, there exists $A \subseteq V(G)$ such that for every vertex $v \in V(G)$,*

$$75\log(2\Delta) < |N_G(v) \cap A| < \frac{125}{c}\log^{1+\epsilon}(2\Delta).$$

Proof. We construct a random subset A of $V(G)$ as described below. Each $v \in V(G)$ is independently chosen into A with probability $\frac{100\log^{1+\epsilon}(2\Delta)}{c\Delta}$. For a vertex $v \in V(G)$, let X_v be a random variable that denotes $|N_G(v) \cap A|$. Then, $\mu_v := E[X_v] = \frac{100\log^{1+\epsilon}(2\Delta)}{c\Delta}d_G(v) \geq 100\log(2\Delta)$. Since $d_G(v) \leq \Delta$, we also have $\mu_v \leq \frac{100\log^{1+\epsilon}(2\Delta)}{c}$. Let B_v denote the event that $|X_v - \mu_v| \geq \frac{\mu_v}{4}$. Applying Theorem 3 with $\delta = 1/4$, we get $Pr[B_v] = Pr[|X_v - \mu_v| \geq \frac{\mu_v}{4}] \leq 2e^{-\frac{\mu_v}{48}} \leq 2e^{-\frac{100\log(2\Delta)}{48}} = 2e^{-\frac{100\ln(2\Delta)}{48\ln 2}} < \frac{2}{(2\Delta)^3}$. The event B_v is mutually independent of all but those events B_u where $N_G(u) \cap N_G(v) \neq \emptyset$. Hence, every event B_v is mutually independent of all but at most Δ^2 other events. Applying Lemma 2 with $p = Pr[B_v] \leq \frac{2}{(2\Delta)^3}$ and $d = \Delta^2$, we have $4 \cdot \frac{2}{(2\Delta)^3} \cdot \Delta^2 \leq 1$. Thus, there is a non-zero probability that none of the events B_v occur. In other words, for every v, it is possible to have $\frac{3}{4}\mu_v < X_v < \frac{5}{4}\mu_v$. Using the upper and lower bounds of μ_v we computed above, we can say that there exists an A such that, for every v, $75\log(2\Delta) < |N_G(v) \cap A| < \frac{125}{c}\log^{1+\epsilon}(2\Delta)$. ◀

3.2 Graphs with bounded claw number

▶ **Theorem 10.** *Let G be a $K_{1,k}$-free graph with maximum degree Δ having no isolated vertices. Then, $\chi^*_{ON}(G) = O(k^2\log\Delta)$.*

Proof. Consider a proper coloring (such that no pair of adjacent vertices receive the same color) of G, $h : V(G) \to [\Delta+1]$, using $\Delta+1$ colors. Let $C_1, C_2, \ldots, C_{\Delta+1}$ be the color classes given by this coloring G. That is, $V(G) = C_1 \uplus C_2 \uplus \cdots \uplus C_{\Delta+1}$ is the partitioning of the vertex set of G given by the coloring, where each C_i is an independent set. We may assume that the coloring h satisfies the following property: for every $1 < i \leq \Delta+1$, every vertex v in C_i has at least one neighbor in every C_j, where $1 \leq j < i$ (otherwise, we can move v to a color class C_j, $j < i$, in which it has no neighbors without compromising on the 'properness' of the coloring). Since G is $K_{1,k}$-free, we have the following observation.

▶ **Observation 11.** *For every $i \in [\Delta+1]$, a vertex in G has at most $k-1$ neighbors in C_i.*

Let $r = 2\log(4\Delta^2)$. We partition the vertex set of G into three parts, namely V_1, V_2, and V_3 as described below. We have $V_1 := C_1$. If $\Delta > r$, then $V_2 := C_2 \uplus C_3 \uplus \cdots \uplus C_{r+1}$ and $V_3 := C_{r+2} \uplus C_{r+3} \uplus \cdots \uplus C_{\Delta+1}$. Otherwise, $V_2 := C_2 \uplus C_3 \uplus \cdots \uplus C_{\Delta+1}$ and $V_3 := \emptyset$.

The rest of the proof is about constructing a coloring $f : V(G) \to \mathbb{N} \times \mathbb{N}$ that is a CFON* coloring of G. Let $N_1 = \{1, 2, \ldots, r_1\}$, $N_2 = \{r_1+1, r_1+2, \ldots, r_1+r_2\}$, and $N_3 = \{r_1+r_2+1, r_1+r_2+2, \ldots, r_1+r_2+r_3\}$, where $|N_1| = r_1 = (k-1)(k-2)r + k$, $|N_2| = r_2 = e(k-1)r$, and $|N_3| = r_3 = k$. We define three colorings f_1, f_2, and f_3 below.

We begin by describing the coloring $f_1 : V_1 \to N_1$. Let $G[V_1 \cup V_2]$ be the subgraph of G induced on $V_1 \cup V_2$. From Observation 11, every vertex in $G[V_1 \cup V_2]$ has at most $k-1$ neighbors in $V_1 = C_1$. Every vertex in V_2 has at least one neighbor in V_1 due to the property of our coloring h. From Observation 11, we can also say that every vertex in V_1 has at most

$r(k-1)$ neighbors in V_2. Applying Lemma 7 on $G[V_1 \cup V_2]$ with $X = V_1$, $Y = V_2$, $d_X = k-1$ and $d_Y = r(k-1)$, we can say that there is a coloring $f_1 : V_1 \to N_1$ of the vertices of V_1 with $(k-1)(k-2)r + k$ colors such that every vertex in V_2 sees some color exactly once among its neighbors in V_1.

We now describe the coloring $f_2 : V_2 \to N_2$. If $V_3 = \emptyset$, then, $\forall v \in V_2$, $f_2(v) = r_1 + 1$. Suppose $V_3 \neq \emptyset$. For a vertex v in G, let $N_G^{V_2}(v)$ denote the set of neighbors of v in V_2 in the graph G. We construct a hypergraph $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ as follows. We have $\mathcal{E}_2 = \{N_G^{V_2}(v) \; : \; v \in V_3\}$. Consider an arbitrary hyperedge $E \in \mathcal{E}_2$. In the graph G, since every vertex in V_3 has at least one neighbor in every color class C_i, $2 \leq i \leq r+1$, $|E| \geq r$. Using Observation 11, we can say that $|E| \leq (k-1)r$. As $|N_G^{V_2}(v)| \leq N_G(v) \leq \Delta, \forall v \in V(G)$, we have $|E| \leq \Delta$. This also implies that E intersects with at most Δ^2 other hyperedges in $\mathcal{E}_2$. Applying Lemma 6 with $\ell = (k-1)$ and $\Gamma = \Delta^2$, we have $\chi_{CF}(\mathcal{H}_2) \leq e(k-1)r$. Thus, there is a coloring $f_2 : V_2 \to N_2$ of the vertices V_2 such that every vertex in V_3 sees some color exactly once among its neighbors in V_2.

Finally, we describe the coloring $f_3 : V_2 \cup V_3 \to N_3$. From Observation 11, every vertex in $V_2 \cup V_3$ has at most $k-1$ neighbors in $V_1 = C_1$. Since there are no isolated vertices in G, every vertex in V_1 has at least one neighbor in $V_2 \cup V_3$. Applying Lemma 8 with $X = V_1$, $Y = V_2 \cup V_3$, and $t_X = k-1$, we get a coloring $f_3 : V_2 \cup V_3 \to N_3$ of the vertices of $V_2 \cup V_3$ using at most k colors such that every vertex in V_1 sees some color exactly once among its neighbors in $V_2 \cup V_3$.

We are now ready to define the coloring f.

$$f(v) = \begin{cases} (1, f_1(v)), \text{ if } v \in V_1 \\ (f_2(v), f_3(v)), \text{ if } v \in V_2 \\ (1, f_3(v)), \text{ if } v \in V_3 \end{cases}.$$

We now argue that f is indeed a CFON* coloring of G. Consider a vertex $v \in V(G)$. If $v \in V_3$, v sees some color exactly once among its neighbors in V_2 under the coloring f_2. Let u be that neighbor of v in V_2 and $f_2(u)$ be that color that appears exactly once in the neighborhood of v in V_2. Since the codomains of f_1, f_2, and f_3 are pairwise disjoint sets, v does not see the same color among its neighbors in V_1 or in V_2. Further, since $f(u) = (f_2(u), f_3(u))$, the final coloring f only refines the color classes of V_2 given by f_2. Thus, the color $(f_2(u), f_3(u))$ appears exactly once among the neighbors of v in G. The cases when $v \in V_1$ and $v \in V_2$ also follow using similar arguments.

The coloring f uses at most $|N_1| + |N_2||N_3| + |N_3| = (k-1)(k-2)r + k + e(k-1)kr + k$ colors. Since $r = O(\log \Delta)$, this implies that $\chi_{CF}^{ON}(G) = O(k^2 \log \Delta)$. ◀

3.3 Graphs with high minimum degree

When a graph G has high minimum degree, the following theorem gives improved upper bounds for $\chi_{ON}^*(G)$ in terms of its maximum degree.

▶ **Theorem 12.** *Let G be a graph with maximum degree Δ. It is given that every vertex in G has degree at least $\frac{c\Delta}{\log^\epsilon \Delta}$ for some $\epsilon \geq 0$ and c is a constant. Then, $\chi_{ON}^*(G) = O(\log^{1+\epsilon} \Delta)$.*

Proof. Apply Lemma 9 to find an $A \subseteq V(G)$ such that for every $v \in V(G)$, $75\log(2\Delta) < |N_G(v) \cap A| < \frac{125}{c}\log^{1+\epsilon}(2\Delta)$. Construct a hypergraph $\mathcal{H} = (A, \mathcal{E})$ where $\mathcal{E} = \{N_G(v) \cap A \; : \; v \in V(G)\}$. Every $E \in \mathcal{E}$ satisfies $2\log(4\Delta^2) < 75\log(2\Delta) < |E| < \frac{125}{c}\log^{1+\epsilon}(2\Delta)$. Applying Lemma 6 with $r = 75\log(2\Delta)$ and $\ell = \frac{5}{3c}\log^\epsilon(2\Delta)$, we get $\chi_{CF}(\mathcal{H}) \leq \frac{340}{c}\log^{1+\epsilon}(2\Delta)$. It is easy to see that this conflict-free coloring of $\mathcal{H}$ is indeed a CFON* coloring for G. ◀

4 Interval graphs

In this section, we show that the problem of determining the CFON* chromatic number of a given interval graph is polynomial time solvable. It was shown in [4, 27] that, for an interval graph G, $\chi^*_{ON}(G) \leq 3$ and that there exists an interval graph that requires three colors. The complexity of the problem on interval graphs was posed as an open question in the above papers. We show that CFON* coloring is polynomial time solvable. That is, given an interval graph G, in polynomial time we decide whether $\chi^*_{ON}(G)$ is 1, 2 or 3. We state it formally below.

▶ **Theorem 13.** *Given an interval graph G, there is a polynomial time algorithm that determines $\chi^*_{ON}(G)$.*

▶ Remark 14 (Notation). In the introduction, we defined CFON* coloring to be an assignment of colors to a *subset* of the vertices. For the sake of convenience, we will use the color 0 to denote uncolored vertices. That is, we will use an assignment $f : V(G) \to \{0, 1, 2\}$, to denote a coloring that assigns the colors 1 and 2 to some vertices. The vertices that are assigned 0 by f are the "uncolored" vertices. The "color" 0 cannot serve as a unique color in the neighborhood of any vertex.

▶ **Definition 15** (Interval Graphs). *A graph $G = (V, E)$ is called an* interval graph *if there exists a set of intervals on the real line such that the following holds: (i) there is a bijection between the intervals and the vertices and (ii) there exists an edge between two vertices if and only if the corresponding intervals intersect.*

The main ingredient of the algorithm is the use of *multi-chain ordering* property on interval graphs. Before defining the multi-chain ordering property, we look at some prerequisites.

▶ **Definition 16** (Chain Graph [12]). *A bipartite graph $G = (A, B)$ is a* chain graph *if and only if for any two vertices $u, v \in A$, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. If G is a chain graph, it follows that for any two vertices $u, v \in B$, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.*

As a consequence, we can order the vertices in B in the decreasing order of the degrees. We can break ties arbitrarily. If $b_1 \in B$ appears before $b_2 \in B$ in the ordering, then it follows that $N(b_2) \subseteq N(b_1)$.

▶ **Definition 17** (Multi-chain Ordering [7, 12]). *Given a connected graph $G = (V, E)$, we arbitrarily choose a vertex as $v_0 \in V(G)$ and construct distance layers $L_0, L_1, \ldots, L_p$ from v_0. The layer L_i, where $i \in [p]$, represents the set of vertices that are at a distance i from v_0. Note that p here denotes the largest integer such that L_p is non-empty.*

We say that these layers form a multi-chain ordering *of G if for every two consecutive layers L_i and L_{i+1}, where $i \in \{0, 1, \ldots, p-1\}$, we have that the vertices in L_i and L_{i+1}, and the edges connecting these layers form a chain graph.*

▶ **Theorem 18** (Theorem 2.5 of [12]). *All connected interval graphs admit multi-chain orderings.*

We give a characterization of interval graphs that require one color and two colors in polynomial time in Theorem 21 and Theorem 23 respectively. Given an interval graph G, the algorithms decide if G is CFON* colorable using one color or two colors. If G is not CFON* colorable using one color or two colors, we conclude that G is CFON* colorable using three colors (since it is known that for an interval graph G, $\chi^*_{ON}(G) \leq 3$). One of the key ideas used in Theorem 23 (to decide if G can be CFON* colored using two nonzero colors) is sort of a bootstrapping idea. After narrowing down the possibilities, we need to test if a given subgraph can be colored using the colors $\{0,1\}$ so as to obtain a CFON* coloring. To solve this, we use Theorem 21.

Before we proceed to the main theorems of this section, we observe the following on a graph G that admits multi-chain ordering.

▶ **Observation 19.** *If G admits a multi-chain ordering, then every distance layer L_i, for $0 \leq i < p$ contains a vertex v such that $N(v) \supseteq L_{i+1}$.*

Proof. Consider a multi-chain ordering of G, starting with an arbitrary vertex. For any two consecutive distance layers L_i and L_{i+1}, it can be seen that each vertex in L_{i+1} has a neighbor in L_i. This, together with the fact that L_i and L_{i+1} form a chain graph, imply that there is a vertex $v \in L_i$ such that $N(v) \supseteq L_{i+1}$. ◀

▶ **Observation 20.** *In any CFON* coloring of G that uses one color, at most one vertex in each L_i is assigned the color 1.*

Proof. Consider a layer L_i of the graph. As per Observation 19, there is a $v \in L_i$ such that $N(v) \supseteq L_{i+1}$. If two vertices in L_{i+1} are colored 1, then the vertex $v \in L_i$ does not have a uniquely colored neighbor. Hence in all the layers $L_1, L_2, \ldots$ up to the last layer L_p, we have that at most one vertex is assigned the color 1. Since L_0 has only one vertex, the statement is trivially true for L_0. ◀

▶ **Theorem 21.** *Given an interval graph $G = (V, E)$, we can decide in $O(n^5)$ time if $\chi^*_{ON}(G) = 1$.*

Proof. Let $L_0, L_1, \ldots, L_p$ be the distance layers of G constructed from an arbitrarily chosen vertex v_0, satisfying the multi-chain ordering. If there is a CFON* coloring that uses 1 color, then from Observation 20, at most one vertex in each layer is assigned the color 1. There are two possibilities for a layer L_i: either it has no vertices colored 1, or it has exactly one vertex that is colored 1. In the former case, there is a unique coloring for L_i when none of the vertices in L_i are assigned the color 1. In the latter case, we have $|L_i|$ many colorings (for L_i) where each coloring has exactly one vertex with color 1 (and the rest are assigned 0). In total, we have at most $|L_i| + 1$ colorings for each L_i. We call all such colorings *valid*.

The task is to find if there is a sequence of colorings assigned to each layer of G such that we have a CFON* coloring. Notice that the vertices in L_i can possibly have neighbors in the layers L_{i-1}, L_i, and L_{i+1}. The question of deciding whether the vertices in L_i have a uniquely colored neighbor entirely depends on the colorings assigned to these three layers. We say that colorings assigned to three consective layers are *good* if the vertices in the central layer have uniquely colored neighbors. We use a dynamic programming based approach to verify the existence of a CFON* coloring for G.

We now construct a layered companion hypergraph $\mathcal{G} = (V', \mathcal{E})$ with vertices in $p + 1$ layers. Each layer T_i of $\mathcal{G}$ corresponds to the layer L_i of G where $i \in [p] \cup \{0\}$. Each vertex in layer T_i of $\mathcal{G}$ corresponds to a valid coloring of vertices in L_i of G. Hence the number of vertices in each layer T_i of $\mathcal{G}$ is equal to $|L_i| + 1$. We now explain how the hyperedges $\mathcal{E}$ of $\mathcal{G}$ are determined.

For $1 \leq i \leq p-1$, the vertices $x \in T_{i-1}$, $y \in T_i$, $z \in T_{i+1}$ form a hyperedge $\{x, y, z\}$ if the corresponding colorings, when assigned to L_{i-1}, L_i and L_{i+1} respectively, ensures that every vertex in L_i has a uniquely colored neighbor. We also have hyperedges $\{y, z\}$, where $y \in T_0$ and $z \in T_1$ are colorings such that when y and z are assigned to L_0 and L_1 respectively, the vertex in L_0 sees a uniquely colored neighbor. Similarly, we have hyperedges $\{x, y\}$, where

$x \in T_{p-1}$ and $z \in T_p$ are colorings such that when x and y are assigned to L_{p-1} and L_p respectively, all the vertices in L_p see a uniquely colored neighbor.

Since the number of valid colorings is $|L_i| + 1$ for the layer L_i, the total number of valid colorings across all layers is at most $2n$. The total number of potential hyperedges to check is at most $O(n^3)$. Once we fix valid colorings x_{i-1}, x_i, x_{i+1} for L_{i-1}, L_i, L_{i+1} respectively, we can check in $O(|L_i| \cdot n) \leq O(n^2)$ time if $\{x_{i-1}, x_i, x_{i+1}\} \in \mathcal{E}$. Hence we need $O(n^5)$ time to construct $\mathcal{G}$.

To obtain a CFON* coloring for G, we need to construct a sequence of colorings $x_0 \in T_0$, $x_1 \in T_1, \ldots, x_p \in T_p$ such that $\{x_0, x_1\} \in \mathcal{E}$, $\{x_{i-1}, x_i, x_{i+1}\} \in \mathcal{E}$ for all $1 \leq i \leq p-1$, and finally $\{x_{p-1}, x_p\} \in \mathcal{E}$. For this, we use Lemma 22, stated and proved below. Since each $|T_i| = |L_i| + 1 \leq n + 1$, and number of layers is at most n, this takes at most $O(n^4)$ time. The construction of $\mathcal{G}$ takes $O(n^5)$ time and dominates the running time. ◀

▶ **Lemma 22.** *Suppose there is a layered hypergraph $\mathcal{G} = (V', \mathcal{E})$ with layers $T_0, T_1, T_2, \ldots, T_p$, where $|T_i| \leq \alpha$, for $0 \leq i \leq p$ and $p \leq \beta$. Suppose further that all the hyperedges in $\mathcal{E}$ contain one vertex each from three consecutive layers, or contain one vertex each from T_0 and T_1, or contain one vertex each from T_{p-1} and T_p. We can determine if there exists a sequence $x_0 \in T_0$, $x_1 \in T_1, \ldots, x_p \in T_p$ such that $\{x_0, x_1\} \in \mathcal{E}$, $\{x_{i-1}, x_i, x_{i+1}\} \in \mathcal{E}$ for all $1 \leq i \leq p-1$, and finally $\{x_{p-1}, x_p\} \in \mathcal{E}$ in $O(\alpha^3\beta)$ time.*

Proof. We start with the vertices in T_0. For each vertex $x_1 \in T_1$, we store a list of predecessors x_0 such that $\{x_0, x_1\} \in \mathcal{E}$. For $1 \leq i \leq p-1$, we do the following at each vertex $x_i \in T_i$. We look at the list of predecessors stored. If x_{i-1} is a listed predecessor of x_i, then we search for all the hyperedges $\{x_{i-1}, x_i, z\}$, where $z \in T_{i+1}$. If we find such a hyperedge $\{x_{i-1}, x_i, x_{i+1}\} \in \mathcal{E}$, then we store x_i as a predecessor in the list at x_{i+1}. Finally, for each $x_p \in T_p$, we check if there is a listed predecessor $z \in T_{p-1}$ of x_p such that $\{z, x_p\} \in \mathcal{E}$. If there is any such $x_p \in T_p$ for which this holds, then there exists a sequence as desired in the statement of the lemma.

Note that the general step involves going through a list of size at most α at each vertex x_i. For each listed predecessor x_{i-1}, there are potentially at most α hyperedges of the form $\{x_{i-1}, x_i, z\}$ to check, where $z \in T_{i+1}$. We need to do this for all the vertices (at most α of them) of T_i. This gives a time complexity of $O(\alpha^3)$ at the i-th layer. Since there are β layers, the total running time is $O(\alpha^3\beta)$. ◀

We now proceed to the next result that decides in polynomial time whether $\chi_{ON}(G) = 2$.

▶ **Theorem 23** ($\star$). *Given an interval graph G, we can decide in $O(n^{20})$ time if $\chi_{ON}(G) = 2$.*

Sketch of Proof. The idea of this proof is similar to the proof of Theorem 21. For a layer $|L_i|$, we had $|L_i| + 1$ colorings to consider in Theorem 21. Unlike in Theorem 21, we have more colorings to consider since the vertices can get the colors $\{0, 1, 2\}$. We have the following types of colorings in each layer L_i:

Type 1: All the vertices in L_i are assigned the color 0. There is only one coloring of L_i of this type.

Type 2: Exactly one vertex is assigned the color 1 or 2 while the rest are assigned the color 0. The number of colorings is $2|L_i|$.

Type 3: Both the colors 1 and 2 appear exactly once and the rest are assigned the color 0. The number of colorings is $|L_i|(|L_i| - 1) \leq |L_i|^2$.

Type 4: One of the colors 1 or 2 appears at least twice while the other color appears exactly once. The remaining vertices are assigned the color 0.

Type 5: One of the colors 1 or 2 appears at least twice and all the other vertices are assigned the color 0.

Due to space constraints, the full proof is omitted. We describe a proof sketch highlighting the key ideas in the proof below.

- The above 5 types are exhaustive. We cannot have a "Type 6" coloring in L_{i+1} where there are at least two vertices with color 1 and at least two vertices with color 2. This is because Observation 19 implies the existence of a vertex $v \in L_i$ such that $N(v) \supseteq L_{i+1}$. This implies that v does not have a uniquely colored neighbor for such a coloring of L_{i+1}.
- The number of colorings of Types 1, 2, 3 are polynomial in $|L_i|$ while the number of colorings of Types 4 and 5 are exponential in $|L_i|$. Since we cannot consider an exponential number of colorings, we consider a polynomial subset of Type 4 and Type 5 colorings which are representatives of all possible Type 4 and Type 5 colorings.
- Given a Type 4 or Type 5 coloring, the key point is that it is enough to fix the colors of a few vertices that we will refer to as "left-important" and "right-important" vertices. This allows us to restrict the focus onto a reduced number of representative colorings.
- Because of the flexibility offered by the representative colorings, there are some cases where we have to explore further in order to decide if the graph is CFON* colorable using colors from $\{0, 1, 2\}$. This reduces to the problem of testing whether a given subgraph is CFON* colorable using colors from $\{0, 1\}$. We use Theorem 21 (with some minor changes) to accomplish this. This is the last, but critical step that we need to complete the proof.

◀

Using Theorems 21 and 23, we can now infer Theorem 13.

▶ Remark 24. Recently, the work of Gonzalez and Mann [20] (done simultaneously and independently from ours) on mim-width showed that the CFON* coloring problem is polynomial-time solvable on graph classes for which a branch decomposition of constant mim-width can be computed in polynomial time. This includes the class of interval graphs. We note that our work gives a more explicit algorithm without having to go through the machinery of mim-width. We also note that the mim-width algorithm, as presented in [20], requires a running time in excess of $\Omega(n^{300})$. Hence our algorithm is better in this regard as well.

5 Subclasses of Bipartite Graphs

It is known that there exist bipartite graphs G for which $\chi^*_{ON}(G) = \Theta(\sqrt{n})$, where n is the number of vertices of G. Abel et al. [1] showed that it is NP-complete to decide if k colors are sufficient to CFON* color a planar bipartite graph even when $k \in \{1, 2, 3\}$. This implies that CFON* coloring is NP-hard on bipartite graphs as well. In this section, we study CFON* coloring on some subclasses of bipartite graphs namely biconvex graphs and bipartite permutation graphs. We show that CFON* coloring is polynomial time solvable on these classes.

We first define biconvex graphs, followed Lemma 26 by a bound on the CFON* chromatic number. The proof of Lemma 26 is omitted.

▶ **Definition 25** (Biconvex Graph). *We say that an ordering σ of X in a bipartite graph $B = (X, Y, E)$ satisfies the* adjacency property *if for every vertex $y \in Y$, the neighborhood $N(y)$ is a set of vertices that are consecutive in the ordering σ of X. A bipartite graph*

(X, Y, E) *is* biconvex *if there are orderings of* X *(with respect to* Y*) and* Y *(with respect to* X*) that fulfill the adjacency property.*

▶ **Lemma 26** ($\star$). *If* G *is a biconvex graph, then* $\chi^*_{ON}(G) \leq 3$.

▶ **Theorem 27.** *The problem of determining the CFON* chromatic number of a given biconvex graph is solvable in polynomial time.*

Proof. Given a biconvex graph G, we show that $\chi^*_{ON}(G) \leq 3$. We use the fact that every induced subgraph of a biconvex graph admits multi-chain ordering [7, 12]. Let $G = (V, E)$ be a biconvex graph and let $V_0, V_1, \ldots, V_q$ be a partition of vertices $V(G)$ respecting the multi-chain ordering conditions. Similar to interval graphs, we now characterize graphs that require one color and two colors. Note that the algorithms in Theorems 21 and 23 work for biconvex graphs too as the proof is based on the multi-chain ordering property and biconvex bipartite graphs admit multi-chain ordering property. In fact, the proof is a bit simpler because of the fact that each V_i is an independent set and we do not need to take care of the edges within a part V_i, as in the case of interval graphs. ◀

The class of bipartite permutation graphs [7] are a subclass of biconvex, and also admit multi-chain ordering property. Hence it follows from Theorem 27 that the problem is polynomial time solvable on bipartite permutation graphs.

▶ **Corollary 28.** *The problem of determining the CFON* chromatic number of a given bipartite permutation graph is solvable in polynomial time.*

6 Conclusion

In this paper, we study CFON* coloring on claw-free graphs, interval graphs and biconvex graphs.

We first show that if G is a $K_{1,k}$-free graph with maximum degree Δ, then $\chi^*_{ON}(G) = O(k^2 \log \Delta)$. We then show that if the minimum degree of G is $\Omega(\frac{\Delta}{\log^{\epsilon} \Delta})$ for some $\epsilon \geq 0$, then $\chi^*_{ON}(G) = O(\log^{1+\epsilon} \Delta)$. The tightness of these bounds is a natural open question.

We show that CFON* coloring is polynomial time solvable on interval graphs and biconvex graphs, critically using the fact that they admit multi-chain ordering property. Using a similar approach, it can be shown that the full coloring variant of the problem (i.e., CFON coloring) is polynomial time solvable on these graph classes. It is known that CFON* coloring is NP-hard on planar bipartite graphs and there exist bipartite graphs on n vertices that requires $\Theta(\sqrt{n})$ colors. It may be of interest to study the problem on other subclasses of bipartite graphs, such as convex bipartite graphs, chordal bipartite graphs and tree-convex bipartite graphs.

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