Conflict-Free Coloring: Graphs of Bounded Clique Width and Intersection Graphs

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Abstract. Given an undirected graph, a conflict-free coloring (CFON*) is an assignment of colors to a subset of the vertices of the graph such that for every vertex there exists a color that is assigned to exactly one vertex in its open neighborhood. The minimum number of colors required for such a coloring is called the conflict-free chromatic number. The decision version of the CFON* problem is NP-complete even on planar graphs. In this paper, we show the following results.

- The CFON* problem is fixed-parameter tractable with respect to the combined parameters clique width and the solution size.
- We study the problem on block graphs and cographs, which have bounded clique width. For both graph classes, we give tight bounds of three and two respectively for the CFON* chromatic number.
- We study the problem on the following intersection graphs: interval graphs, unit square graphs and unit disk graphs. We give tight bounds of two and three for the CFON* chromatic number for proper interval graphs and interval graphs. Moreover, we give upper bounds for the CFON* chromatic number on unit square and unit disk graphs.
- We also study the problem on split graphs and Kneser graphs. For split graphs, we show that the problem is NP-complete. For Kneser graphs $K(n,k)$, when $n \geq k(k+1)^2+1$, we show that the CFON* chromatic number is $k+1$.

We also study the closed neighborhood variant of the problem denoted by CFCN*, and obtain analogous results in some of the above cases.

1 Introduction

Given an undirected graph $G=(V,E)$, a *conflict-free coloring* is an assignment of colors to a subset of the vertices of G such that every vertex in G has a uniquely colored vertex in its neighborhood. The minimum number of colors required for such a coloring is called the *conflict-free chromatic number*. This problem was introduced in 2002 by Even, Lotker, Ron and Smorodinsky [8], motivated by the frequency assignment problem in cellular networks where base stations and clients communicate with one another. To avoid interference, we require that there exists a base station with a unique frequency in the neighborhood of each client. Since the number of frequencies are limited and expensive, it is ideal to minimize the number of frequencies used.

This problem has been well studied [1,5,11,16,18] for nearly 20 years. Several variants of the problem have been studied. We focus on the following variant of the problem with respect to both closed and open neighborhoods, which are defined as follows.

Definition 1 (Conflict-Free Coloring). *A* CFON* coloring *of a graph $G = (V, E)$ using k colors is an assignment $C : V(G) \rightarrow \{0\} \cup \{1, 2, \ldots, k\}$ such that for every $v \in V(G)$, there exists a color $i \in \{1, 2, \ldots, k\}$ such that $|N(v) \cap C^{-1}(i)| = 1$. The smallest number of colors required for a CFON* coloring of G is called the CFON* chromatic number of G, denoted by $\chi^*_{ON}(G)$.*

The closed neighborhood variant, CFCN* coloring*, is obtained by replacing the open neighborhood $N(v)$ by the closed neighborhood $N[v]$ in the above. The corresponding chromatic number is denoted by $\chi^*_{CN}(G)$.*

In the above definition, vertices assigned the color 0 are treated as "uncolored". Hence in a CFON* coloring (or CFCN* coloring), no vertex can have a vertex colored 0 as its uniquely colored neighbor. The *CFON* problem* (resp. *CFCN* problem*) is to compute the minimum number of colors required for a CFON* coloring (resp. CFCN* coloring) of a graph. Abel et al. in [1] showed that both the problems are NP-complete even for planar graphs. They also showed that eight colors are sufficient to CFON* color planar graphs, which was improved to four colors [12]. Further these problems have been studied on outerplanar graphs [4], and intersection graphs like string graphs, circle graphs [13], disk graphs, square graphs and interval graphs [9]. Continuing this line of work, we study these problems on various restricted graph classes such as block graphs, cographs, intervals graphs, unit square graphs, unit disk graphs, Kneser graphs and split graphs.

The parameterized complexity of conflict-free coloring, for both neighborhoods, has been of recent research interest. They are fixed-parameter tractable (FPT) when parameterized by tree width [2, 5], distance to cluster (distance to disjoint union of cliques) [17] and neighborhood diversity [11]. Further, with respect to distance to threshold graphs there is an additive approximation algorithm in FPT-time [17].[4]

We study CFON* and CFCN* problems for the parameter clique width, which generalizes all the above parameters. Specifically, for every graph G, $cw(G) \leq 3 \cdot 2^{tw(G)-1}$, where $tw(G)$ and $cw(G)$ denote the tree width of G and the clique width of G respectively [7]. Graphs with distance to cluster at most $k \in \mathbb{N}$, have clique width of at most $O(2^k)$ [19]. We show that the CFON* and CFCN* problems are FPT with respect to the combined parameters clique

[4] Some of the above FPT results are shown for the "full-coloring variant" of the problem (as defined in Definition 2). Our clique width result can also be adapted for the full-coloring variant.

width and the number of colors used. Note that the previously mentioned FPT-results [2, 5, 11, 17] do not additionally need the solution size as a parameter.

1.1 Results

- In Section 3, we show fixed-parameter tractable algorithms for both CFON* CFCN* problems with respect to the combined parameters clique width w and the solution size k, that runs in $2^{O(w3^k)}n^{O(1)}$ time where n is the number of vertices of G.
- In Section 4, we discuss the results on block graphs and cographs. Both the graph classes are solvable in polynomial time, which follows from the clique width result.
  - For block graphs G, we show that $\chi^*_{ON}(G) \leq 3$. We show a block graph G that requires three colors making the above bound tight.
  - For cographs, we show that two colors are sufficient for a CFON* coloring. We also characterize cographs for which one color suffices.
- In Section 5, we show that for interval graphs G, $\chi^*_{ON}(G) \leq 3$. We show an interval graph that requires three colors making the above bound tight. Moreover, two colors are sufficient to CFON* color proper interval graphs. We also show that the CFCN* problem is polynomial time solvable on interval graphs.
- In Section 6, we study the problem on geometric intersection graphs like unit square graphs and unit disk graphs.
  We show that $\chi^*_{ON}(G) \leq 27$ for unit square graphs G. For unit disk graphs G, we show that $\chi^*_{ON}(G) \leq 51$. No upper bound was previously known.
- In Section 7, we study both the problems on Kneser graphs and split graphs.
  - We show that $k+1$ colors are sufficient to CFON* color the Kneser graphs $K(n,k)$, when $n \geq 3k-1$. We also show that $\chi^*_{ON}(K(n,k)) \geq k+1$ when $n \geq k(k+1)^2+1$, thereby proving that $\chi^*_{ON}(K(n,k)) = k+1$ when $n \geq k(k+1)^2+1$.
    We also show that k colors are sufficient to CFCN* color a Kneser graph $K(n,k)$, when $n \geq 2k+1$.
  - On split graphs, we show that the CFON* problem is NP-complete and the CFCN* problem is polynomial time solvable.

2 Preliminaries

Throughout the paper, we assume that the graph G is connected. Otherwise, we apply the algorithm on each component independently. We also assume that G does not contain any isolated vertices as the CFON* problem is not defined for an isolated vertex. We use $[k]$ to denote the set $\{1, 2, \ldots, k\}$ and $C : V(G) \rightarrow \{0\} \cup [k]$ to denote the color assigned to a vertex. A *universal vertex* is a vertex that is adjacent to all other vertices of the graph. In some of our algorithms and proofs, it is convenient to distinguish between vertices that are intentionally left uncolored, and the vertices that are yet to be assigned any color. The assignment of color 0 is used to denote that a vertex is left “uncolored”.

To avoid clutter and to simplify notation, we use the shorthand notation vw to denote the edge $\{v, w\}$. The open neighborhood of a vertex $v \in V(G)$ is the set of vertices $\{w : vw \in E(G)\}$ and is denoted by $N(v)$. Given a conflict-free coloring C, a vertex $w \in N(v)$ is called a *uniquely colored neighbor* of v if $C(w) \neq 0$ and $\forall x \in N(v) \setminus \{w\}, C(w) \neq C(x)$. The closed neighborhood of v is the set $N(v) \cup \{v\}$, denoted by $N[v]$. The notion of uniquely colored neighbor in the closed neighborhood variant is analogous to the open neighborhood variant, and is obtained by replacing $N(v)$ by $N[v]$. We sometimes use the mapping $h : V \to V$ to denote the uniquely colored neighbor of a vertex. We also extend C for vertex sets by defining $C(V') = \bigcup_{v \in V'} C(v)$ for $V' \subseteq V(G)$. To refer to the multi-set of colors used in V', we use $C_{\{\!\{\}\!\}}(V')$. The difference between $C_{\{\!\{\}\!\}}(V')$ and $C(V')$ is that we use multiset union in the former.

In many of the sections, we also refer to the full coloring variant of the conflict-free coloring problem, which is defined below.

Definition 2 (Conflict-Free Coloring – Full Coloring Variant). *A* CFON coloring *of a graph $G = (V, E)$ using k colors is an assignment $C : V(G) \to \{1, 2, \ldots, k\}$ such that for every $v \in V(G)$, there exists an $i \in \{1, 2, \ldots, k\}$ such that $|N(v) \cap C^{-1}(i)| = 1$. The smallest number of colors required for a CFON coloring of G is called the CFON chromatic number of G, denoted by $\chi_{ON}(G)$.*

The corresponding closed neighborhood variant is denoted CFCN coloring, and the chromatic number is denoted $\chi_{CN}(G)$.

A full conflict-free coloring, where all the vertices are colored with a non-zero color, is also a partial conflict-free coloring (as defined in Definition 1) while the converse is not true. It is clear that one extra color suffices to obtain a full coloring variant from a partial coloring variant. However, it is not always clear if the extra color is actually necessary.

For the theorems marked ($\star$), the full proofs are omitted due to space constraints.

3 FPT with Clique Width and Number of Colors

In this section, we study the conflict-free coloring problem with respect to the combined parameters clique width $cw(G)$ and number of colors k. We present FPT algorithms for both the CFON* and CFCN* problems.

Definition 3 (Clique width [7]). *Let $w \in \mathbb{N}$. A w-expression Φ defines a graph G_Φ where each vertex receives a label from $[w]$, using the following four recursive operations with indices $i, j \in [w]$, $i \neq j$:*

1. *Introduce, $\Phi = v(i)$: G_Φ is a graph consisting a single vertex v with label i.*
2. *Disjoint union, $\Phi = \Phi' \oplus \Phi''$: G_Φ is a disjoint union of $G_{\Phi'}$ and $G_{\Phi''}$.*
3. *Relabel, $\Phi = \rho_{i \to j}(\Phi')$: G_Φ is the graph $G_{\Phi'}$ where each vertex labeled i in $G_{\Phi'}$ now has label j.*

4. Join, $\Phi = \eta_{i,j}(\Phi')$: G_Φ is the graph $G_{\Phi'}$ with additional edges between each pair of vertices u of label i and v of label j.

The clique width *of a graph G denoted by* cw(G) *is the minimum number w such that there is a w-expression Φ that defines G.*

In the following, we assume that a w-expression Ψ of G is given. There is an FPT-algorithm that, given a graph G and integer w, either reports that $\mathrm{cw}(G) > w$ or outputs a $(2^{3w+2} - 1)$-expression of G [15].

A w-expression Ψ is an *irredundant w-expression of G*, if no edge is introduced twice in Ψ. Given a w-expression of G, it is possible to get an irredundant w-expression of G in polynomial time [7]. For a coloring of G, a vertex v is said to be *conflict-free dominated* by the color c, if exactly one vertex in $N(v)$ is assigned the color c. In general, a vertex v is said to be conflict-free dominated by a set of colors S, if each color in S conflict-free dominates v. Also, a vertex v is said to *miss the color c* if there exists no vertex in $N(v)$ that is assigned the color c. In general, a vertex v is said to miss a set of colors T, if every color in T is missed by v.

Now, we prove the main theorem of this section.

Theorem 4. *Given a graph G, a w-expression of G and an integer k, it is possible to decide if $\chi^*_{ON}(G) \le k$ in $2^{O(w3^k)} n^{O(1)}$ time.*

Proof. We give a dynamic program that works bottom-up over a given irredundant w-expression Ψ of G. For each subexpression Φ of Ψ and a coloring $C : V(G_\Phi) \to \{0, 1, \ldots, k\}$ of G_Φ, we have a boolean table entry $d[\Phi; N; M]$ with

$$N = n_{1,0}, \ldots, n_{1,k}, \ldots, n_{w,0}, \ldots, n_{w,k}, \text{ and}$$

$$M = M_1, \ldots, M_w \quad \text{where for every } a \in [w], \quad M_a = m_{a,S_1,T_1}, \ldots, m_{a,S_{3^k},T_{3^k}}$$

where S_ℓ, T_ℓ are all the possible disjoint subsets of the set of colors $[k]$. Note that there are 3^k many disjoint subsets $S_\ell, T_\ell \in [k]$.

Given some vertex-coloring of G_Φ, values of M and N have the following meaning.

N**:** For each label $a \in [w]$ and color $q \in \{0\} \cup [k]$, the variable $n_{a,q} \in \{0, 1, 2\}$. Let $n^\star_{a,q}$ be the number of vertices with label a that are colored q. Then $n_{a,q}$ is equal to $n^\star_{a,q}$ when limited to a maximum of two, in other words $n_{a,q} = \min\{2, n^\star_{a,q}\}$.

M**:** For each label $a \in [w]$, and disjoint sets $S, T \subseteq [k]$, the variable $m_{a,S,T} \in \{0, 1\}$. The variable $m_{a,S,T}$ is equal to 1 if there is at least one vertex v with label a which is conflict-free dominated by exactly colors S and the set of colors that misses v is exactly T. If there is no such vertex, then $m_{a,S,T}$ is equal to 0.

For each subexpression Φ of Ψ, the boolean entry $d[\Phi; N; M]$ is set to TRUE if and only if there exists a vertex-coloring $C : V(G_\Phi) \to \{0\} \cup [k]$ that satisfies the variables $n_{a,q}$ and $m_{a,S,T}$, for each label $a \in [w]$, color $q \in \{0\} \cup [k]$ and disjoint subsets $S, T \subseteq [k]$. To decide if k colors are sufficient to CFON* color G, we consider the expression Ψ with $G_\Psi = G$. We answer 'yes' if and only if there exists an entry $d[\Psi; N; M]$ set to TRUE where $m_{a,\{\},T} = 0$ for each $a \in [w]$ and for each $T \subseteq [k]$. This means there exists a coloring such that there is no label $a \in [w]$ with a vertex v that is not conflict-free dominated.

Now, we show how to compute $d[\Phi; N; M]$ at each operation.

1. $\Phi = v(i)$.
   The graph G_Φ represents a node with one vertex v that is labelled $i \in [w]$. For each color $q \in \{0\} \cup [k]$, we set the entry $d[\Phi; N; M] =$ TRUE if and only if $n_{i,q} = 1$, $m_{i,\{\},[k]} = 1$ and all other entries of N and M are 0.
2. $\Phi = \Phi' \oplus \Phi''$.
   The graph G_Φ results from the disjoint union of graphs $G_{\Phi'}$ and $G_{\Phi''}$.
   We set $d[\Phi; N; M] =$ TRUE if and only if there exist entries $d[\Phi'; N'; M']$ and $d[\Phi''; N''; M'']$ such that $d[\Phi'; N'; M'] =$ TRUE, $d[\Phi''; N''; M''] =$ TRUE and the following conditions are satisfied:
   (a) For each label $a \in [w]$ and color $q \in \{0\} \cup [k]$, $n_{a,q} = \min\{2,\ n'_{a,q} + n''_{a,q}\}$.
   (b) For each label $a \in [w]$ and disjoint $S, T \subseteq [k]$, $m_{a,S,T} = \min\{1,\ m'_{a,S,T} + m''_{a,S,T}\}$.

   We may determine each table entry of $d[\Phi; N, M]$ for every N, M as follows. We initially set $d[\Phi; N, M]$ to FALSE for all N, M. We iterate over all combinations of table entries $d[\Phi'; N'; M']$ and $d[\Phi''; N''; M'']$. For each combination of TRUE entries $d[\Phi'; N'; M']$ and $d[\Phi''; N''; M'']$, we update the corresponding entry $d[\Phi; N; M]$ to TRUE. The corresponding entry $d[\Phi; N; M]$ has variables $n_{a,q}$ which is the sum of $n'_{a,q}$ and $n''_{a,q}$ limited by two, and variables $m_{a,S,T}$ which is the sum of $m'_{a,S,T}$ and $m''_{a,S,T}$ limited by one. Thus, to compute every entry for $d[\Phi; ;]$ we visit at most $(3^{w(k+1)}2^{w3^k})^2$ combinations of table entries and for each of those compute $w(k+1) + w3^k$ values for M and N.
3. $\Phi = \rho_{i \to j}(\Phi')$.
   The graph G_Φ is obtained from the graph $G_{\Phi'}$ by relabelling the vertices of label i in $G_{\Phi'}$ with label j where $i, j \in [w]$. Hence, $n_{i,q} = 0$ for each $q \in \{0\} \cup [k]$ and $m_{i,S,T} = 0$ for each disjoint $S, T \subseteq [k]$.
   We set $d[\Phi; N; M] =$ TRUE if and only if there exists an entry $d[\Phi'; N'; M']$ such that $d[\Phi'; N'; M'] =$ TRUE in $G_{\Phi'}$ that satisfies the following conditions:
   (a) For each color $q \in \{0\} \cup [k]$, each label $a \in [w] \setminus \{i, j\}$ and disjoint $S, T \subseteq [k]$, $n_{a,q} = n'_{a,q}$ and $m_{a,S,T} = m'_{a,S,T}$.
   (b) For each color $q \in \{0\} \cup [k]$, $n_{j,q} = \min\{2,\ n'_{i,q} + n'_{j,q}\}$ and $n_{i,q} = 0$.
   (c) For each disjoint $S, T \subseteq [k]$, $m_{j,S,T} = \min\{1,\ m'_{i,S,T} + m'_{j,S,T}\}$ and $m_{i,S,T} = 0$.

   We may determine each table entry of $d[\Phi; N; M]$ for every N, M as follows. We initially set $d[\Phi; N; M]$ to FALSE for all N, M. We iterate over all the TRUE table entries $d[\Phi'; N'; M']$, and for each such entry we update the corresponding entry $d[\Phi; N; M]$ to TRUE, if applicable. To compute every entry for $d[\Phi; ;]$ we visit at most $3^{w(k+1)}2^{w3^k}$ table entries $d[\Phi'; ;]$ and for each of those compute $w(k+1) + w3^k$ values for M and N.

4. $\Phi = \eta_{i,j}(\Phi')$.
   The graph G_Φ is obtained from the graph $G_{\Phi'}$ by connecting each vertex with label i with each vertex with label j where $i, j \in [w]$. Consider a vertex v labelled i in $G_{\Phi'}$ and let v contribute to the variable $m'_{i,\widehat{S},\widehat{T}}$, which is v is conflict-free dominated by exactly $\widehat{S}$ and the set of colors that misses v is exactly $\widehat{T}$. After this operation, the vertex v may contribute to the variable $m_{i,S,T}$ in G_Φ where the choice of the set S in G_Φ depends on the colors assigned to the vertices labelled j in $G_{\Phi'}$.
   We set $d[\Phi; N; M] = \text{TRUE}$ if and only if there exists an entry $d[\Phi'; N'; M']$ such that $d[\Phi'; N'; M'] = \text{TRUE}$ in $G_{\Phi'}$ that satisfies the following conditions:
   (a) For each label $a \in [w]$ and color $q \in \{0\} \cup [k]$, $n_{a,q} = n'_{a,q}$.
   (b) For each label $a \in [w] \setminus \{i, j\}$ and disjoint $S, T \subseteq [k]$, $m_{a,S,T} = m'_{a,S,T}$.
   (c) For the label i and disjoint $S, T \subseteq [k]$, $m_{i,S,T} = 1$ if and only if there are disjoint subsets $\widehat{S}, \widehat{T} \subseteq [k]$ with $m'_{i,\widehat{S},\widehat{T}} = 1$ such that
      i. For each color $q \in S \cap \widehat{S}$, variable $n'_{j,q} = 0$.
      ii. For each color $q \in S \setminus \widehat{S}$, variable $n'_{j,q} = 1$.
      iii. For each color $q \in \widehat{S} \setminus S$, variable $n'_{j,q} \geq 1$.
      iv. $S \setminus \widehat{S} \subseteq \widehat{T}$ and $T \subseteq \widehat{T}$.
      v. For each color $q \in \widehat{T} \setminus (T \cup S)$, $n'_{j,q} = 2$.
   (d) For the label j, entry $m_{j,S,T}$ is computed in a symmetric fashion by swapping the labels i and j in (c).
   It can be observed that each TRUE table entry $d[\Phi'; N'; M']$ sets exactly one entry $d[\Phi; N; M]$ to TRUE. We can determine each table entry of $d[\Phi; N; M]$ as follows. We initially set $d[\Phi; N, M]$ to FALSE for all N, M. We iterate over all the TRUE table entries $d[\Phi'; N'; M']$, and for each such entry we update the corresponding entry $d[\Phi; N; M]$ to TRUE, if applicable. To compute every entry for $d[\Phi; ;]$ we visit at most $3^{w(k+1)} 2^{w3^k}$ table entries $d[\Phi'; ;]$ and for each of those compute $w(k+1) + w3^k$ values for M and N.

We described the recursive formula at each operation, that computes the value of each entry $d[; ;]$. The correctness of the algorithm easily follows from the description of the algorithm. The DP table consists of $3^{w(k+1)} 2^{w3^k}$ entries at each node of the w-expression. The running time is dominated by the operations at the disjoint union node that requires $O(3^{2w(k+1)} 2^{2w3^k} w(k + 1 + 3^k) n^{O(1)})$ time. □

Similarly, we obtain the following result for the CFCN* problem:

Theorem 5 ($\star$). *Given a graph G, a w-expression and an integer k, it is possible to decide if $\chi^*_{CN}(G) \leq k$ in $2^{O(w3^k)} n^{O(1)}$ time.*

By modifying the above algorithm, it is possible to obtain FPT algorithms for the full coloring variants (CFON and CFCN) of the problem. We merely have to restrict the entries of the dynamic program to entries without color 0.

Theorem 6. *The CFON and the CFCN problems are FPT when parameterized by the combined parameters clique width and the solution size.*

4 Block Graphs and Cographs

In this section, we study the problems on block graphs and cographs. Note that block graphs have clique width at most 3, and cographs have clique width at most 2. Hence, CFON* and CFCN* problems are polynomial time solvable on block graphs and cographs by Theorems 4 and 5 respectively. However, we present direct proofs for these problems on these graph classes. In particular we show that $\chi^*_{ON}(G) \leq 3$ and $\chi^*_{CN}(G) \leq 2$, for block graphs G. We show a block graph G such that $\chi^*_{ON}(G) = 3$, making the above bound tight. Next, we show that $\chi^*_{ON}(G), \chi^*_{CN}(G) \leq 2$, for cographs G.

Definition 7 (Block Graph). *A* block graph *is a graph in which every 2-connected component is a clique.*

For the CFON* problem, we give a tight upper bound of 3, in the following sense: we present a graph (see Fig. 1) that is not CFON*-colorable with colors $\{0, 1, 2\}$, where 0 is the dummy-color. Complementing this result, we show that there is an algorithm that colors a given block-graph with colors $\{1, 2, 3\}$, thus without the need of a dummy-color 0.

Lemma 8 ($\star$). *If G is a block graph, $\chi_{ON}(G) \leq 3$, hence $\chi^*_{ON}(G) \leq 3$.*

Proof (Proof Sketch). We give a constructive algorithm that given a block graph G outputs a CFON-coloring C using at most three colors $1, 2, 3$. For convenience, let us also specify a mapping h that maps each vertex $v \in G$ to one of its uniquely colored neighbors $w \in N(v)$. We use the fact that block-graphs are exactly the diamond-free chordal graphs (a diamond is a K_4 with one edge removed) [3]. As usual, we assume that G is connected and contains at least one edge uv. Color $C(u) = 1$ and $C(v) = 2$. Color every vertex $w \in (N(u) \cup N(v)) \setminus \{u, v\}$ with $C(w) = 3$. Assign $h(w) = v$ for every $w \in N(v)$, and assign $h(w) = u$ for every $w \in N(u) \setminus N(v)$.

Let G_v contain every connected component of $G \setminus \{u, v\}$ that contains a vertex from $N(v)$. Similarly, let G_u contain every connected component of $G \setminus \{u, v\}$ that contains a vertex from $N(u) \setminus N(v)$.

Claim ($\star$). The sets $V(G_u)$ and $V(G_v)$ are disjoint.

We color every vertex $x \in V(G_v)$ in distance $2, 3, 4, 5, 6, 7, \ldots$ from v in graph G_v with colors $1, 2, 3, 1, 2, 3, \ldots$ periodically. We assign $h(x)$ for $x \in V(G_v)$ in distance $i \geq 2$ to v to an arbitrary neighbor $y \in N(x)$ that has distance $i - 1$ to v in graph G_v. Similarly we color every vertex $x \in V(G_u)$ in distance $2, 3, 4, 5, 6, 7, \ldots$ from u in G_u with colors $2, 1, 3, 2, 1, 3, \ldots$ periodically. Again, let $h(x)$ for $x \in V(G_u)$ in distance $i \geq 2$ to u map to an arbitrary neighbor $y \in N(x)$ in distance $i - 1$ to u in graph G_u. $\square$

Lemma 9. *There is block graph G with $\chi^*_{ON}(G) > 2$.*

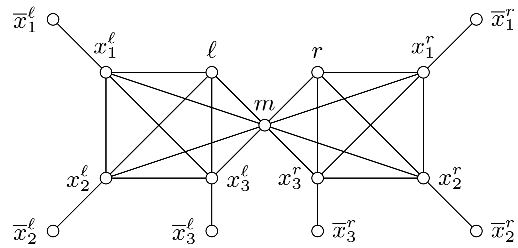

Fig. 1. A block graph G with $\chi^*_{ON}(G) > 2$.

Proof. Let G have vertex set $\{\ell, m, r\} \cup \bigcup_{i\in\{1,2,3\}} \{x_i^\ell, \overline{x}_i^\ell, x_i^r, \overline{x}_i^r\}$, see also Fig. 1. Let the edge set be defined by the set of maximal cliques $\{x_1^s, x_2^s, x_3^s, s, m\}$ and $\{x_s^i, \overline{x}_s^i\}$ for every $s \in \{\ell, r\}$ and $i \in \{1, 2, 3\}$. It is easy to see that G is a block graph. To prove that $\chi^*_{ON}(G) > 2$, assume, for the sake of contradiction, that there is χ^*_{ON}-coloring $C : V \to \{0, 1, 2\}$. Then there is a mapping h on V that assigns each vertex $v \in V(G)$ its uniquely colored neighbor $w \in N(v)$. Note that x_i^s, for $s \in \{\ell, r\}$ and $i \in \{1, 2, 3\}$, has to be colored 1 or 2, since it is the only neighbor of $\overline{x}_i^s$. Further, we may assume that $h(m) \in \{\ell, x_1^\ell\}$ and $C(h(m)) = 2$ because of symmetry.

First consider that $h(m) = \ell$ and $C(\ell) = 2$. Then $C(x_i^s) = 1$ for every $s \in \{\ell, r\}$ and $i \in \{1, 2, 3\}$. It follows that $h(\ell) = m$ and hence $C(m) = 2$. Then however $C_{\{\!\{\}\!\}}(N(x_1^\ell)) \supseteq \{\!\{1, 1, 2, 2\}\!\}$, a contradiction.

Thus it remains to consider that $h(m) = x_1^\ell$ and $C(x_1^\ell) = 2$. Then $C(x_i^s) = 1$ for every x_i^s with $(s, i) \in \{\ell, r\} \times [3] \setminus (\ell, 1)$. It follows that $h(r) = m$ and hence $C(m) = 2$. Then however $C_{\{\!\{\}\!\}}(N(\ell)) = \{\!\{1, 1, 2, 2\}\!\}$, also a contradiction.

Since both cases lead to a contradiction, it must be that $\chi^*_{ON}(G) > 2$. $\square$

Since a block graph G have clique width at most 3, and since $\chi^*_{ON}(G) \leq 3$, we may use Theorem 4 to decide the CFON* problem for block graphs in polynomial time.

Corollary 10. *For block graphs, CFON* is polynomial time solvable.*

By observing that the number of colors required is constant, we have the following analogous result on the CFCN* problem. However, we also present a direct proof using a characterization of block graphs G with $\chi^*_{CN}(G) = 1$.

Theorem 11 ($\star$). *If G is a block graph, then $\chi^*_{CN}(G) \leq 2$. The CFCN* problem is polynomial time solvable on block graphs.*

We now consider the problem on cographs, and obtain Theorem 13, the proof of which is omitted.

Definition 12 (Cograph [6]). *A graph G is a* cograph *if G consists of a single vertex, or if it can be constructed from a single vertex graph using the disjoint union and complement operations.*

Theorem 13 ($\star$). *The CFON* and the CFCN* problems are polynomial time solvable on cographs.*

Since constant bounds for the partial coloring variants imply constant bounds for the full coloring variants and since block graphs and cographs have clique width at most 3, we have the following.

Theorem 14. *The CFON and the CFCN problems are polynomial time solvable on block graphs and cographs.*

5 Interval Graphs

In this section, we show three colors are sufficient and sometimes necessary to CFON* color an interval graph. For *proper interval graphs*, we show that two colors are sufficient. We also show that the CFCN* problem is polynomial time solvable on interval graphs.

Definition 15 (Interval Graph). *A graph $G = (V, E)$ is an* interval graph *if there exists a set $\mathcal{I}$ of intervals on the real line such that there is a bijection $f : V \to \mathcal{I}$ satisfying the following: $\{v_1, v_2\} \in E$ if and only if $f(v_1) \cap f(v_2) \neq \emptyset$.*

For an interval graph G, we refer to the set of intervals $\mathcal{I}$ as the *interval representation* of G. An interval graph G is a *proper interval graph* if it has an interval representation $\mathcal{I}$ such that no interval in $\mathcal{I}$ is properly contained in any other interval of $\mathcal{I}$. An interval graph G is a *unit interval graph* if it has an interval representation $\mathcal{I}$ where all the intervals are of unit length. It is known that the class of proper interval graphs and unit interval graphs are the same [10].

Lemma 16 (⋆). *If G is an interval graph, then $\chi^*_{ON}(G) \leq 3$.*

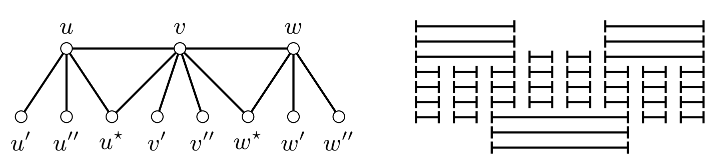

Fig. 2. On the left hand side, we have the graph G', and on the right hand side we have an interval graph representation of G, a graph where $\chi_{ON}(G) > 3$. The graph G is obtained by replacing each vertex $u, v, w, u^\star, v^\star$ of G' with a 3-clique and replacing $u', u''.v', v'', w', w''$ by a 4-clique.

The bound of $\chi^*_{ON}(G) \leq 3$ for interval graphs is tight. In particular, there is an interval graph G (see Fig. 2) that cannot be colored with three colors when excluding the dummy-color 0. That shows the stronger result $\chi_{ON}(G) > 3$, which implies that $\chi^*_{ON}(G) > 2$.

Lemma 17 (⋆). *There is an interval graph G such that $\chi_{ON}(G) > 3$ (and thus $\chi^*_{ON}(G) \geq 3$).*

Lemma 18. *If G is a proper interval graph, then $\chi^*_{ON}(G) \leq 2$.*

Proof. Let $\mathcal{I}$ be a unit interval representation of G. We denote the left endpoint of an interval I by $L(I)$. We assign $C : \mathcal{I} \to \{1, 2, 0\}$ which will be a CFON* coloring.

At each iteration i, we pick two intervals $I_1^i, I_2^i \in \mathcal{I}$. The interval I_1^i is the interval whose $L(I_1^i)$ is the least among intervals for which C has not been assigned. The interval I_2^i is a neighbor of I_1^i, whose $L(I_2^i)$ is the greatest. It might be the case that C has been already assigned for all neighbors of I_1^i. This can happen only in the very last iteration of the algorithm. Depending on this, we have the following two cases.

- **Case 1: I_1^i has neighbors for which C is unassigned**.
  We assign $C(I_1^i) = 1$ and $C(I_2^i) = 2$. All other intervals adjacent to I_1^i and I_2^i are assigned the color 0.
  Now, we argue that C is a CFON* coloring. The intervals I_1^i and I_2^i act as the uniquely colored neighbors for each other. All intervals that are assigned 0 are adjacent to either I_1^i or I_2^i, and thus will have a uniquely colored neighbor. Notice that for every iteration i, the vertices I_1^i (or I_2^i) and I_1^{i+1} (or I_2^{i+1}) will have the same color. This is fine as there is no interval that intersects both I_1^i and I_1^{i+1}.
- **Case 2: C is already assigned for all the neighbors of I_1^i**.
  As mentioned before, this can happen only during the last iteration $i = j$. In this case, I_1^j is the only interval for which C is yet to be assigned. Choose an interval $I_m \in N(I_2^{j-1}) \cap N(I_1^j)$. Such an I_m exists, else $\mathcal{I}$ is disconnected. We reassign $C(I_1^{j-1}) = 0$, $C(I_2^{j-1}) = 1$, $C(I_m) = 2$ and assign $C(I_1^j) = 0$.
  The assignment of colors in iterations $1 \leq i \leq j-2$ are unchanged. Though $C(I_1^{j-1})$ is changed to 0, this does not affect any interval, since there are no intervals which depend only on I_1^{j-1} for their uniquely colored neighbor. If there was such an interval, this would contradict the choice of I_1^{j-1}.
  For the intervals I_2^{j-1} and I_1^j, we have the interval I_m as the uniquely colored neighbor and for the interval I_m, we have the interval I_2^{j-1} as the uniquely colored neighbor. □

It is known [9] that 2 colors suffice to CFCN* color an interval graph. We show that the CFCN* problem is polynomial time solvable on interval graphs using a characterization.

Theorem 19 ($\star$). *CFCN* problem is polynomial time solvable on interval graphs.*

6 Unit Square and Unit Disk Intersection Graphs

Unit square (respectively, unit disk) intersection graphs are intersection graphs of unit sized squares (resp., disks) in the Euclidean plane. It is shown in [9] that $\chi_{CN}^*(G) \leq 4$ for a unit square intersection graph G. They also showed that $\chi_{CN}^*(G) \leq 6$ for a unit disk intersection graph G. We study the CFON* problem on these graphs and get the following constant upper bounds. To the best of our knowledge, no upper bound was previously known on unit square and unit disk graphs for CFON* coloring. Due to space constraints, the proofs of the following theorems are omitted.

Theorem 20 ($\star$). *If G is a unit square intersection graph, then $\chi^*_{ON}(G) \leq 27$.*

Theorem 21 ($\star$). *If G is a unit disk intersection graph, then $\chi^*_{ON}(G) \leq 51$.*

7 Kneser Graphs and Split Graphs

In this section, we study the CFON* and the CFCN* colorings of Kneser graphs and split graphs. For Kneser graphs $K(n,k)$, we show that $\chi^*_{ON}(K(n,k)) = k+1$ when $n \geq k(k+1)^2+1$ and show bounds for $\chi^*_{CN}(K(n,k))$. For split graphs, we show that CFON* problem is NP-complete and CFCN* problem is polynomial time solvable.

Definition 22 (Kneser graph). *The Kneser graph $K(n,k)$ is the graph whose vertices are $\binom{[n]}{k}$, the k-sized subsets of $[n]$, and the vertices x and y are adjacent if and only if $x \cap y = \emptyset$ (when x and y are viewed as sets).*

Theorem 23 ($\star$). *$\chi^*_{ON}(K(n,k)) \leq k+1$, for $n \geq 3k-1$. Further when $n \geq k(k+1)^2+1$, $\chi^*_{ON}(K(n,k)) = k+1$.*

It is easy to see that a proper coloring of a graph G is also a CFCN* coloring. Since $\chi(K(n,k)) \leq n-2k+2$ [14], we have that $\chi^*_{CN}(K(n,k)) \leq n-2k+2$. We show the following:

Theorem 24 ($\star$). *$\chi^*_{CN}(K(n,k)) \leq n-2k+1$, for $2k+1 \leq n \leq 3k-1$. For the case when $n \geq 3k$, we have $\chi^*_{CN}(K(n,k)) \leq k$.*

Definition 25 (Split Graph). *A graph $G=(V,E)$ is a split graph if there exists a partition of $V = K \cup I$ such that the graph induced by K is a clique and the graph induced by I is an independent set.*

Theorem 26 ($\star$). *The CFON* problem is NP-complete on split graphs.*

Theorem 27. *The CFCN* problem is polynomial time solvable on split graphs.*

The proof of Theorem 27 is through a characterization. We first show that for split graphs G, $\chi^*_{CN}(G) \leq 2$. Then we characterize split graphs G for which $\chi^*_{CN}(G) = 1$ thereby proving Theorem 27.

Lemma 28. *If $G=(V,E)$ is a split graph, then $\chi^*_{CN}(G) \leq 2$.*

Proof. Let $V = K \cup I$ be a partition of vertices into a clique K and an independent set I. We use $C : V \rightarrow \{1,2,0\}$ to assign colors to the vertices of V. Choose an arbitrary vertex $u \in K$ and assign $C(u) = 2$. The remaining vertices (if any) in $K \setminus \{u\}$ are assigned the color 0. For every vertex $v \in I$, we assign $C(v) = 1$. Each vertex in I will have itself as the uniquely colored neighbor and every vertex in K will have the vertex u as the uniquely colored neighbor. $\square$

We now characterize split graphs that are CFCN* colorable using one color.

Lemma 29. *Let $G = (V, E)$ be a split graph with $V = K \cup I$, where K and I are the clique and independent sets respectively. We have $\chi^*_{CN}(G) = 1$ if and only if at least one of the following is true: (i) G has a universal vertex, or (ii) $\forall v \in K, |N(v) \cap I| = 1$.*

Proof. We first prove the reverse direction. If there exists a universal vertex $u \in V$, then we assign the color 1 to u and assign the color 0 to all vertices in $V \setminus \{u\}$. This is a CFCN* coloring.

Suppose[5] $\forall v \in K, |N(v) \cap I| = 1$. We assign the color 1 to each vertex in I and color 0 to the vertices in K. Each vertex in I acts as the uniquely colored neighbor for itself and for its neighbor(s) in K.

For the forward direction, let $C : V \to \{1, 0\}$ be a CFCN* coloring of G. We further assume that $\exists y \in K$, $|N(y) \cap I| \neq 1$ and show that there exists a universal vertex. We assume that $|K| \geq 2$ and $|I| \geq 1$ (if either assumption is violated, G has a universal vertex). We first prove the following claim.

Claim. Exactly one vertex in K is assigned the color 1.

Proof. Suppose not. Let two vertices $v, v' \in K$ be such that $C(v) = C(v') = 1$. Then none of the vertices in K have a uniquely colored neighbor.

Suppose if all vertices in K are assigned the color 0. For vertices in I to have a uniquely colored neighbor, each vertex in I has to be assigned the color 1. By assumption, $\exists y \in K$ such that $|N(y) \cap I| \neq 1$. This means that y does not have a uniquely colored neighbor. □

Now we show that there is a universal vertex in K.

By the above claim, there is a unique vertex $v \in K$ such that $C(v) = 1$. We will show that v is a universal vertex. Suppose not. Let $w' \notin N(v) \cap I$. For w' to have a uniquely colored neighbor, either w' or one of its neighbors in K has to be assigned the color 1. The latter is not possible because v is the lone vertex in K that is colored 1. If $C(w') = 1$, then its neighbor(s) in K does not have a uniquely colored neighbor because of the vertices w' and v. Hence, v is a universal vertex. □

From Lemmas 28 and 29, we get Theorem 27.

8 Conclusion

We gave an FPT algorithm for conflict-free coloring for the combined parameters clique width w and number of colors k. Since the problem is NP-hard for constant number of colors k, it is unlikely to be FPT with respect to k only. However an interesting open question is whether this result can be strengthened to an FPT algorithm for parameter clique width w only. To the best of our knowledge, it

[5] This case also captures the case when K is empty.

is open whether there is some bound of any conflict-free chromatic number by the clique width. If there exists such a bound, our algorithm would also be a fixed-parameter tractable algorithm for parameter w only.

Further we showed a constant upper bound of conflict-free chromatic numbers for several graph classes. For most of them we established matching or almost matching lower and upper bounds for their conflict-free chromatic numbers. For unit square and square disk graphs there still is a wide gap, and it would be interesting to improve those bounds.

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References

1. Zachary Abel, Victor Alvarez, Erik D. Demaine, Sndor P. Fekete, Aman Gour, Adam Hesterberg, Phillip Keldenich, and Christian Scheffer. Conflict-free coloring of graphs. *SIAM Journal on Discrete Mathematics*, 32(4):2675–2702, 2018.
2. Akanksha Agrawal, Pradeesha Ashok, Meghana M. Reddy, Saket Saurabh, and Dolly Yadav. FPT algorithms for conflict-free coloring of graphs and chromatic terrain guarding. *CoRR*, abs/1905.01822, 2019.
3. Hans-Jürgen Bandelt and Henry Martyn Mulder. Distance-hereditary graphs. *J. Comb. Theory, Ser. B*, 41(2):182–208, 1986.
4. Sriram Bhyravarapu and Subrahmanyam Kalyanasundaram. Combinatorial bounds for conflict-free coloring on open neighborhoods. In *Graph-Theoretic Concepts in Computer Science - 46th International Workshop, WG 2020, Leeds, UK, June 24-26, 2020, Revised Selected Papers*, pages 1–13, 2020.
5. Hans L. Bodlaender, Sudeshna Kolay, and Astrid Pieterse. Parameterized complexity of conflict-free graph coloring. In *Algorithms and Data Structures - 16th International Symposium, WADS 2019, Edmonton, AB, Canada, August 5-7, 2019, Proceedings*, pages 168–180, 2019.
6. Derek G. Corneil, Helmut Lerchs, and L. Stewart Burlingham. Complement reducible graphs. *Discrete Applied Mathematics*, 3(3):163–174, 1981.
7. Bruno Courcelle and Stephan Olariu. Upper bounds to the clique width of graphs. *Discret. Appl. Math.*, 101(1-3):77–114, 2000.
8. Guy Even, Zvi Lotker, Dana Ron, and Shakhar Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM J. Comput.*, 33(1):94–136, January 2004.
9. Sándor P. Fekete and Phillip Keldenich. Conflict-free coloring of intersection graphs. In *28th International Symposium on Algorithms and Computation, ISAAC 2017, December 9-12, 2017*, volume 92, pages 31:1–31:12, 2017.
10. Frdric Gardi. The roberts characterization of proper and unit interval graphs. *Discrete Mathematics*, 307(22):2906 – 2908, 2007.
11. Luisa Gargano and Adele A. Rescigno. Complexity of conflict-free colorings of graphs. *Theor. Comput. Sci.*, 566(C):39–49, February 2015.

12. Fei Huang, Shanshan Guo, and Jinjiang Yuan. A short note on open-neighborhood conflict-free colorings of graphs. *SIAM Journal on Discrete Mathematics*, 34(3):2009–2015, 2020.
13. Chaya Keller, Alexandre Rok, and Shakhar Smorodinsky. Conflict-free coloring of string graphs. *Discrete & Computational Geometry*, pages 1–36, 2020.
14. László Lovász. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A*, 25(3):319 – 324, 1978.
15. Sang-il Oum and Paul D. Seymour. Approximating clique-width and branch-width. *J. Comb. Theory, Ser. B*, 96(4):514–528, 2006.
16. Janos Pach and Gabor Tardos. Conflict-free colourings of graphs and hypergraphs. *Combinatorics, Probability and Computing*, 18(5):819–834, 2009.
17. I. Vinod Reddy. Parameterized algorithms for conflict-free colorings of graphs. *Theor. Comput. Sci.*, 745:53–62, 2018.
18. Shakhar Smorodinsky. Conflict-free coloring and its applications. In *GeometryIntuitive, Discrete, and Convex*, pages 331–389. Springer, 2013.
19. Manuel Sorge and Mathias Weller. The graph parameter hierarchy. https://manyu.pro/assets/parameter-hierarchy.pdf. Accessed: 2021-03-09.