Review of Linear Algebra

Vector space $(\mathbb{V}, +, \cdot)$

① $\underline{x} + \underline{y} = \underline{y} + \underline{x} \quad \forall \underline{x}, \underline{y} \in \mathbb{V}$

② $\underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z}$

③ $\exists\ \underline{0} \in \mathbb{V}$ st $\underline{0} + \underline{x} = \underline{x} \quad \forall \underline{x} \in \mathbb{V}$.

④ for each $\underline{x}$, $\exists\ (-\underline{x})$ st $\underline{x} + (-\underline{x}) = \underline{0}$

⑤ $\alpha(\beta \underline{x}) = (\alpha\beta)\underline{x} \quad \forall\ \alpha, \beta \in \mathbb{R}$ & $\underline{x} \in \mathbb{V}$

⑥ $\alpha(\underline{v}_1 + \underline{v}_2) = \alpha \underline{v}_1 + \alpha \underline{v}_2$

$(\alpha + \beta)\underline{v}_1 = \alpha \underline{v}_1 + \beta \underline{v}_1$

⑦ $1 \cdot \underline{v} = \underline{v}$

Eg : $\mathbb{R}^n$, $\mathbb{R}^k$, $\mathbb{R}^{n\times k}$ } All vector spaces

$\longrightarrow$ dim =

$\mathbb{Q}^k$ Not a vector space over $\mathbb{R}$

S^n_+ : set of all $n\times n$ symmetric PSD matrices

A, B PSD

$\underline{x}^T A \underline{x} \geq 0 \quad \underline{x}^T B \underline{x} \geq 0 \quad \forall \underline{x}$

$\underline{x}^T (A+B) \underline{x} \geq 0 \quad \forall \underline{x}$

Not a vector space

S^n : Set of $n\times n$ symmetric matrices

Examples (continued)

① set of all $n \times n$ matrices

$$A^{(i,j)} = \begin{bmatrix} 0 & \cdots & 1 & \\ 0 & & 1 & \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

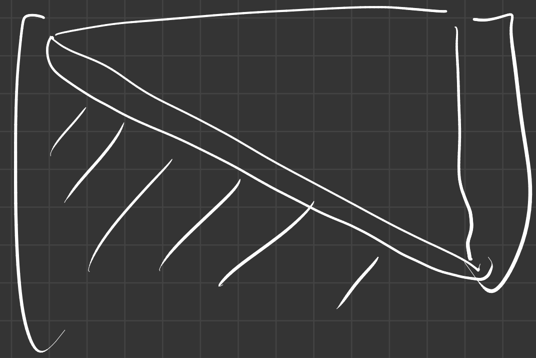
(i,j) th entry

$$B = \sum_{i,j} b_{ij} A^{(i,j)}$$

$$\dim(\mathbb{R}^{n \times n}) = n^2$$

② set of all $n \times n$ symmetric matrices S^n

$$\dim(S^n) = \frac{n(n+1)}{2}$$



$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix} \quad - \quad - \quad \begin{bmatrix} 0 & \cdots & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \ddots & 1 \\ & & \\ 1 & & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$$

3 Set of polynomials of degree $\leq n$ & real coefficients

$$\mathbb{P}_2 = \{ a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R} \}$$

$$\alpha(p_1(x) + p_2(x)) = \alpha p_1(x) + \alpha p_2(x) \qquad \{1, x, x^2\}$$

$$(\alpha_1 + \alpha_2) p(x) = \alpha_1 p(x) + \alpha_2 p(x)$$

$$\dim(\mathbb{P}_n) = n+1$$

3.5 What about the set of all polynomials of degree $= n$

4 Set of complex nos.

$$a + ib \qquad \{1, i\}$$

$$\dim(\mathbb{C}) = 2$$

Subspace, test for subspaces

V is a v/s & $S \subseteq V$

If S is also a vector space, then S is called a subspace of V.

* Only need to test if S is closed under l.c.

$\underline{v}_1, \underline{v}_2 \in S \Rightarrow \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 \in S$

Linear independence

$$\{\underline{v}_1, \underline{v}_2 \cdots \underline{v}_m\} \in \mathbb{V}$$

$$\sum_{i=1}^{m} c_i \underline{v}_i = \underline{0} \quad \text{iff} \quad c_1 = c_2 \cdots = c_m = 0$$

Span

$$\text{Span}\{\underline{v}_1 \cdots v_m\} = \left\{ \underline{v} = \sum_{i=1}^{m} \alpha_i \underline{v}_i : \alpha_i \in \mathbb{R} \right\}$$

What is a basis for a vector space? Is it unique?

$\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{V}$ if $v_1 \ldots v_n$ are linearly independent & span $\mathbb{V}$

Dimension of a vector space

$\dim(\mathbb{V}) = |\text{Basis for } \mathbb{V}|$

Four fundamental subspaces associated with a matrix

A : m x n matrix

1. Row space : Span (rows)

2. Column space : Span (cols)

3. Right null space $NS(A) = \{ \underline{v} \in \mathbb{R}^n : A\underline{v} = \underline{0} \}$

4. Left null space $\{ \underline{v} \in \mathbb{R}^n : A^T\underline{v} = \underline{0} \}$

Rank and nullity

$$\text{Rank}(A) + \text{Nullity}(A) = \#\text{ cols}$$

Compute the rank, nullity, column space and right null space:

① $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$\text{rank}(A) = 1$

$\text{nullity} = 2 - 1 = 1$

$\text{ColSp}(A) = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$

$\text{Rt NS}(A) = \left\{ \alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$

Permutations and determinant

$$[n] = \{1, 2, \ldots, n\}$$

σ

$$(1, 2, 3)$$
$$(1\ 3\ 2)$$
$$(2\ 1\ 3)$$
$$(2\ 3\ 1)$$
$$(3\ 2\ 1)$$
$$(3\ 1\ 2)$$

__Defn:__ σ is a permutation on $[n]$ if it is a bijection on $[n]$

$$(1\ 2\ 3\ 4) \longmapsto (4\ 1\ 3\ 2)$$

$$(4\ 2\ 3\ 1)$$

$$(4\ 1\ 3\ 2)$$

$$(1\ \underline{2}\ 3\ \underline{4})$$
$$(\underline{1}\ \underline{4}\ 3\ 2)$$
$$(4\ 1\ 3\ 2)$$

$$(1\ \underline{2}\ \underline{3}\ 4)$$
$$(\underline{1}\ 3\ 2\ \underline{4})$$
$$(4\ \underline{3}\ 2\ \underline{1})$$
$$(4\ 1\ \underline{2}\ \underline{3})$$
$$(4\ 1\ \underline{3}\ \underline{2})$$

Fact: The number of pairwise swaps required is always odd or always even

$$\text{sign}(\sigma) = \begin{cases} +1 & \text{if even} \\ -1 & \text{if odd} \end{cases}$$

Defn:

$$\det(A) = \sum_{\sigma\,:\,\text{permutations}(n)} \text{sign}(\sigma)\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$\text{sign}(1,2) = 1 \longrightarrow \sigma(1) = 1 \quad \sigma(2) = 2$

$\text{sign}(2,1) = -1 \qquad \sigma(1) = 2 \quad \sigma(2) = 1$

$$\det(A) = +1 \times a_{1,1}\, a_{2,2} + (-1)\, a_{12}\, a_{21} = a_{11}a_{22} - a_{21}a_{12}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} \\ & - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\ & - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} \end{aligned}$$

$\text{sign}(1, 2, 3) = 1$

$\text{sign}(1\ 3\ 2) = -1$

$\text{sign}(2\ 1\ 3) = -1$

$\text{sign}(2\ 3\ 1) = 1$

$\text{sign}(3\ 2\ 1) = -1$

$\text{sign}(3\ 1\ 2) = 1$

$$\begin{aligned} = {} & a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ & - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ & + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

Computational complexity $= \Theta(n! \times n)$

Row operations and determinant

1. Subtracting scaled row from another

$$R_2' \leftarrow R_2 - \alpha R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \longmapsto A'$$

$$\det(A') = \det(A)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

2. Scaling a row

$$R_2' \leftarrow \alpha R_2$$

$$\det(A') = \alpha \det(A)$$

3. Exchanging rows / columns

$$R_2' \leftarrow R_1 , \quad R_1' \leftarrow R_2$$

$$\det(A') = -\det(A)$$

Computing the determinant

$$A \xrightarrow{RREF} A'$$

$$\det(A') = \begin{cases} 1 & \text{if } A \text{ is full rank} \\ 0 & \text{else} \end{cases}$$

$$\underbrace{c_1 \, c_2 \cdots c_m}_{\text{Scaling factors for type-2 op}} \quad (-1)^{\#\text{row sp}} \det(A) = \det(A')$$

$$\det(A) = \frac{(-1)^{\#\text{swp}}}{c_1 \, c_2 \cdots c_m}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\det(A) = 2$$

$$\xrightarrow{R_2 \Leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2' = \frac{1}{-2} R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det = \frac{(-1)^1}{(-1/2)}$$

$$= +2$$

Gram-Schmidt orthogonalization

$$\{\underline{v}_1 \; \cdots \; \underline{v}_m\}$$

$$\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}$$

$$\underline{u}_2' = \underline{v}_2 - \langle \underline{v}_2, \underline{u}_1 \rangle \underline{u}_1$$

$$\underline{u}_2 = \frac{\underline{u}_2'}{\|u_2'\|}$$

$$u_3' = v_3 - \langle v_3, u_2 \rangle u_2 - \langle v_3, u_1 \rangle u_1$$

$$\vdots$$

Eigenvalues and eigenvectors

$A \in \mathbb{R}^{n \times n}$

λ is an eigenvalue of A if $A\underline{v} = \lambda \underline{v}$ for some $\underline{v} \neq \underline{0}$

eigenvector

Spectrum : set of all eigenvalues of A

Characteristic eqn : $\det(A - \lambda I) = 0$

roots of $\det(A - \lambda I) \to$ eigenvalues

Computing eigenvalues and eigenvectors

Computing eigenvectors: λ_i

$$A\underline{v} = \lambda_i \underline{v}$$

$$(A - \lambda_i I)\,\underline{v} = \underline{0}$$

Does every nxn matrix have n real eigenvalues?

Examples

① $\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$ $\lambda_i = 0, \quad i = 1, 2, \ldots n$

Set of eigenvectors : $\mathbb{R}^n \setminus \{\underline{0}\}$

② $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\lambda_1 = 3, \quad \lambda_2 = 0, \quad \lambda_3 = 0$

$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Diagonalizability

$A \in \mathbb{R}^{n \times n}$ is diagonalizable if $\exists\, P \in \mathbb{R}^{n \times n}$ invertible & D diagonal

st $A = P^{-1} D P$

What can go wrong?

① A may not have n real eigenvalues

② If A has n distinct eigenvalues, it is diagonalizable

③ If we can find n linearly independent eigenvectors, done.

$\det(A - \lambda I)$

Is every matrix diagonalizable?

Symmetric matrices and diagonalizability

A is symmetric if $A = A^T$.

① All symmetric matrices are diagonalizable

Say $\lambda_1 \neq \lambda_2$ are eigenvalues of A

$\underline{v}_1 \quad \underline{v}_2$

$$\underline{v}_1^T A \underline{v}_2 = \underline{v}_1^T (\lambda_2 \underline{v}_2) = \lambda_2 v_1^T v_2$$

$$\parallel$$

$$v_2^T A^T v_1 = v_2^T A v_1 = \underline{v}_2 (\lambda_1 \underline{v}_1) = \lambda_1 v_1^T v_2$$

This can happen only if $\underline{v}_1^T \underline{v}_2 = 0$

$\Rightarrow$ Eigenvectors corresponding to distinct eigenvalues are orthogonal

(2) Suppose λ is a repeated eigenvalue.

Arithmetic multiplicity $= 2$

Can we choose $\{v_1, v_2\}$ orthogonal eigenvectors?

$\{\underline{v} : A\underline{v} = \lambda \underline{v}\} \rightarrow$ Set of all eigenvectors corresp to λ

Forms a vector space

(3) If A is orthogonally diagonalizable, then A is symmetric

$$A = PDP^{-1}$$

$$A = PDP^T$$

$$A^T = (PDP^T)^T = PD^TP^T$$

$$= PDP^T$$

Positive semidefinite and positive definite matrices

A : symmetric is said to be positive semidefinite if all eigenvalues of A are ≥ 0

$$\lambda_{max} = \max_{\underline{v} \neq \underline{0}} \frac{\underline{v}^T A \underline{v}}{\underline{v}^T \underline{v}} \longrightarrow \text{largest eigenvalue of } A$$

$$= \max_{\underline{v} : \|\underline{v}\| = 1} \underline{v}^T A \underline{v}$$

Proof : Say $\underline{v}_1, \underline{v}_2 - \underline{v}_n$ are orthonormal eigenvectors of A

$\lambda_1 \geq \cdots \geq \lambda_n$

$$\underline{v} = \sum_{i=1}^{n} \alpha_i \underline{v}_i$$

$$\|\underline{v}\|^2 = \underline{v}^T \underline{v} = \left(\sum_{i=1}^{n} \alpha_i v_i\right)^T \left(\sum_{j=1}^{n} \alpha_j v_j\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j v_i^T v_j = \sum_{i=1}^{n} \alpha_i^2$$

$$\underline{v}^T A \underline{v} = \underline{v}^T \left(A \sum_{i=1}^{n} \alpha_i v_i \right) = \underline{v}^T \left(\sum_{i=1}^{n} \alpha_i A v_i \right)$$

$$= v^T \left(\sum_{i=1}^{n} \alpha_i \lambda_i v_i \right)$$

$$= \left(\sum_{j=1}^{n} \alpha_j v_j \right)^T \left(\sum_{i=1}^{n} \alpha_i \lambda_i v_i \right)$$

$$= \sum_{i=1}^{n} \alpha_i^2 \lambda_i$$

$$= \sum_{i=1}^{n} \beta_i \lambda_i$$

$\beta_i \geq 0 \ \forall i$

$\sum_{i=1}^{n} \beta_i = 1$

$\downarrow$

Maximized when

$\beta_1 = 1$

For a P.S.D matrix, $\underline{v}^T A \underline{v} \geq 0$

$$\underline{v}^T A \underline{v} \leq (v^T v) \lambda_{max}$$

A is positive definite if all eigenvalues are > 0.

$$\max_{A \text{ is PSD}} \log\det(I + A)$$

- The set of all $n \times n$ PSD matrices is denoted S_+
- The set of all $n \times n$ PD matrices : S_{++}

Claim : The set of all PSD matrices is closed

Square root of a positive semidefinite matrix

$$A = P D P^T$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$D^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_n} \end{bmatrix}$$

$$(P D^{1/2} P^T)^2 = P D^{1/2} P^T P \; D^{1/2} P^T$$

$$= P D^{1/2} D^{1/2} P^T = P D P^T = A$$

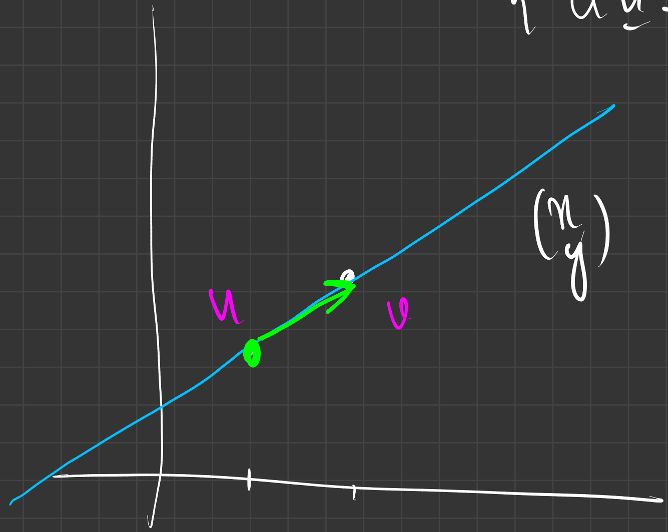
$$P D^{1/2} P^T = A^{1/2}$$

Line

$\underline{u}, \underline{v} \in R^n$

The straight line passing through $\underline{u}$ & $\underline{v}$

$= \{ \alpha \underline{u} + (1-\alpha)\underline{v} : \alpha \in \mathbb{R} \}$



$\underline{x} = \underline{u} + (1-\alpha)(\underline{v} - \underline{u})$

$= \alpha \underline{u} + (1-\alpha)\underline{v}$

$(x_1, y_1) \quad (x_2, y_2)$

$$\frac{(y - y_2)}{(x - x_2)} = \frac{(y_1 - y_2)(1-\alpha)}{(x_1 - x_2)(1-\alpha)}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + (1-\alpha)\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$$

$(1, 2) \quad (2, 3)$

$\downarrow \quad \downarrow$

$\underline{u} \quad v$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (1-\alpha)\begin{pmatrix} 2-1 \\ 3-2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (1-\alpha)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$=$

$\{ \alpha \underline{u} + (1-\alpha)\underline{v} : \alpha \in [0,1] \} \longrightarrow$ line segment joining $\underline{u}$ & $\underline{v}$

Convex set: A is convex if $\underline{u}, \underline{v} \in A$

$$\Rightarrow \quad \alpha \underline{u} + (1-\alpha)\underline{v} \in A \quad \forall\, \alpha \in [0,1]$$