Quiz-2, CS1340: 2-Apr-18, 7:15pm-8:25pm. ROLL NO. CS1340

Note: Please fill-in-the-blanks with correct answers. Additionally, note the key arguments in arriving at your answer in the box provided below the blank. Illegible writings and those outside the appropriate blank/box **will be ignored**. Please attempt the solution on a rough sheet first, and then make an appropriate fair copy on this sheet. Please use the notations, definitions, terminology etc. from your textbook. Do not forget to fill in your ROLL NO.

1. Consider the following recursion:

$$a_n = \frac{10^{(3^n(n-2))} a_{n-1}^6}{a_{n-2}^9}, n \geq 2, n \in \mathbb{N}. \tag{1}$$

The solution of (1) under the conditions $a_0 = 10, a_1 = 10$ is given by $10^{\left(\frac{n^3}{6}+\frac{3n^2}{2}-\frac{7n}{3}+1\right)3^n}$ [1].

Let $b_n \equiv \log_{10} a_n$, then (1) is same as:	Substituting in recursion gives: $\alpha = \frac{1}{6}$, $\beta = \frac{3}{2}$
$b_n = 6b_{n-1} - 9b_{n-2} + (n+2)3^n$.	Soln. of homogenous part: $(\gamma n + \delta)3^n$.
Since 3 is double root of ch. eqn.,	Initial conditions $b_0 = 1, b_1 = 1$ give:
the particular soln: $n^2(\alpha n + \beta)3^n$.	$\gamma = -7/3$, $\delta = 1$.

[4 Marks]

2. Consider the sequence $\{a_n\}$ defined by:

$$a_n = \sum_{m=1}^{n} \underbrace{\sum_{k_1 \in \mathbb{N}} \cdots \sum_{k_m \in \mathbb{N}}}_{\ni k_1 + \ldots + k_m = n} k_1 k_2 \ldots k_{m-1} k_m, \; n \in \mathbb{N} \tag{2}$$

The generating function for $\{a_n\}$ is given by $G(z) \equiv \frac{z}{(1-3z+z^2)}$ [2].

a_n is a sum of m-way convolutions of	Hence, $G(z) = F(z) + F(z)^2 + \cdots + F(z)^n + \cdots$
the sequence $1, 2, \ldots, n, \ldots$,	$= F(z)/1-F(z)$
for which the generating function is $F(z) = \frac{z}{(1-z)^2}$.	$= z/(1-3z+z^2)$.

[4 Marks]

Let $\phi_1 \equiv \frac{3+\sqrt{5}}{2}, \phi_2 \equiv \frac{3-\sqrt{5}}{2}$. A simplified expression, involving n, ϕ_1, ϕ_2, for $a_n = \frac{\phi_1^n - \phi_2^n}{\sqrt{5}}$.

Note that, $G(z) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi_1 z} - \frac{1}{1-\phi_2 z}\right) \xrightarrow{n^{th}\text{ coeff}}$	$\phi_1^n - \phi_2^n / \sqrt{5}$.

[1 Mark]

[1] Fill in this blank with an appropriately simplified expression in n.

[2] Fill in this blank with an appropriately simplified expression in z. Marks will NOT be awarded if the expression is not simplified.

3. Consider the following twist to the usual "Towers of Brahma" problem: There are three pegs. On the first peg, disks of n different sizes (e.g., size-1, ... size-n) are present such that disks of smaller size are placed above those of bigger size. The number of disks of size-i is m_i (which need not be unity). Assume that these m_i disks of size-i are all absolutely indistinguishable from each other. Note that the previous two statements are true for all $i = 1, \ldots, n$. Also, assume that the other two pegs are empty. If in one move only one disk can be transfered from a peg to another peg, then the minimum number of moves required to transfer all the disks present on the first peg to the second peg such that after any move a disk of lesser size is not placed below a disk of bigger size is $\sum_{i=1}^{n} 2^{n-i} m_i$ [3].

Let $T(m_1, \ldots, m_n)$ denote the min. moves; $T(m_1, \ldots, m_n) = 2T(m_1, \ldots, m_{n-1}, 0) + m_n$ $\vdots$ $T(m_1, m_2, 0, \ldots 0) = 2T(m_1, 0, \ldots, 0) + m_2$	Multiplying i^{th} eqn. by 2^{i-1} & telescoping the sum, we get: $T(m_1, \ldots, m_n) = \sum_{i=1}^{n} 2^{n-i} m_i$, since $T(m_1, 0, \ldots 0) = m_1$.

[2 Marks]

Now, consider the alternate case where the disks of equal size are distinguishable and it is required that after all the disks are transfered to the second peg, the order in which they existed on the first peg is maintained. However, during the intermediate moves the order need not be maintained as long as no lower-sized disk is placed below a bigger-sized disk. In this case, the minimum number of moves required for transferring the disks from the first peg to the second is $\sum_{i=1}^{n} 2^{n-i+1} m_i - 1$ [4].

Important observation is that the previous algo., reverses only the order of the biggest disk. Order of rest of disks is preserved.	Let $A(m_1, \ldots, m_n)$ be min. moves, then: $A(m_1, \ldots, m_n) = 2T(m_1, \ldots, m_n - 1) + 1$, where T is same as above. Solving gives the result.

[4 Marks]

Replacement question from Quiz-1:

1. Let r, n be two natural numbers such that $r \le n$. Then, the statement $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$ is true [5].

LHS is no. of bit strings of length '$n+1$' with '$r+1$' ones. $\therefore$ the last one must occur at $k = r+1$ & $r+2 \cdots$ & $n+1$.	Also, in first $k-1$ positions 'r' ones $\Rightarrow$ LHS $= \sum_{k=r+1}^{n+1} {}^{k-1}C_r = \sum_{j=r}^{n} {}^{j}C_r$.

[2 Marks]

[3] Fill in this blank with an appropriately simplified expression involving $n, m_1, \ldots, m_n$. Do NOT fill this blank with a recurrence relation etc. It is enough to show your derivation in the box; you need not argue why your answer is indeed the minimum.

[4] Fill in this blank with an appropriately simplified expression involving $n, m_1, \ldots, m_n$. Do NOT fill this blank with a recurrence relation etc. It is enough to show your derivation in the box; you need not argue why your answer is indeed the minimum.

[5] Fill in this blank with 'true' or 'false'. Marks will be awarded only if a combinatorial argument is provided in the box.