SPECTRAL THEOREM FOR COMPACT SELF-ADJOINT OPERATORS

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Contents

Introduction

Let $T : V \to V$ be a normal matrix on a finite dimensional complex vector space V. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigenvalues of T and M_i $(i = 1, 2 \ldots, n)$ be the corresponding eigenspaces and $P_i : V \to V$ be the orthogonal projections onto M_i. Then by the finite dimensional Spectral theorem, we have $I = \sum_{k=1}^{n} P_k$ and $T = \sum_{k=1}^{n} \lambda_k P_k$.

In these lectures we see that this result can be extended to a particular class of operators on infinite dimensional Hilbert spaces, which resembles finite dimensional operators in some sense.

1. Bounded Operators

In this section we define bounded linear operators between Hilbert spaces and discuss some properties and examples. Throughout we consider only Complex Hilbert spaces. Until other wise specified, all Hilbert spaces are assumed to be infinite dimensional.

Definition 1.1. Let $T : H_1 \to H_2$ be linear. Then T is said to be bounded if and only if $T(B)$ is bounded in H_2 for every bounded subset B of H_1.

If H_1 and H_2 are Hilbert spaces and $T : H_1 \to H_2$ is a bounded operator, then we denote this by $T \in \mathcal{B}(H_1, H_2)$. If $H_1 = H_2 = H$, then $\mathcal{B}(H_1, H_2)$ is denoted by $\mathcal{B}(H)$. For $T \in \mathcal{B}(H)$, the null space and range space are denoted by $N(T)$ and $R(T)$ respectively. The unit sphere of a Hilbert space H is denoted by S_H. The following conditions are equivalent for a linear operator to be bounded.

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Theorem 1.2. *Let $T : H_1 \to H_2$ be linear. Then the following are equivalent;*

(1) *T is bounded*
(2) *T is continuous at 0*
(3) *T is uniformly continuous*
(4) *there exists an $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in H_1$*

Definition 1.3. If T is bounded, then by Theorem 1.2, $\sup\limits_{x \in S_{H_1}} \|Tx\| < \infty$. This quantity is called the norm of T and is denoted by $\|T\|$.

We have the following equivalent formulae for computing the norm of a bounded linear operator.

Theorem 1.4. *Let $T \in \mathcal{B}(H_1, H_2)$. Then the following are equivalent;*

(1) $\|T\| = \sup\{\|Tx\| : x \in S_{H_1}\}$
(2) $\|T\| = \sup\{\|Tx\| : x \in H_1, \|x\| \leq 1\}$
(3) $\|T\| = \sup\{\dfrac{\|Tx\|}{\|x\|} : x \in H_1\}$
(4) $\|T\| = \inf\{k > 0 : \|Tx\| \leq k\|x\| \text{ for all } x \in H_1\}$.

Remark 1.5. The statement (4) of the above Theorem gives a geometric interpretation of the norm a bounded operator as follows: $\|T\|$ is the radius of the smallest ball contating the image of the unit ball in H_1.

For every bounded linear operator there is an another bounded linear operator associated with it in the following way.

Definition 1.6. Let $T \in \mathcal{B}(H_1, H_2)$. Then there exists a unique operator from H_2 into H_1, denoted by T^* such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x \in H_1, y \in H_2.$$

This operator T^* is called the **adjoint** of T.

We have the following properties of T^*.

(1) $(T^*)^* = T$
(2) $\|T^*\| = \|T\|$
(3) if $S \in \mathcal{B}(H_2, H_3)$, then $(ST)^* = T^*S^*$
(4) if $R \in \mathcal{B}(H_1, H_2)$, then $(R + T)^* = R^* + T^*$
(5) $(\alpha T)^* = \bar{\alpha} T^*$

Remark 1.7. let $S, T \in \mathcal{B}(H_1, H_2)$ and $\alpha \in \mathbb{C}$. Then

(a) $\|S + T\| \leq \|S\| + \|T\|$
(b) $\|\alpha T\| = |\alpha|\, \|T\|$
(c) $\|T^*T\| = \|T\|^2 = \|TT^*\|$.

Exercise 1.1. *Let $T \in \mathcal{B}(H)$. Then show that*

$$\|T\| = \sup\{|\langle Tx, y\rangle| : x, y \in S_H\}.$$

Exercise 1.2. *Let $T \in \mathcal{B}(H_1, H_2)$. Then*

(1) $N(T) = R(T^*)^\perp$
(2) $N(T^*) = R(T)^\perp$
(3) $\overline{R(T)} = N(T^*)^\perp$
(4) $\overline{R(T^*)} = N(T)^\perp$

(5) $\overline{N(T^*T)} = \overline{N(T)}$
(6) $\overline{R(TT^*)} = \overline{R(T)}$.

1.3. **Examples.**

(1) (identity operator) Let H be a complex Hilbert space. Let I be the identity map on H. Then $\|I\| = 1$.
(2) (right shift operator) Let $R : \ell^2 \to \ell^2$ be given by
$$R(x_1, x_2, \dots) = (0, x_1, x_2, \dots) \quad \text{for all } (x_1, x_2, \dots) \in \ell^2.$$
Then we can show that $\|Rx\| = \|x\|$ for all $x \in \ell^2$. Hence R is bounded and $\|R\| = 1$. One can check that $R^*(y_1, y_2, y_3, \dots) = (y_2, y_3, \dots)$ for all $(y_n) \in \ell^2$. Note that $\|R^*\| = 1$.
(3) (matrix) Let H_1 be a finite dimensional Hilbert space and H_2 be a Hilbert space. Let $T : H_1 \to H_2$ be linear. Then T is bounded. To see this, let $\{\phi_1, \phi_2, \phi_3, \dots \phi_n\}$ be an orthonormal basis for H_1. If $x \in H_1$, then $x = \sum_{k=0}^{\infty} \langle x, \phi_k \rangle \phi_k$. Hence $Tx = \sum_{k=0}^{\infty} \langle x, \phi_k \rangle T\phi_k$. Using the Cauchy-Schwarz inequality, we can show that
$$\|Tx\| \leq \left(\sum_{k=1}^{n} \|T\phi_j\|^2 \right)^{\frac{1}{2}} \|x\| \text{ for all } n$$
That is $\|T\| \leq \left(\sum_{k=1}^{n} \|T\phi_j\|^2 \right)^{\frac{1}{2}}$.

Exercise 1.4. *Solve the following.*

(a) *(diagonal matrix) Let $D : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator whose matrix with respect to the standard orthonormal basis of $\mathbb{C}^n$ is the diagonal matrix with entries $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Show that $\|T\| = \max_{1 \leq j \leq n} \{|\lambda_j|\}$.*
(b) *(diagonal operator) Let H be a separable Hilbert space with an orthonormal basis $\{\phi_n\}$ and (λ_n) be a bounded sequence of complex numbers. Define $T : H \to H$ by*
$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n, \quad \textit{for all } x \in H.$$
Show that T is bounded and $\|T\| = \sup\{|\lambda_n| : n \in \mathbb{N}\}$.
(c) *(multiplication operator)Let $g \in \big(C[0,1], \|\cdot\|_\infty\big)$. Define $M_g : L^2[0,1] \to L^2[0,1]$ by*
$$M_g(f) = gf, \quad \textit{for all } f \in L^2[0,1].$$
Show that M_g is bounded and $\|M_g\| = \|g\|_\infty$.
(d) *find adjoint of each operator in (a), (b) and (c).*

Definition 1.8. Let $T \in \mathcal{B}(H)$. Then T is said to be

(1) **self-adjoint** if $T = T^*$
(2) **normal** if $T^*T = TT^*$
(3) **unitary** if $T^*T = TT^* = I$
(4) **isometry** if $\|Tx\| = \|x\|$ for all $x \in H$ (equivalently $T^*T = I$)
(5) **orthogonal projection** if $T^2 = T = T^*$.

Remark 1.9. If $T \in \mathcal{B}(H)$, then T^*T and TT^* are self-adjoint operators. Also $T = A + iB$, where $A = \dfrac{T+T^*}{2}$ and $B = \dfrac{T-T^*}{2i}$. It can be easily checked that $A = A^*$ and $B = B^*$. This decomposition is called the **cartesian decomposition** of T. It can be easily verified that T is normal if and only if $AB = BA$.

Definition 1.10. Let H be a separable Hilbert space with an orthonormal basis $\phi_1, \phi_2, \phi_3, \ldots$. Then the matrix of T with respect to $\phi_1, \phi_2, \phi_3, \ldots$ is given by (a_{ij}) where $a_{ij} = \langle T\phi_j, \phi_i\rangle$ for $i, j = 1, 2, 3 \ldots, n$.

Exercise 1.5.
(1) *Find the matrix of the right shift operator with respect to the standard orthonormal basis of* ℓ^2
(2) *show that if* (a_{ij}) *is a matrix of* T *with respect to an orthonormal basis, then* $(\overline{a_{ji}})$ *is the matrix of* T^* *with respect to the same basis.*

In order to get a spectral theorem analogous to the finite dimensional case, we have to look for bounded operators with the similar properties of the finite dimensional operators. One such property is that

If H_1, H_2 are finite dimensional complex Hilbert spaces and $T : H_1 \to H_2$ is linear then:

For every bounded set $S \subseteq H_1$, $T(S)$ is pre compact in H_2 under T.(*)

For a set to be compact in a metric space, we have the following;

Theorem 1.11. *Let X be a metric space and $S \subseteq X$. Then the following conditions are equivalent.*
(1) *S is compact*
(2) *S is sequentially compact*
(3) *S is totally bounded and complete.*

The following example shows that this property depends on the dimension of the range of the operator.

Let H_1 and H_2 be infinite dimensional Hilbert spaces. Let $w \in H_1$ and $z \in H_2$ be a fixed vectors. Define $T : H_1 \to H_2$ by

$$Tx = \langle x, w\rangle z, \text{ for all } x \in H_1.$$

As the range of the operator is one dimensional, T maps bounded sets into pre compact sets. This can be generalized to a bounded operator whose range is finite dimensional as follows:

Let $w_1, w_2, \ldots w_n \in H_1$ and $z_1, z_2, \ldots, z_n \in H_2$ be fixed vectors. Define $T : H_1 \to H_2$ given by

$$Tx = \sum_{j=1}^{n} \langle x, w_j\rangle z_j, \text{ for all } x \in H_1. \tag{1.1}$$

Then T is linear, bounded and has finite dimensional range. Such operators are called as finite rank operators.

Definition 1.12. Let $T \in \mathcal{B}(H)$. Then T is called a **finite rank** (rank n say) if $R(T)$ is finite dimensional.

Note 1.6. We have seen that any operator given by Equation (1.1) is a finite rank operator and has the property $(*)$. Can we express every finite rank operator as in (1.1).

Theorem 1.13. *Let $K : H_1 \to H_2$ be a bounded operator of rank n. Then there exists vectors $v_1, v_2, \dots v_n \in H_1$ and vectors $\phi_1, \phi_2, \dots, \phi_n \in H_2$ such that for every $x \in H_1$, we have*

$$Kx = \sum_{j=1}^{n} \langle x, v_i \rangle \phi_i.$$

The vectors $\phi_1, \phi_2, \dots, \phi_n$ may be chosen to be any orthonormal basis for $R(K)$.

Proof. Let $\phi_1, \phi_2, \dots, \phi_n$ be an orthonormal basis for $R(K)$. Then

$$Kx = \sum_{i=1}^{n} \langle Kx, \phi_i \rangle \phi_i, \quad \text{for every } x \in H_1.$$

For each i, the functionals $f_i(x) = \langle Kx, \phi_i \rangle$ is a bounded linear functional on H_1. Now by the Riesz Representation theorem, there exists a unique $v_i \in H_1$ such that $f_i(x) = \langle x, v_i \rangle$ and $\|f_i\| = \|v_i\|$ for each i. Hence the result follows. □

Remark 1.14. In the above theorem, the representation of K is not unique as it depends on the orthonormal basis and hence on the vectors $\{v_j\}_{j=1}^n$.

Can this happen for operators whose range and domain are infinite dimensional?.

2. Compact Operators

In this section we discuss the properties of operators which are analogues of the finite dimensional operators. In other words we describe the infinite dimensional operators which have the property $(*)$.

Definition 2.1. Let $T : H_1 \to H_2$ be a linear operator. Then T is said to be **compact** if for every bounded set $S \subseteq H_1$, the set $\overline{T(S)}$ is compact in H_2.

Example 2.2. Every $m \times n$ matrix corresponds to a compact operator.

Example 2.3. Every bounded finite rank operator is compact.

Notation: The set of all compact operators from H_1 into H_2 is denoted by $\mathcal{K}(H_1, H_2)$ and if $H_1 = H_2 = H$, then $\mathcal{K}(H)$.

Remark 2.4.
1. Every compact operator is bounded.
   The converse need not be true. For example consider the identity operator $I : H \to H$. Clearly I is bounded. Then I is compact if and only if dimension of H is finite.
2. An isometry is compact if and only if it is a finite rank operator.
3. Restriction of a compact operator to a closed subspace is again compact
4. An orthogonal projection onto a closed subspace of a Hilbert space is compact if and only if it is of finite rank.
5. Let $T \in \mathcal{B}(H)$ be a compact operator which is not a finite rank operator. Then $R(T)$ cannot be closed.

In view of Theorem 1.11, the definition of a compact operator can be described as follows.

Let $T : H_1 \to H_2$ be a bounded operator and $B := \{x \in H : \|x\| \leq 1\}$. Then the following conditions are equivalent.

1. $\overline{T(B)}$ is compact

(2) For every bounded sequence $(x_n) \subseteq H_1$, (Tx_n) has a convergent subsequence in H_2
(3) T maps bounded sets into totally bounded sets.

2.1. **Properties.**

Theorem 2.5. *Let $T_1, T_2 : H_1 \to H_2$ be compact operators and $\alpha \in \mathbb{C}$. Then*
(1) *αT_1 is compact*
(2) *$T_1 + T_2$ is compact.*

Proof. Proof of (1) is obvious.

For the proof of (2), let $(x_n) \subseteq H$ be a bounded sequence. Since T_1 is compact, $T_1 x_n$ has a subsequence $T_1 x_{n_k}$ which is convergent, say $T_1 x_{n_k} \to y$. Now x_{n_k} is bounded sequence. Since T_2 is compact, there exists a subsequence $(x_{n_{k_l}})$ of (x_{n_k}) such that $T_2 x_{n_{k_l}}$ is convergent. Note that $T_1 x_{n_{k_l}}$ is convergent. Therefore $(T_1 + T_2)x_{n_{k_l}}$ is convergent. Hence $T_1 + T_2$ is compact. □

From Theorem 2.5, we can conclude that $\mathcal{K}(H_1, H_2)$ is a vector subspace of $\mathcal{B}(H_1, H_2)$.

Theorem 2.6. *Let $A : H_1 \to H_2$ be compact and $B : H_3 \to H_1$, $C : H_2 \to H_3$ are bounded. Then CA and AC are compact.*

Proof. Let $(x_n) \subseteq H_1$ be a bounded sequence. As A is compact, there exists a subsequence (x_{n_k}) of (x_n) such that Ax_{n_k} is convergent. Since C is bounded, CAx_{n_k} is also convergent. Hence CA is compact.

Now

$$\begin{aligned} B \text{ is bounded} &\Rightarrow (Bx_n) \text{ is bounded} \\ &\Rightarrow (ABx_n) \text{ has a convergent subsequence, since A is compact} \\ &\Rightarrow AB \text{ is compact.} \end{aligned}$$

□

Conclusion: From the above two results one can conclude that the set of all compact operator on H is a two sided ideal in $\mathcal{B}(H)$, the space of all bounded operators on H.

Remark 2.7. By Theorem 2.6, if $T \in \mathcal{K}(H)$, then $T^2 \in \mathcal{K}(H)$. But the converse need not be true.

Exercise 2.2. *Let $T : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$ given by*

$$T(x, y) = (0, x), \quad (x, y) \in \ell^2 \oplus \ell^2$$

is not compact (Note that $T^2 = 0$).

Theorem 2.8. *Let $T \in B(H_1, H_2)$. Then*
(1) *T is compact $\Leftrightarrow$ T^*T or TT^* is compact*
(2) *T is compact $\Leftrightarrow$ T^* is compact*

Proof. **Proof of (1)**:
If T is compact, then T^*T is compact by Theorem 2.6. To prove the converse, assume that $T^*T : H_1 \to H_1$ is compact. If $(x_n) \subseteq H_1$ is a bounded sequence with bound $M > 0$, then T^*Tx_n has a convergent subsequence namely $T^*Tx_{n_k}$,

say $T^*Tx_{n_k} \to y$. For notational convenience we denote the subsequence $(x_{n_k})$by (x_n).

For $n > m$, we have

$$\begin{aligned}\|Tx_n - Tx_m\|^2 &= \langle T(x_n - x_m), T(x_n - x_m)\rangle \\ &= \langle T^*T(x_n - x_m), (x_n - x_m)\rangle \\ &\le \|T^*T(x_n - x_m)\| \, \|x_n - x_m\| \\ &\le 2M \, \|T^*T(x_n - x_m)\|.\end{aligned}$$

Hence (Tx_n) is Cauchy and hence convergent.

Similarly, $TT^* \in \mathcal{K}(H_2)$.

Proof of (2):
Let $(z_n) \subseteq H_2$ be a bounded sequence. As TT^* is compact, (TT^*z_n) has a convergent subsequence, say $TT^*z_{n_k}$ converging to z. Now for $k > l$,

$$\begin{aligned}\|T^*z_{n_k} - T^*z_{n_l}\|^2 &= \langle TT^*(z_{n_k} - z_{n_l}), z_{n_k} - z_{n_l}\rangle \\ &\le \|TT^*(z_{n_k} - z_{n_l})\| \, \|z_{n_k} - z_{n_l}\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

That is $(T^*z_{n_k})$ is Cauchy, hence convergent. Thus T^* is compact.

By the above argument, T^* is compact implies that $T^{**} = T$ is compact. □

Theorem 2.9. *$\mathcal{K}(H)$ is a closed in $B(H)$.*

Proof. Let (K_n) be a sequence of compact operators converging to K. Let $M > 0$ be such that $\|K_n\| \le M$ for all n. Our aim is to show that K is compact. Let (x_i) be a bounded sequence in H. Let (x_i^1) be a subsequence of (x_i) be such that $(K_1x_i^1)$ is convergent. Let $(x_i^2) \subseteq (x_i^1)$ such that $(K_2x_i^2)$ is convergent. Let $(x_i^3) \subseteq x_i^2$ be such that $(K_3x_i^3)$ is convergent. Continuing this process, let (x_i^n) be a subsequence of (x_i^{n-1}) such that $(K_nx_i^n)$ is convergent.

The sequence $(z_i) = (x_i^i)$ is a subsequence of (x_i). Also for each n, except the first n terms, (z_i) is a subsequence of (x_i^n) such that (K_nz_i) is convergent.

Now for all i, j and n we have,

$$\begin{aligned}\|Kz_i - Kz_i\| &= \|(K - K_n)z_i + K_nz_i - K_nz_j - (K - K_n)z_j\| \\ &\le \|K - K_n\| \Big(\|z_i\| + \|z_j\|\Big) + \|K_n(z_i - z_j)\|.\end{aligned}$$

That is (Kz_i) is Cauchy, hence convergent as H is a Hilbert space. This concludes that K is compact. □

Lemma 2.10. *Let K be a compact operator on a separable Hilbert space H and suppose that $(T_n) \subseteq B(H)$ and $T \in \mathcal{B}(H)$ are such that for each $x \in H$, the sequence $T_nx \to Tx$. Then $T_nK \to TK$ in the norm of $B(H)$.*

Proof. Suppose that $\|T_nK - TK\| \nrightarrow 0$. Then there exists a $\delta > 0$ and a subsequence $\{T_{n_j}K\}$ such that

$$\|T_{n_j}K - TK\| > \delta.$$

Choose unit vectors (x_{n_i}) of H such that

$$\|(T_{n_j}K - TK)(x_{n_i})\| > \delta.$$

Since K is compact, we get a subsequence (x_{n_j}) of (x_{n_i}) such that Kx_{n_j} is convergent. Assume that $Kx_{n_j} \to y$. Then

$$\delta < \|(T_{n_j}K - TK)x_{n_j}\| \leq \|(T_{n_i} - T)(Kx_{n_j} - y)\| + \|(T_{n_j} - T)y\| \tag{2.1}$$

Since $Kx_{n_j} \to y$, there exists n such that for $n_j > n$,

$$\|Kx_{n_j} - y\| < \frac{\delta}{8C}.$$

Also as $T_{n_j}y \to Ty$ for ach $y \in H$, there exists m such that $n_j > m$ implies

$$\|(T - T_{n_j})y\| < \frac{\delta}{4}.$$

Since $(T_n) \subseteq \mathcal{B}(H)$ is bounded, then $\|T_n\| \leq C$ and $\|Tx\| = \lim_{n\to} \|T_n x\| \leq C$. Hence $\|T - T_{n_j}\| \leq 2C$. Now from Equation 2.1,

$$\delta < \|(T_{n_j}K - TK)x_{n_j}\| < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2},$$

a contradiction. □

Theorem 2.11. *Every compact operator on a separable Hilbert space H is a norm limit of a sequence of finite rank operators. In other words , the set of finite rank operators is dense in the space of compact operators.*

Proof. Let $\{\phi_n : n \in \mathbb{N}\}$ be an orthonormal basis for H and $H_n := \text{span}\{\phi_k\}_{k=1}^n$. Then the orthogonal projections $P_n : H \to H$ defined by

$$P_n x = \sum_{j=1}^{n} \langle x, \phi_j \rangle \phi_j$$

has the property that $P_n x \to x$ for each $x \in H$.

Now, if K is compact, then by Lemma 2.10, it follows that $P_n K \to K$ in the operator norm of $B(H)$. Here $R(P_n K) \subseteq R(P_n) = H_n$ is finite dimensional. □

Example 2.12. Let $H = \ell^2$ and $\{e_n\}$ be the standard orthonormal basis of H. Define $D : H \to H$ by

$$D(x_1, x_2, x_3, \dots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots), \text{ for all } (x_n) \in H.$$

Then D is bounded. Next, we show that $D \in \mathcal{K}(H)$. Define $D_n : H \to H$ by $D_n x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j$. Then D_n is finite rank bounded operator and $D_n \to D$ as $n \to \infty$. Hence by theorem 2.11, D is compact.

Example 2.13. Let $T = RD$ and $S = DR$ where R is the right shift operator and D is as in example 2.12. Then both T and S are compact.

Example 2.14. Let $k(\cdot, \cdot) \in L^2[a, b]$. Define $K : L^2[a, b] \to L^2[a, b]$ by

$$(Kf)(s) = \int_a^b k(s,t) f(t) dt, \text{ for all } f \in L^2[a, b].$$

It can be verified that $K \in \mathcal{B}(L^2[a,b])$. Let $\{\phi_n : n \in \mathbb{N}\}$ be an orthonormal basis for $L^2[a,b]$. Then $\psi_{m,n}(s,t) = \phi_n(s)\phi_m(t)$ for all $s,t \in [a,b]$ and for all $m,n \in \mathbb{N}$ forms an orthonormal basis for $L^2([a,b] \times [a,b])$. Hence

$$k(s,t) = \sum_{m,n=1}^{\infty} \langle k(s,t), \psi_{m,n}(s,t)\rangle \psi_{m,n}(s,t).$$

Let

$$k_N(s,t) = \sum_{m,n=1}^{N} \langle k(s,t), \psi_{m,n}(s,t)\rangle \psi_{m,n}(s,t).$$

Now define $K_N : L^2[a,b] \to L^2[a,b]$ by

$$(K_N f)(s) = \int_a^b k_N(s,t) f(t)dt, \text{ for all } f \in L^2[a,b].$$

Note that K_N is a finite rank operator and $K_N \to K$ as $N \to \infty$. Hence by Theorem 2.11, $K \in K(L^2[a,b])$.

3. The Spectral Theorem

Definition 3.1. A complex number $\lambda \in \mathbb{C}$ is called an eigenvalue of $T \in \mathcal{B}(H)$ if there exists a vector $0 \neq x \in H$ such that $Tx = \lambda x$. The vector x is called an eigenvector for T corresponding to the eigenvalue λ. Equivalently λ is an eigenvalue of T iff $T - \lambda I$ is not one-to-one.

Example 3.2. Let $H = \ell^2$. Define $T : H \to H$ by

$$T(x_1, x_2, x_3, \dots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots), \quad (x_1, x_2, x_3, \dots) \in H.$$

Let $\{e_n : n \in \mathbb{N}\}$ denote the standard orthonormal basis for H. Then $Te_n = \frac{1}{n} e_n$. Hence $\{\frac{1}{n}\}$ is a set of eigenvalues of T with corresponding eigenvectors e_n.

Exercise 3.1. *Let $R : \ell^2 \to \ell^2$ be given by*

$$R(x_1, x_2, \dots) = (0, x_1, x_2, \dots), \quad (x_1, x_2, \dots) \in \ell^2.$$

Find the eigenvalues and eigenvectors of R.

Exercise 3.2. *Show that the operator $T : \ell^2 \to \ell^2$ defined by*

$$T(x_1, x_2, \dots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \dots), \quad (x_1, x_2, \dots)$$

is compact but has no eigenvalues.

3.3. Self-adjoint Operators.

Definition 3.3. Let $T \in \mathcal{B}(H)$. If $T = T^*$, then T is called **self-adjoint**.

The operators in Example 3.2 is self-adjoint, where as the operator in Exercise 3.1 is not.

Exercise 3.4. *Let $M : L^2[0,1] \to L^2[0,1]$ be given by*

$$(Mf)(t) = tf(t), \quad f \in L^2[0,1], \ t \in [0,1].$$

Show that M is self-adjoint and has no eigenvalue.

Proposition 3.4. *Let $T \in \mathcal{B}(H)$ be self-adjoint. Then*

(1) *eigenvalues of T are real*

(2) *eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. **Proof of 1**:
Let λ be an eigenvalue of T and x be the corresponding eigen vector. Then $Tx = \lambda x$.

$$\lambda\|x\|^2 = \lambda \langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \bar{\lambda}\, \|x\|^2.$$

Since $x \neq 0$, we have $\lambda = \bar{\lambda}$.
Proof of 2:
Let λ and μ be distinct eigenvalues of T and x and y be the corresponding eigenvectors. Then $Tx = \lambda x$ and $Ty = \mu y$. Now

$$\mu \langle x, y\rangle = \langle x, \mu y\rangle = \langle x, Ty\rangle = \langle Tx, y\rangle = \lambda\langle x, y\rangle.$$

Since $\lambda, \mu \in \mathbb{R}$ and distict, it follows that $\langle x, y\rangle = 0$. □

Theorem 3.5. *Let $T \in \mathcal{B}(H)$ be self-adjoint. Then* $\|T\| = \sup\limits_{\|x\|=1} |\langle Tx, x\rangle|$.

Proof. Let $m = \sup\limits_{\|x\|=1} |\langle Tx, x\rangle|$. Note that $|\langle Tx, x\rangle| \leq m$ for each $x \in H$ with $\|x\| = 1$. If $\|x\| = 1$, then by the Cauchy-Schwartz-Bunyakovsky inequality,

$$|\langle Tx, x\rangle| \leq \|Tx\|\,\|x\| = \|Tx\|.$$

Hence $m \leq \|T\|$.

To prove the other way, let $x, y \in H$. Then $\langle T(x \pm y), x \pm y\rangle = \langle Tx, x\rangle + 2\mathrm{Re}\langle Tx, y\rangle + \langle Ty, y\rangle$. Therefore,

$$\begin{aligned} 4\mathrm{Re}\langle Tx, y\rangle &= \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \\ &\leq |\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle| \\ &\leq m\Big(\|x+y\|^2 + \|x-y\|^2\Big) \\ &= 2m\,(\|x\|^2 + \|y\|^2) \quad \text{(by the parallelogram law)}. \end{aligned}$$

Now $\langle Tx, y\rangle = |\langle Tx, y\rangle| \exp(i\theta)$ for some real θ. Substituting $x\exp(i\theta)$ in the above equation, we get

$$\begin{aligned} 4\mathrm{Re}\langle Tx\exp(-i\theta), y\rangle &\leq 2m(\|x\|^2 + \|y\|^2) \\ 4|\langle Tx, y\rangle| &\leq 2m(\|x\|^2 + \|y\|^2). \end{aligned}$$

Substituting $y = \dfrac{\|x\|}{\|Tx\|}Tx$ in place of y in the above equation, we get $\|Tx\| \leq m\,\|x\|$. Hence $\|T\| = m$. □

Corollary 3.6. *Let $T \in \mathcal{B}(H)$ be self-adjoint. If $\langle Tx, x\rangle = 0$, for each $x \in H$. Then $T = 0$.*

Theorem 3.7. *Let $T \in \mathcal{B}(H)$ be self-adjoint.*

(1) *Let $\lambda = \inf_{\|x\|=1} \langle Tx, x\rangle$. If there exists an $x_0 \in H$ such that $\|x_0\| = 1$ and $\lambda = \langle Tx_0, x_0\rangle$, then λ is an eigenvalue of T with corresponding eigenvector x_0.*
(2) *Let $\mu = \sup_{\|x\|=1} \langle Tx, x\rangle$. If there exists a $x_1 \in H$, $\|x_1\| = 1$ such that $\mu = \langle Tx_1, x_1\rangle$, then μ is eigenvalue value of T with corresponding eigenvector x_1.*

Proof. Let $\alpha \in \mathbb{C}$ and $v \in H$, by defintion of λ, we have

$$\langle T(x_0 + \alpha v), x_0 + \alpha v\rangle \geq \lambda\langle x_0 + \alpha v, x_0 + \alpha v\rangle.$$

Hence

$$\langle Tx_0, x_0\rangle + \bar{\alpha}\langle Tx_0, v\rangle + \alpha\langle Tv, x_0\rangle + |\alpha|^2\langle Tv, v\rangle \geq \lambda\langle x_0, x_0\rangle + \lambda\bar{\alpha}\langle x_0, v\rangle + \lambda\alpha\langle x_0, v\rangle + \lambda|\alpha|^2\langle v, v\rangle.$$

That is

$$\langle Tx_0, x_0\rangle + 2\,\alpha\mathrm{Re}\,\langle Tx_0, v\rangle + |\alpha|^2\langle Tv, v\rangle \geq \lambda\langle x_0, x_0\rangle + \lambda|\alpha|^2\langle v, v\rangle + 2\,\mathrm{Re}\,\alpha\lambda\langle x_0, v\rangle.$$

Substituting $\lambda = \langle Tx_0, x_0\rangle$ and taking $\alpha = r\,\overline{\langle v, (T - \lambda I)x_0\rangle}$, $r \in \mathbb{R}$, we can conclude that $\langle v, (T - \lambda I)x_0\rangle = 0$ for each $v \in H$. Hence $Tx_0 = \lambda x_0$.

The proof of the other statement follows by taking $-T$ in place of T. □

Theorem 3.8. *If $T \in \mathcal{B}(H)$ is compact and self-adjoint, then at least one of the numbers $\|T\|$ or $-\|T\|$ is an eigenvalue.*

Proof. By Theorem 3.7, there exists a sequence $(x_n) \subseteq H$ with $\|x_n\| = 1$ for each n such that and $\lambda \in \mathbb{R}$ such that $\langle Tx_n, x_n\rangle \to \lambda$, where $\lambda = +\|T\|$ or $\lambda = -\|T\|$. Now

$$\begin{aligned}\|Tx_n - \lambda x_n\|^2 &= \|Tx_n\|^2 + \lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \\ &\leq 2\lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \to 0 \text{ as } n \to \infty.\end{aligned}$$

Since T is compact, there exists a sub sequence (Tx_{n_j}) of (Tx_n) such that $Tx_{n_j} \to y$. So $Tx_{n_j} - \lambda x_{n_j} \to 0$ as $n \to \infty$. That is $x_n \to \frac{1}{\lambda}y$. Hence $y = \lim Tx_{n_j} = \frac{1}{\lambda}Ty$. Hence λ is an eigen value. □

Corollary 3.9. *If $T \in \mathcal{K}(H)$ be self-adjoint, then $\max_{\|x\|=1} |\langle Tx, x\rangle| = \|T\|$.*

Exercise 3.5. *Let $K : L^2[0,1] \to L^2[0,1]$ be defined by*

$$(Kf)(t) = \int_0^t f(s)ds.$$

Check that K is non self-adjoint, compact and does not possesses eigenvalues.

Definition 3.10. A closed subspace M of H is said to be **invariant** under $T \in \mathcal{B}(H)$ if and only if $T(M) \subseteq M$. If both M and $M^{\perp}$ are invariant under T, then we say that M is a **reducing** subspace for T.

Exercise 3.6. *M is invariant under T iff $M^{\perp}$ is invariant under T^*.*

Exercise 3.7. *Let $T \in \mathcal{B}(H)$. Let M be a closed subspace of a Hilbert space and $P : H \to H$ be an orthogonal projection such that $R(P) = M$. Then*

1. *M is invariant under T iff $TP = PTP$*
2. *M reduces T iff $TP = PT$.*

Observations

1. If T is a self-adjoint operator, then every invariant subspace is reducing.
2. every eigen space corresponding to a particular eigenvalue is invariant under the operator. In particular, if the operator is self-adjoint every eigen space reduce the operator.

Note 3.8. If T is compact operator and M is a closed subspace of H, then the restriction operator $T|_M$ is also compact.

Theorem 3.11 (The Spectral theorem)**.** *Suppose $T \in \mathcal{K}(H)$ be self-adjoint. Then there exists a system of orthonormal vectors $\phi_1, \phi_2, \phi_3, \dots$ of eigenvectors of T and corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots$ such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$,*

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, \phi_k \rangle \phi_k \text{ for all } x \in H.$$

If (λ_n) is infinite, then $\lambda_n \to 0$ as $n \to \infty$. The series on the right hand side converges in the operator norm of $\mathcal{B}(H)$.

Proof. We prove the theorem step by step.

Step 1: Construction of eigen vectors

We use Theorem 3.8 repeatedly for constructing eigenvalues and eigenvectors.

Let $H_1 = H$ and $T_1 = T$. Then by Theorem 3.8, there exists an eigenvalue λ_1 of T_1 and an eigenvector ϕ_1 such that $\|\phi_1\|$ and $|\lambda_1| = \|T_1\|$. Now span$\{\phi_1\}$ is a closed subspace of H_1, hence by the projection theorem, $H_1 = \text{span}\{\phi_1\} \oplus^{\perp} \text{span}\{\phi_1\}^{\perp}$. Let $H_2 = \text{span}\{\phi_1\}^{\perp}$. Clearly H_2 is a closed subspace of H_1 and $T(H_2) \subseteq H_2$.

Now let $T_2 = T_1|_{H_2}$. Then T_2 is a compact and self-adjoint operator in $\mathcal{B}(H_2)$. If $T_2 = 0$, then there is nothing to prove. Assume that $T_2 \neq 0$. Then by Theorem 3.8, there exists an eigenvalue λ_2 of T_2 with $|\lambda_2| = \|T_2\|$and a corresponding eigenvector ϕ_2 with $\|\phi_2\| = 1$. Since T_2 is a restriction of T_1, $|\lambda_2| = \|T_2\| \leq \|T_1\| = |\lambda_1|$. By the construction ϕ_1 and ϕ_2 are orthonormal. Now let $H_3 = \text{span}\{\phi_1, \phi_2\}^{\perp}$. Clearly $H_3 \subseteq H_2$. It is easy to show that $T(H_3) \subseteq H_3$. The operator $T_3 = T|_{H_3}$ is compact and self-adjoint. Hence by Theorem 3.8, there exists an eigenvalue λ_3 of T_3 and a corresponding eigenvector ϕ_3 with $\|\phi_3\| = 1$. Here $|\lambda_3| = \|T_3\|$. Hence $|\lambda_3| \leq |\lambda_2| \leq |\lambda_1|$.

Proceeding in this way, either after some stage n, $T_n = 0$ or there exists a sequence λ_n of eigenvalues of T and corresponding vector ϕ_n with $\|\phi_n\| = 1$ and $|\lambda_n| = \|T_n\|$. Also $|\lambda_{n+1}| \leq |\lambda_n|$ for each n.

Step 2: If (λ_n) is infinite, then $\lambda_n \to 0$

If $\lambda_n \nrightarrow 0$, there exists an $\epsilon > 0$ such that $|\lambda_n| \geq \epsilon$ for infinitely many n' s. If $n \neq m$,

$$\|T\phi_n - T\phi_m\|^2 = \|\lambda_n\phi_n - \lambda_m\phi_m\|^2 = \lambda_m^2 + \lambda_n^2 > \epsilon^2.$$

This shows that $(T\phi_n)$ has no convergent subsequence, a contradiction to the compactness of T. Hence $\lambda_n \to 0$ as $n \to \infty$.

Step 3: Representation of T

Here we consider two cases

Case 1: $T_n = 0$ **for some** n

Let $x_n = x - \sum_{k=1}^{n} \langle x, \phi_k \rangle \phi_k$. Then $x_n \perp \phi_i$ for $1 \le i \le n$. Hence

$$0 = T_n x_n = Tx - \sum_{k=1}^{n} \lambda_k \langle x, \phi_k \rangle \phi_k.$$

That is $Tx = \sum_{k=1}^{n} \lambda_k \langle x, \phi_k \rangle \phi_k$.

Case 2: $T_n \neq 0$ for infinitely many n

For $x \in H$, by Case 1, we have

$$\begin{aligned} \|Tx - \sum_{k=1}^{n} \lambda_k \langle x, \phi_k \rangle \phi_k| = \|T_n x_n\| &\le \|T_n\| \, \|x_n\| \\ &= |\lambda_n| \, \|x_n\| \le |\lambda_n| \, \|x\| \\ &\to 0. \end{aligned}$$

Hence $Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, \phi_k \rangle \phi_k$. □

Observations:

(a) $N(T)^{\perp} = \overline{\text{span}}\{\phi_1, \phi_2, \dots\}$

(b) By the projection theorem $H = N(T) \oplus^{\perp} N(T)^{\perp} = N(T) \oplus^{\perp} \overline{R(T)}$. The spectral theorem guarantees the existstence of the orthonormal basis for $\overline{R(T)}$. Hence $R(T)$ is separable. In addition if $N(T)$ is separable, then H is separable.

(c) $H = N(T) \oplus_{k=1}^{\infty} G_i$, where $G_n = H_n^{\perp} = \oplus_{i=1}^{n-1} N(T - \lambda_i I)$.

(d) $T|_{N(T)^{\perp}} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n, \dots)$.

(e) each λ_j is repeated in the sequence $\{\lambda_n\}$ exactly $p_j = \dim N(T - \lambda_j I)$ times. Since the sequence $\lambda_n \to 0$, each λ_j appears finitely many times. Let $\lambda_j = \lambda_{n_i}$, $i = 1, 2, \dots p$. Then $N(T - \lambda_j I) = \text{span}\{\phi_{n_1}, \dots, \phi_{n_p}\}$. If this is not possible, then there exists a $0 \neq v \in N(T - \lambda_j I)$ such that $v \perp \phi_{n_i}$ where $1 \le i \le p$. If $k \neq n_i$ $1 \le i \le p$, $v \perp \phi_k$ as $\lambda_j \neq \lambda_k$. Hence

$$\lambda_j v = Tv = \sum_k \lambda_k \langle v, \phi_k \rangle \phi_k = 0$$

which is impossible because $\lambda_j \neq 0$ and $v \neq 0$.

Example 3.12. Define $D : \ell^2 \to \ell^2$ by

$$D(x_1, x_2, \dots) = (x_1, \frac{x_2}{2}, \dots) \quad (x_1, x_2, \dots) \in \ell^2.$$

Then $T \in \mathcal{K}(H)$ and self-adjoint. Note that $De_n = \frac{1}{n} e_n$. Hence $\frac{1}{n}$ is an eigenvalue with an eigen vector e_n. Also $\frac{1}{n} \to 0$ as $n \to \infty$. We can also represent D as $D(x) = \sum_{n=0}^{\infty} \frac{1}{n} \langle x, e_n \rangle e_n$ for all $x = (x_n) \in \ell^2$.

Exercise 3.9. (1) *Let $K \in \mathcal{K}(H)$. Show that $R(K-\lambda I)$ is closed for each $\lambda \in \mathbb{C}\setminus\{0\}$.*

The converse of the spectral theorem is also true.

Theorem 3.13. *Suppose that $\phi_1, \phi_2, \dots$ is a sequence of orthogonal vectors in H and (λ_k) be a sequence of real numbers such that (λ_k) is finite or converges to 0. Then the operator defined by*

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, \phi_k\rangle \phi_k$$

is compact and self-adjoint.

Proof. We consider the following cases.
Case 1: (λ_k) is finite
Consider $Tx = \sum_{k=1}^{n} \lambda_k \langle x, \phi_k\rangle \phi_k$. Then

$$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \sum_{k=1}^{n} |\lambda_k|^2 |\langle x, \phi_k\rangle|^2 \\ &\leq \max_k |\lambda_k| \|x\|^2.\end{aligned}$$

This shows that $T \in \mathcal{B}(H)$ and T is a finite rank operator. Hence T is compact.
Case 2: (λ_k) is infinite and $\lambda_k \to 0$ as $n \to \infty$.
By the Spectral Theorem, $\|Tx\|^2 = \sum_n |\lambda_n|^2 |\langle x, \phi_n\rangle|^2 \leq \max_{k\geq n} |\lambda_k| \|x\|^2$. Hence $\|T\| < \infty$. Now define $T_n x = \sum_{k=1}^{n} \lambda_k \langle x, \phi_k\rangle \phi_k$. Then

$$\begin{aligned}\|T - T_n\|^2 &= \sup_{\|x\|=1} \| \sum_{k=n+1} \lambda_k \langle x, \phi_k\rangle \phi_k\|^2 \\ &\leq \sup_{k>n} |\lambda_k|^2 \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

As each T_n is finite rank and hence compact, T is compact. It easy to verify that $T = T^*$. □

3.10. Second form of the Spectral Theorem.

Definition 3.14. An orthonormal system $\phi_1, \phi_2, \dots$ of eigenvectors of $T \in \mathcal{B}(H$ with corresponding non zero eigen values $\lambda_1, \lambda_2, \dots$ is called a **basic system of eigenvalues and eigenvectors** of T if

$$Tx = \sum_{k} \lambda_k \langle x, \phi_k\rangle \phi_k$$

.

The Spectral Theorem guarantees the existence of a basic system of eigenvalues and eigenvectors for a compact self-adjoint operator.

Theorem 3.15. *Let T be a compact self-adjoint operator and μ_j be the set of all non zero eigenvalues of T and let P_j be the orthogonal projection onto $N(T - \mu_j I)$. Then*

(1) $P_jP_k = 0, \quad j \neq k$
(2) $T = \sum_j \mu_j P_j$, *where the convergence of the series is with respect to the norm of* $\mathcal{B}(H)$.
(3) *For each* $x \in H$,

$$x = P_0 x + \sum_j P_j x,$$

where P_0 *is the orthogonal projection onto* $N(T)$.

Proof. Let $\{\phi_n\}, \ \{\lambda_n\}$ be a basic system of eigen vectors and eigen values of T. For each k, a subset of $\{\phi_n\}$ is an orthonormal basis for $N(T - \mu_k I)$ say, ϕ_{n_i} $1 \leq i \leq p$. Then $P_k x = \sum_{i=1}^{p} \langle x, \phi_{n_i} \rangle \phi_{n_i}$ since each λ_n is μ_k, it follows that

$$x = P_0 x + \sum P_k x \quad \text{and} \quad Tx = \sum_k \lambda_k P_k x.$$

Furthermore $P_jP_k = 0, \ \ j \neq k$, since $N(T - \mu_j I) \perp N(T - \mu_k I)$.

If (λ_n) is an infinite sequence, then

$$\begin{aligned} \|T - \sum_{k=1}^{n} \mu_k P_k\|^2 &= \sup_{\|x\|=1} \|Tx - \sum_{k=1}^{n} \mu_k P_k x\|^2 \\ &\leq \sup_{\|x\|=1} \sum_{j \geq n} \lambda_j^2 |\langle x, \phi_j \rangle|^2 \\ &\leq \sup_{j>n} |\lambda_j|^2 \to 0 \text{ as } n \to \infty. \end{aligned}$$

□