Laplace Transforms

G. Ramesh

21 Sep 2015

Outline

1 Lecture-IV

LAPLACE TRANSFORM OF DERIVATIVES

Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function with exponential order α. Assume that f' is piecewise continuous. Then

- $\mathcal{L}(f')$ exists and
- $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$.

Laplace transform of Derivatives

Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function with exponential order α. Assume that f' is piecewise continuous. Then

- $\mathcal{L}(f')$ exists and
- $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$.

Example

1. Let $f(t) = \sin^2(t)$. Find $\mathcal{L}(f)$?

LAPLACE TRANSFORM OF DERIVATIVES

Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function with exponential order α. Assume that f' is piecewise continuous. Then

- $\mathcal{L}(f')$ exists and
- $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$.

Example

1. Let $f(t) = \sin^2(t)$. Find $\mathcal{L}(f)$?
   Ans: $\mathcal{L}(f) = \dfrac{2}{s(s^2+4)}$.
2. Find $\mathcal{L}(t\sin(wt)),\ t \in [0, \infty)$?.

Laplace transform of Derivatives

Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function with exponential order α. Assume that f' is piecewise continuous. Then

- $\mathcal{L}(f')$ exists and
- $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$.

Example

1. Let $f(t) = \sin^2(t)$. Find $\mathcal{L}(f)$?
   Ans: $\mathcal{L}(f) = \dfrac{2}{s(s^2+4)}$.
2. Find $\mathcal{L}(t\sin(wt)),\ t \in [0, \infty)$?.
   Ans: $\mathcal{L}(f) = \dfrac{2ws}{(s^2+w^2)^2}$.

GENERALIZATION

THEOREM

Let $f, f', f'', \cdots, f^{n-1}$ *be continuous on* $[0, \infty)$ *and* f^j $(j = 1, 2, \ldots, n)$ *be of exponential order* α. *Then* f^n *is piecewise continuous and*

$$\mathcal{L}(f^n) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) -, \ldots, -f^{n-1}(0).$$

GENERALIZATION

THEOREM

Let $f, f', f'', \cdots, f^{n-1}$ be continuous on $[0, \infty)$ and f^j $(j = 1, 2, \ldots, n)$ be of exponential order α. Then f^n is piecewise continuous and

$$\mathcal{L}(f^n) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) -, \ldots, -f^{n-1}(0).$$

Example

Solve the IVP: $y' + 4y = e^t, \; y(0) = 2.$

GENERALIZATION

THEOREM

Let $f, f', f'', \cdots, f^{n-1}$ be continuous on $[0, \infty)$ and f^j $(j = 1, 2, \ldots, n)$ be of exponential order α. Then f^n is piecewise continuous and

$$\mathcal{L}(f^n) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) -, \ldots, -f^{n-1}(0).$$

Example

Solve the IVP: $y' + 4y = e^t, \; y(0) = 2.$

- $\mathcal{L}(y) = \frac{9}{5}(\frac{1}{s+4}) + \frac{1}{5}(\frac{1}{s-1})$

GENERALIZATION

THEOREM

Let $f, f', f'', \cdots, f^{n-1}$ be continuous on $[0, \infty)$ and f^j $(j = 1, 2, \ldots, n)$ be of exponential order α. Then f^n is piecewise continuous and

$$\mathcal{L}(f^n) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) -, \ldots, -f^{n-1}(0).$$

Example

Solve the IVP: $y' + 4y = e^t, \ y(0) = 2.$

- $\mathcal{L}(y) = \frac{9}{5}(\frac{1}{s+4}) + \frac{1}{5}(\frac{1}{s-1})$
- $y(t) = \frac{9}{5}e^{-4t} + \frac{1}{5}e^t.$

LAPLACE TRANSFORMS FOR INTEGRALS

THEOREM

Let $f : [0, \infty) \to \mathbb{R}$ be a piecewise continuous function with exponential order α. Then

$$\mathcal{L}\left(\int_0^t f(u)du\right) = \frac{\mathcal{F}(s)}{s}, \quad (s > 0, \; s > \alpha)$$

Hence $\mathcal{L}^{-1}\left(\frac{\mathcal{F}(s)}{s}\right) = \int_0^t f(u)du$.

Laplace transforms for integrals

Theorem

Let $f : [0, \infty) \to \mathbb{R}$ be a piecewise continuous function with exponential order α. Then

$$\mathcal{L}\left(\int_0^t f(u)du\right) = \frac{\mathcal{F}(s)}{s}, \quad (s > 0,\ s > \alpha)$$

Hence $\mathcal{L}^{-1}\left(\frac{\mathcal{F}(s)}{s}\right) = \int_0^t f(u)du$.

Example-1

Find $\mathcal{L}^{-1}\left(\frac{1}{s(s^2 + w^2)}\right)$

LAPLACE TRANSFORMS FOR INTEGRALS

THEOREM

Let $f : [0, \infty) \to \mathbb{R}$ be a piecewise continuous function with exponential order α. Then

$$\mathcal{L}\left(\int_0^t f(u)du\right) = \frac{\mathcal{F}(s)}{s}, \quad (s > 0, \ s > \alpha)$$

Hence $\mathcal{L}^{-1}\left(\frac{\mathcal{F}(s)}{s}\right) = \int_0^t f(u)du$.

EXAMPLE-1

Find $\mathcal{L}^{-1}\left(\frac{1}{s(s^2+w^2)}\right)$

Answer: $f(t) = \frac{1-\cos(wt)}{w^2}$.

EXAMPLE-2

Find $\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+w^2)}\right)$

EXAMPLE-2

Find $\mathcal{L}^{-1}\left(\dfrac{1}{s^2(s^2+w^2)}\right)$

Answer: $f(t) = \dfrac{t - \frac{\sin(wt)}{w}}{w^2}$.

DIFFERENTIATION OF LAPLACE TRANSFORMS

THEOREM

Let $f : [0, \infty) \to \mathbb{R}$ be piecewise continuous and has exponential order α. Then

$$\frac{d}{ds}(\mathcal{F}(s)) = (-1)(tf(t)).$$

Hence $\mathcal{L}^{-1}(tf(t)) = (-1)\frac{d}{ds}(\mathcal{F}(s))$.

EXAMPLES

- $\mathcal{L}\Big(t\cos(wt)\Big) = \dfrac{s^2 - w^2}{s^2 + w^2}.$
- $\mathcal{L}\Big(t\sin(wt)\Big) = \dfrac{2ws}{s^2 + w^2}.$

Exercises

Find the inverse Laplace transforms of the following:

- $\mathcal{L}^{-1}\left(\log(\frac{s+a}{s+b})\right)$
- $\mathcal{L}^{-1}\left(\log(\frac{s^2+a^2}{s^2+b^2})\right)$

Integration of Laplace transform

Let $f : [0, \infty) \to \mathbb{R}$ be a piecewise continuous function with exponenetial order α. Assume that $\lim_{t \to 0+} \frac{f(t)}{t}$ exists. Then

$$\int_s^\infty F(u)du = \mathcal{L}\left(\frac{f(t)}{t}\right) \quad (s > \alpha).$$

Examples

1. $\mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{x^2+1} dx = \frac{\pi}{2} - \tan^{-1}(s) = \tan^{-1}(\frac{1}{s}), \quad s > 0$

Integration of Laplace transform

Let $f : [0, \infty) \to \mathbb{R}$ be a piecewise continuous function with exponenetial order α. Assume that $\lim_{t \to 0+} \frac{f(t)}{t}$ exists. Then

$$\int_s^\infty F(u)du = \mathcal{L}\left(\frac{f(t)}{t}\right) \quad (s > \alpha).$$

Examples

1. $\mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{x^2+1}dx = \frac{\pi}{2} - \tan^{-1}(s) = \tan^{-1}(\frac{1}{s}), \quad s > 0$
2. $\mathcal{L}\left(\frac{\sinh wt}{t}\right) =$

Integration of Laplace transform

Let $f : [0, \infty) \to \mathbb{R}$ be a piecewise continuous function with exponenetial order α. Assume that $\lim\limits_{t \to 0+} \dfrac{f(t)}{t}$ exists. Then

$$\int_s^{\infty} F(u)du = \mathcal{L}\left(\frac{f(t)}{t}\right) \quad (s > \alpha).$$

Examples

1. $\mathcal{L}\left(\dfrac{\sin t}{t}\right) = \int_s^{\infty} \frac{1}{x^2+1} dx = \frac{\pi}{2} - \tan^{-1}(s) = \tan^{-1}(\frac{1}{s}), \quad s > 0$
2. $\mathcal{L}\left(\dfrac{\sinh wt}{t}\right) = \frac{1}{2} \ln \frac{s+w}{s-w} \quad (s > w).$

CONVOLUTION PRODUCT

Let $f, g : [0, \infty) \to \mathbb{R}$ be two functions. Then the convolution of f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

if the above integral exists.

Convolution product

Let $f, g : [0, \infty) \to \mathbb{R}$ be two functions. Then the convolution of f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

if the above integral exists.

If f, g are piecewise continuous, the the above integral exists.

Example

Let $f(t) = e^t$ and $g(t) = t$. Then

$$(f * g)(t) = e^t - t - 1.$$

Laplace transform

If $f, g : [0, \infty) \to \mathbb{R}$ are piecewise continuous with exponential order α, then

$$\mathcal{L}(f * g) = \mathcal{L}(f).\mathcal{L}(g)(s > \alpha)$$

Laplace transform

If $f, g : [0, \infty) \to \mathbb{R}$ are piecewise continuous with exponential order α, then

$$\mathcal{L}(f * g) = \mathcal{L}(f).\mathcal{L}(g)(s > \alpha)$$

- $\mathcal{L}(e^{at} * e^{bt}) = \frac{1}{s-a}\frac{1}{s-b} \ \ (s > a, \ s > b)$
- $\mathcal{L}^{-1}\left(\frac{1}{s-a}\frac{1}{s-b}\right) = e^{at} * e^{bt}$.

1. Find $\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{1}{s-1}\right)$
2. Find $\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)$
3. Solve the integral equation

$$y(t) = t + \int_0^t y(\tau)\sin(t-\tau)d\tau.$$

Partial Fractions

Useful in finding the inverse Laplace transform when it is difficult to recognize that a given function is a Laplace transform of a known function.

Partial Fractions

Useful in finding the inverse Laplace transform when it is difficult to recognize that a given function is a Laplace transform of a known function.

Example

$$\mathcal{F}(s) = \frac{1}{(s-2)(s-3)};\ (s > 3)$$

Partial Fractions

Useful in finding the inverse Laplace transform when it is difficult to recognize that a given function is a Laplace transform of a known function.

Example

$$\mathcal{F}(s) = \frac{1}{(s-2)(s-3)};\ (s > 3)$$

Write $\mathcal{F}(s) = \dfrac{A}{s-2} + \dfrac{B}{s-3}$ (A, B are constants)

PARTIAL FRACTIONS

Useful in finding the inverse Laplace transform when it is difficult to recognize that a given function is a Laplace transform of a known function.

EXAMPLE

$$\mathcal{F}(s) = \frac{1}{(s-2)(s-3)};\ (s > 3)$$

Write $\mathcal{F}(s) = \dfrac{A}{s-2} + \dfrac{B}{s-3}$ (A, B are constants)

Substituting $s = 2$, we get $A = -1$.

Partial Fractions

Useful in finding the inverse Laplace transform when it is difficult to recognize that a given function is a Laplace transform of a known function.

Example

$\mathcal{F}(s) = \dfrac{1}{(s-2)(s-3)}$; $(s > 3)$

Write $\mathcal{F}(s) = \dfrac{A}{s-2} + \dfrac{B}{s-3}$ (A, B are constants)

Substituting $s = 2$, we get $A = -1$.

$s = 3$ gives $B = 1$.

Hence $\mathcal{F}(s) = \dfrac{1}{s-3} - \dfrac{1}{s-2}$.

LINEAR FACTORS

Let $\mathcal{F}(s) = \dfrac{P(s)}{Q(s)}$, where P and Q are polynomials in s such that

1. degree of P is less than or equals to the degree of Q
2. P and Q have no common factors.

LINEAR FACTORS

Let $\mathcal{F}(s) = \dfrac{P(s)}{Q(s)}$, where P and Q are polynomials in s such that

1. degree of P is less than or equals to the degree of Q
2. P and Q have no common factors.

Then

1. For each factor of the form $as + b$ of $Q(s)$, there corresponds a partial fraction of the form $\dfrac{A}{as + b}$ (A is a contsant)

LINEAR FACTORS

Let $\mathcal{F}(s) = \dfrac{P(s)}{Q(s)}$, where P and Q are polynomials in s such that

1. degree of P is less than or equals to the degree of Q
2. P and Q have no common factors.

Then

1. For each factor of the form $as + b$ of $Q(s)$, there corresponds a partial fraction of the form $\dfrac{A}{as+b}$ (A is a contsant)
2. For each repeated linear factor of the form $(as + b)^n$ of $Q(s)$, there corresponds a partial fraction of the form $\dfrac{A_1}{as+b} + \dfrac{A_2}{(as+b)^2} + \cdots + \dfrac{A_n}{(as+b)^n}$($A_i's$ are contsants)

QUADRATIC FACTORS

1. For each factor of the form $as^2 + bs + c$ of $Q(s)$, there corresponds a partial fraction of the form $\dfrac{As + B}{as^2 + bs + c}$ ($A,\ B$ are contsants)

QUADRATIC FACTORS

1. For each factor of the form $as^2 + bs + c$ of $Q(s)$, there corresponds a partial fraction of the form $\dfrac{As + B}{as^2 + bs + c}$ (A, B are contsants)
2. For each repeated factor of the form $(as^2 + bs + c)^n$ of $Q(s)$, there corresponds a partial fraction of the form

$$\frac{A_1 s + B_1}{as^2 + bs + c} + \frac{A_2 s + B_2}{(as^2 + bs + c)^2} + \cdots + \frac{A_n s + B_n}{(as^2 + bs + c)^n}$$

(A_i, $B_i's$ are contsants)

EXAMPLES

Find $\mathcal{L}^{-1}\left(\dfrac{s+1}{s^2(s-1)}\right)$; $s > 1$

Examples

Find $\mathcal{L}^{-1}\left(\frac{s+1}{s^2(s-1)}\right)$; $s > 1$

- $\dfrac{s+1}{s^2(s-1)} = \dfrac{As+B}{s^2} + \dfrac{C}{s-1}$

EXAMPLES

Find $\mathcal{L}^{-1}\left(\frac{s+1}{s^2(s-1)}\right)$; $s > 1$

- $\frac{s+1}{s^2(s-1)} = \frac{As+B}{s^2} + \frac{C}{s-1}$
- substituting $s = 0, 1, 2$, we get $B = -1,\ C = 2,\ A = 1$

EXAMPLES

Find $\mathcal{L}^{-1}\left(\dfrac{s+1}{s^2(s-1)}\right)$; $s > 1$

- $\dfrac{s+1}{s^2(s-1)} = \dfrac{As+B}{s^2} + \dfrac{C}{s-1}$
- substituting $s = 0, 1, 2$, we get $B = -1,\ C = 2,\ A = 1$
- $\mathcal{L}^{-1}\left(\dfrac{s+1}{s^2(s-1)}\right) = 1 - t + 2e^t.$

Examples

Find $\mathcal{L}^{-1}\left(\frac{s+1}{s^2(s-1)}\right)$; $s>1$

- $\frac{s+1}{s^2(s-1)} = \frac{As+B}{s^2} + \frac{C}{s-1}$
- substituting $s=0,1,2$, we get $B=-1,\ C=2,\ A=1$
- $\mathcal{L}^{-1}\left(\frac{s+1}{s^2(s-1)}\right) = 1-t+2e^t$.

Find $\mathcal{L}^{-1}\left(\frac{2s^2}{(s^2+1)(s-1)^2}\right)$; $s>1$

EXAMPLES

Find $\mathcal{L}^{-1}\left(\dfrac{s+1}{s^2(s-1)}\right)$; $s > 1$

- $\dfrac{s+1}{s^2(s-1)} = \dfrac{As+B}{s^2} + \dfrac{C}{s-1}$
- substituting $s = 0, 1, 2$, we get $B = -1,\ C = 2,\ A = 1$
- $\mathcal{L}^{-1}\left(\dfrac{s+1}{s^2(s-1)}\right) = 1 - t + 2e^t$.

Find $\mathcal{L}^{-1}\left(\dfrac{2s^2}{(s^2+1)(s-1)^2}\right)$; $s > 1$

Solution: $-\cos(t) + e^t + te^t$.

Solve the IVP:

$$y'' + 3y' + 2y = t + 1, \quad y(0) = 1, \; y'(0) = 0.$$

Solve the IVP:

$$y'' + 3y' + 2y = t + 1, \;\; y(0) = 1, \; y'(0) = 0.$$

- $\mathcal{L}(y) = \dfrac{s^3 + 3s^2 + s + 1}{s^2(s+1)(s+2)}; \; s > 0$

Solve the IVP:

$$y'' + 3y' + 2y = t + 1, \;\; y(0) = 1, \; y'(0) = 0.$$

- $\mathcal{L}(y) = \dfrac{s^3 + 3s^2 + s + 1}{s^2(s+1)(s+2)}; \; s > 0$
- $\dfrac{s^3 + 3s^2 + s + 1}{s^2(s+1)(s+2)} = \dfrac{As + b}{s^2} + \dfrac{C}{s+1} + \dfrac{D}{s+2}$

Solve the IVP:

$$y'' + 3y' + 2y = t + 1, \;\; y(0) = 1, \; y'(0) = 0.$$

- $\mathcal{L}(y) = \dfrac{s^3 + 3s^2 + s + 1}{s^2(s+1)(s+2)}; \; s > 0$
- $\dfrac{s^3 + 3s^2 + s + 1}{s^2(s+1)(s+2)} = \dfrac{As + b}{s^2} + \dfrac{C}{s+1} + \dfrac{D}{s+2}$
- $A = \frac{-1}{4}, \; B = \frac{1}{2}, \; C = 2, \; D = \frac{-3}{4}$

Solve the IVP:

$$y'' + 3y' + 2y = t + 1, \quad y(0) = 1, \ y'(0) = 0.$$

- $\mathcal{L}(y) = \dfrac{s^3 + 3s^2 + s + 1}{s^2(s+1)(s+2)}; \ s > 0$
- $\dfrac{s^3 + 3s^2 + s + 1}{s^2(s+1)(s+2)} = \dfrac{As + b}{s^2} + \dfrac{C}{s+1} + \dfrac{D}{s+2}$
- $A = \frac{-1}{4}, \ B = \frac{1}{2}, \ C = 2, \ D = \frac{-3}{4}$
- $y(t) = \frac{t}{2} - \frac{1}{4} + 2e^{t} - \frac{3}{4}e^{-2t}.$

Periodic Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic with period L if $f(x) = f(x + L)$ for all $x \in \mathbb{R}$

PERIODIC FUNCTIONS

A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic with period L if $f(x) = f(x + L)$ for all $x \in \mathbb{R}$

EXAMPLES

1. the functions $\cos(x), \sin(x)$ are periodic with period 2π
2. the functions $\tan(x)$ and $\cot(x)$ are periodic with period π
3. A constant function is periodic with any period
4. $f(x) = x^n,\ x \in \mathbb{R}$ $(n \in \mathbb{N})$ is not periodic
5. the functions $e^x, \cosh(x)$ are not periodic

- If L is a period for a periodic function, then nL is a period for each $n \in \mathbb{Z}$

- If L is a period for a periodic function, then nL is a period for each $n \in \mathbb{Z}$
- the smallest period is called the fundamental period for the function.
- 2π is the fundamental period for $\cos(x)$ and $\sin(x)$
- a constant function has no fundamental period.

- If L is a period for a periodic function, then nL is a period for each $n \in \mathbb{Z}$
- the smallest period is called the fundamental period for the function.
- 2π is the fundamental period for $\cos(x)$ and $\sin(x)$
- a constant function has no fundamental period.

THEOREM

Let f be a periodic function with period L and $\mathcal{L}(f) = \mathcal{F}(s)$. Then

$$\mathcal{F}(s) = \frac{1}{1 - e^{sL}} \int_0^L e^{-st} f(t) dt.$$

THANK YOU