INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

DEPARTMENT OF MATHEMATICS

ASSIGNMENT 1

MA 4020 : Linear algebra — Max Marks: 65

1. Show that the set of all positive real numbers forms a vector space over $\mathbb{R}$ if the sum of x and y is defined to be the usual product xy and α times x is defined to be x^α. [5]

2. Consider the vectors $x_1 = (1, 3, 2)$ and $x_2 = (-2, 4, 3)$ in $\mathbb{R}^3$. Show that the span of $\{x_1, x_2\}$ is $\{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 - 7y_2 + 10y_3 = 0\}$. Show that the subspace can also be written as [5]

$$\{(\alpha, \beta, (-\alpha + 7\beta)/10) : \alpha, \beta \in \mathbb{R}\}$$

3. (a) Show that, for any set $A \subset V$, $\mathrm{Sp}(A)$ is the intersection of all subspaces of V containing A. [3]

   (b) Let S and T be subspaces of V. Then prove that $S \cup T$ is a subspace if and only if either $S \subseteq T$ or $T \subseteq S$. [2]

4. Find all maximal linearly independent subset of $\{x_1, x_2, ..., x_5\}$ where $x_1 = (1, 1, 0, 1), x_2 = (1, 2, -1, 0), x_3 = (1, 0, 1, 2)$,$x_4 = (0, 1, 1, 1)$ and $x_5 = (2, 0, 2, 4)$ in $\mathbb{R}^4$. [5]

5. If $\mathrm{Sp}(A) = S$, then show that no proper subset of A generates S if and only if A is linearly independent. [5]

6. (a) For what values of $\alpha \in \mathbb{R}$ are the vectors $(0, 1, \alpha), (\alpha, 1, 0)$, and $(1, \alpha, 1)$ in $\mathbb{R}^3$ are linearly independent.

   (b) Determine all the values of α, β for which the vectors $(\alpha, \beta, \beta, \beta), (\beta, \beta, \alpha, \beta), (\beta, \beta, \alpha, \beta)$ and $(\beta, \beta, \beta, \alpha)$ of $\mathbb{R}^4$ are linearly dependent.

7. Show that $f_1(t) = 1, f_2(t) = t - 2$ and $f_3(t) = (t - 2)^2$ form a basis for $\mathcal{P}_3$, where $\mathcal{P}_3$ is the all polynomials of degree ≤ 2. Express $3t^2 - 5t + 4$ as a linear combination of f_1, f_2, f_3. [5]

8. Extend $A = \{1, 1, 1..., 1)\}$ to a basis of $\mathbb{R}^n$. [5]

9. Find a basis for each of the following subspaces of $\mathbb{R}^4$. [5]

   (a) $S_1 = \{(x_1.x_2, x_3, x_4) : x_1 - 2x_3 + x_4 = 0\}$ [2]

   (b) $S_2 = \{(x_1.x_2, x_3, x_4) : x_1 + x_2 - x_3 = 0, x_2 + 2x_3 - x_4 = 0, 2x_1 + 3x_2 - x_4 = 0\}$. [3]

10. Let F be a finite field with q elements and V an n-dimensional vector space over F. Show that $|V| = q^n$. [5]

11. Let S, T and W be subspaces. If $W \subseteq T$, prove that $S + W \subseteq S + T$. Is the converse true? When is $S + T = S \cup T$. [5]

12. Let S and T be subspaces of a vector space V with $d(S) = 2$, $d(T) = 3$ and $d(V) = 5$. Find the minimum and maximum possible values of $d(S + T)$ and show that every (integer) value between these can be attained. [5]

13. Let S and T be subspaces of $\mathbb{R}^4$ given by [5]

$$S = \{(x_1, x_2, x_3, x_4) : 3x_1 + x_2 + x_3 + x_4 = 0, x_1 - x_3 + 2x_4 = 0\}$$

and

$$T = \{(x_1, x_2, x_3, x_4) : 5x_1 + 2x_2 + 3x_3 = 0, x_1 - x_3 + x_4 = 0\}$$

   (a) Obtain a basis each for $S \cap T$, S, T and $S + T$.

   (b) Verify the modular law for S and T.

   (c) Extend the basis of $S + T$ you obtained in (a) to form a basis for $\mathbb{R}^4$.

   (d) Express $S + T$ and $S \cap T$ in the same form as S and T.