Backtracking and Look-Ahead Decoding Algorithms for Improved Successive Cancellation Decoding Performance of Polar Codes

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Two Decoding Algorithms for Polar Codes

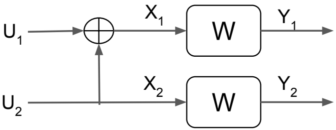
- Both the algorithms use $O(N)$ memory.

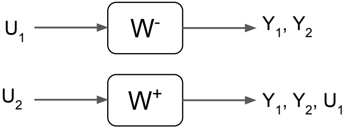
- Successive Cancellation Back-Tracking (SC-BT)
  - Performs close to ML for short block lengths $N = 32, 64$.
  - Improved performance over SCD for larger block lengths $N = 128, 256$.
  - Computation complexity varies with error patterns.
  - Small average complexity at large SNR and high complexity at small SNRs.

- Successive Cancellation Look-Ahead (SC-LA)
  - Tractable complexity. $O(\frac{2^D}{D} N \log N)$ where D is the parameter of the algorithm.
  - Improved performance over successive cancellation decoding.
  - Can be extended to list decoding.

Polarization of 2-channels (Arıkan 2008)

W is binary DMC

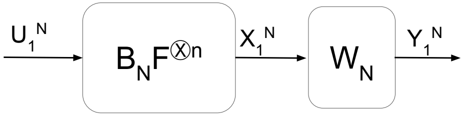




$$\begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{F}$$

$$I(W^+) \geq I(W) \geq I(W^-)$$

Extension to Polarization of $N = 2^n$-channels (Arıkan 2008)



- B_N is a bit reversal permutation matrix.
- $F^{\otimes n}$ is n-time Kronecker product of F.
- W_N is N-independant uses of channel W.

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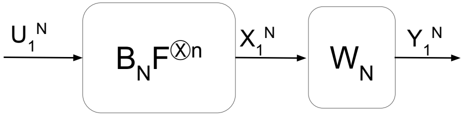

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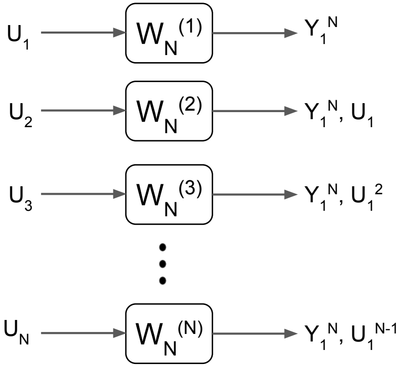

N Bit Channels

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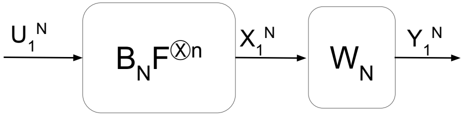

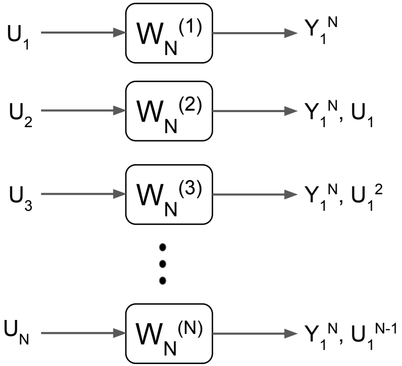

N Bit Channels

- B_N is a bit reversal permutation matrix.
- $F^{\otimes n}$ is n-time Kronecker product of F.
- W_N is N-independant uses of channel W.

- $\mathcal{F}$ is the set of frozen symbols, for any $i \in \mathcal{F}$, $U_i = 0$.
- $|\mathcal{F}^c| = K$, the dimension of the code.
- $\mathcal{F}$ carefully chosen such that $\sum_{i \in \mathcal{F}^c} P_e(W_N^{(i)}) \leq \epsilon_N$

Successive Cancellation Decoding (SCD)

$$\hat{u}_i = \begin{cases} \arg\max_{u_i \in \{0,1\}} W_N^{(i)}(y_1^N, \hat{u}_1^{i-1} | u_i) & i \notin \mathcal{F} \\ 0 & i \in \mathcal{F} \end{cases}$$

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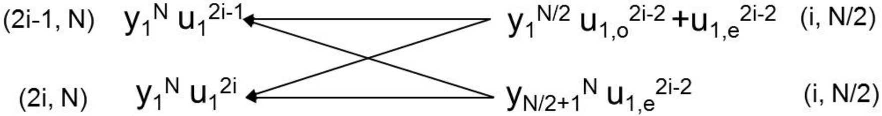
Recursive Metric Computation: For $i \in [\frac{N}{2}]$,



- Need two metrics from $\frac{N}{2}$ level with index i to compute metric at level N for indices $2i-1$ and $2i \implies O(N \log N)$ computation.

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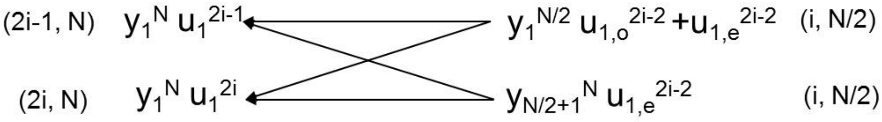


- Need two metrics from $\frac{N}{2}$ level with index i to compute metric at level N for indices $2i-1$ and $2i \implies O(N \log N)$ computation.
- The recursive definition assumes that $U_1^N \sim \text{uniform}(\mathbb{F}_2^N)$. This is not true due to frozen symbols.

Impact of Uniformity Assumption (UA)

$$\underbrace{P(y_1^N u_1^{i-1} | u_i)}_{\text{UA metric}} = \sum_{u_{i+1}^N \in \mathbb{F}_2^{N-i}} 2P(y_1^N | u_1^N) P(u_1^N)$$

Impact of Uniformity Assumption (UA)

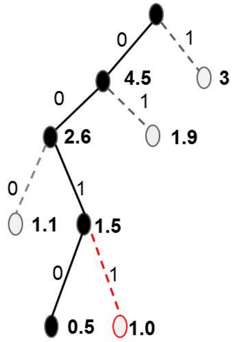
$$
\begin{aligned}
\underbrace{P(y_1^N u_1^{i-1} | u_i)}_{\text{UA metric}} &= \sum_{u_{i+1}^N \in \mathbb{F}_2^{N-i}} 2P(y_1^N | u_1^N) P(u_1^N) \\
&= \frac{1}{2^{N-1}} \sum_{u_{i+1}^N \in \mathbb{F}_2^{N-i}} P(y_1^N | u_1^N) \\
&= \underbrace{\frac{1}{2^{N-1}} \sum_{u_{i+1}^N \in \mathbb{F}_2^{N-i} : u_{i+1,\mathcal{F}}^N = 0} P(y_1^N | u_1^N)}_{\textit{actual}} + \text{noise}
\end{aligned}
$$

Decoding Algorithms as Search over Binary Tree

- Binary Tree of depth N with 2^N leaf nodes.
- Each node at depth i can be represented by u_1^i.
- Metric $M_i(u_1^i) = W_N^{(i)}(y_1^N, u_1^{i-1}|u_i)$

$$\begin{aligned} M_i(u_1^i) &= \frac{1}{2^{N-1}} \sum_{u_{i+1}^N \in \mathbb{F}_2^{N-i}} P(y_1^N | u_1^N) \\ &= M_{i+1}(u_1^i, 0) + M_{i+1}(u_1^i, 1) \quad \text{(Sum Metric)} \end{aligned}$$

- Metric at a node is sum of the child metrics.



$\hat{u}_1^4 = (0, 0, 1, 0)$

Decoding Algorithms: Search over Binary Tree

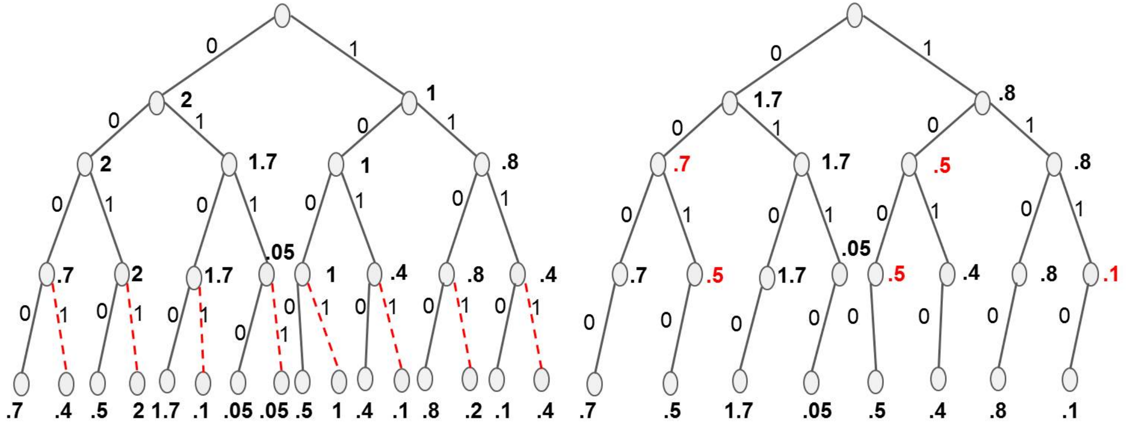
- Max Metric

$$M_i(u_1^i) \quad = \quad \max_{u_{i+1}^N \in \mathbb{F}_2^{N-i}} P(y_1^N | u_1^N) = \max\{M_{i+1}(u_1^i, 0), M_{i+1}(u_1^i, 1)\}$$

- At step i, we compute $M_i(\hat{u}_1^{i-1}, 0), M_i(\hat{u}_1^{i-1}, 1)$

$$\hat{u}_i = \arg \max_{u_i \in \{0,1\}} M_i(\hat{u}_1^{i-1}, u_i) \text{ for } i \notin \mathcal{F}$$

Decoding Algorithms: Search over Binary Tree

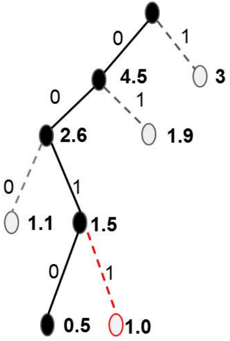


UA Metric vs Actual Metric (under Max Metric)

Decoding Algorithms: Search over Binary Tree

- 4-th symbol is assumed to be frozen

Successive Cancellation Decoding
(Arıkan, 2008)



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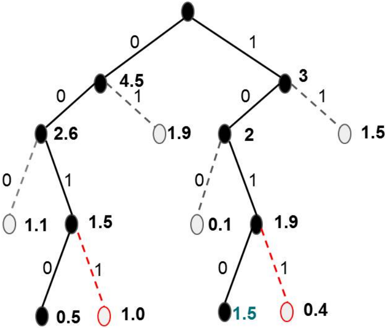
Succesive Cancellation List Decoding (SCLD)
(Tal & Vardy 2015)

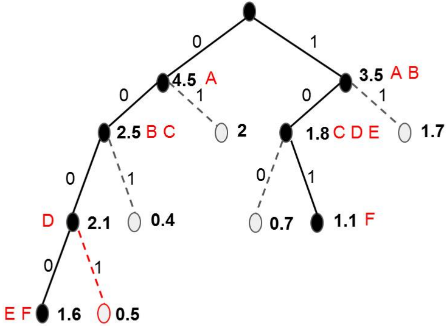


List size $L = 2$, $\hat{u}_1^4 = (1, 0, 1, 0)$

Output of SCLD is more likely (1.5) to be the message vector than the SCD output vector (0.5).

Decoding Algorithms: Search over Binary Tree

Stack Decoding
(Niu & Chen, 2012)



Stack size $D = 2$

Decoding Algorithms: Search over Binary Tree

Stack Decoding
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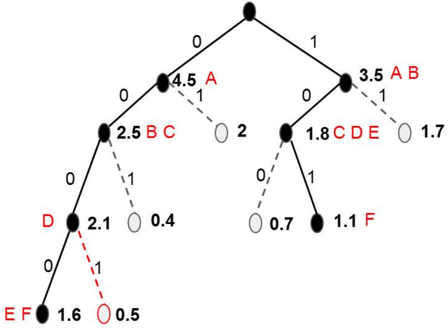
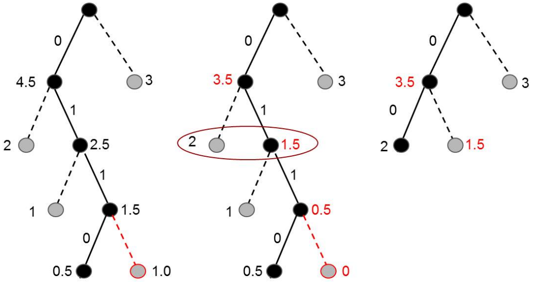


Stack size $D = 2$

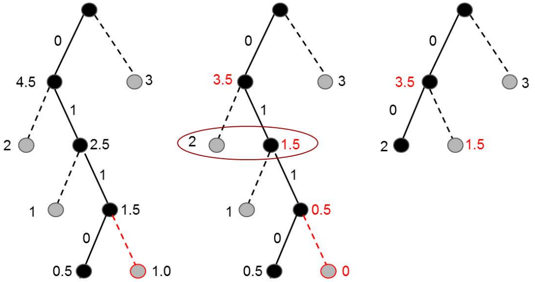
- SC Hybrid decoding (2013) by Chen, Liu, Lin that combines principles of stack decoding and SCLD.
- Sequential Decoding (2014) by Miloslavskaya & Trifonov and Score function (2018) by Trifonov introduce carefully designed bias functions to metrics.
- Partitioned SC List Decoding (2017) by Hashemi, Mondelli et. al. Trades memory for performance.

Successive Cancellation with Back-Tracking Algorithm (SC-BT)



- For $i \in \mathcal{F}$, set $M_i(\hat{u}_1^{i-1}, 1) = 0$.
- Backtrack the tree to update the metric at parent nodes. $M_{i-1}(\hat{u}_1^{i-1}) = M_i(\hat{u}_1^{i-1}, 0) + M_i(\hat{u}_1^{i-1}, 1)$
- Select new best path and continue the algorithm.
- Can select backtracking height D to be small in order to reduce computation at the cost of performance.

Successive Cancellation with Back-Tracking Algorithm (SC-BT)



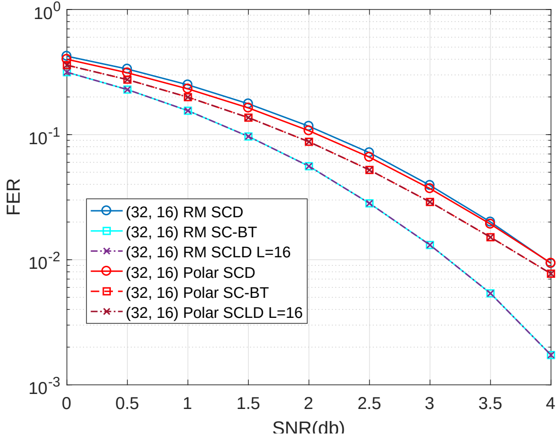
- For $i \in \mathcal{F}$, set $M_i(\hat{u}_1^{i-1}, 1) = 0$.
- Backtrack the tree to update the metric at parent nodes. $M_{i-1}(\hat{u}_1^{i-1}) = M_i(\hat{u}_1^{i-1}, 0) + M_i(\hat{u}_1^{i-1}, 1)$
- Select new best path and continue the algorithm.
- Can select backtracking height D to be small in order to reduce computation at the cost of performance.

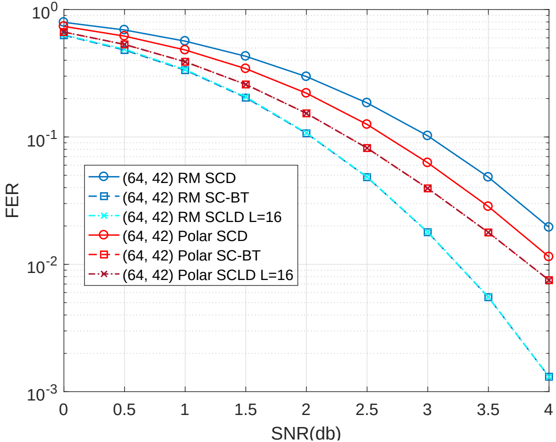
Complexity

	0 db		4 db	
D	max	mean	max	mean
4	105	$\simeq 71$	93	$\simeq 81$
8	1841	84	175	$\simeq 81$
10	5330	113	142	$\simeq 81$
16	10000	318	284	$\simeq 81$
32	10000	8496	322	$\simeq 81$

Table: Number of reflections observed for $(N = 256, K = 128)$, when backtracking is allowed to a height D.

Evaluation of SC-BT Decoding





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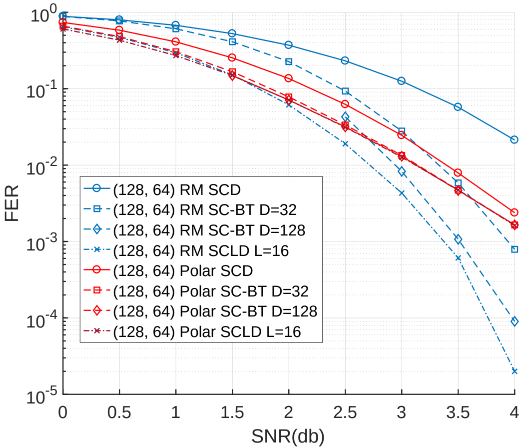
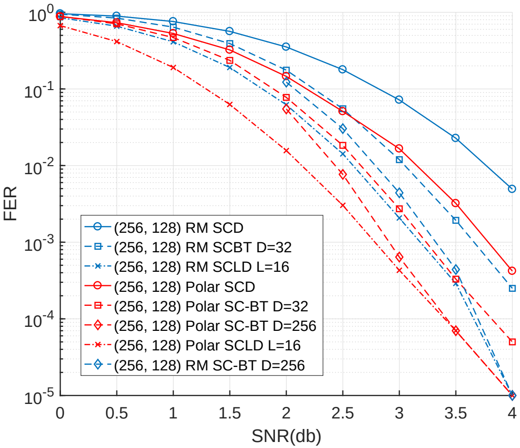
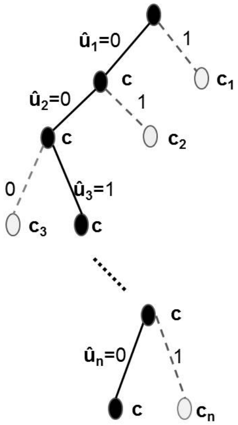




Successful SC-BT results in ML codeword (under Max Metric, $D = N$)



- It is clear that $c \geq c_i$ for all $i \in [N]$.
- Suppose $\exists \tilde{u}_1^N \neq \hat{u}_1^N$ such that

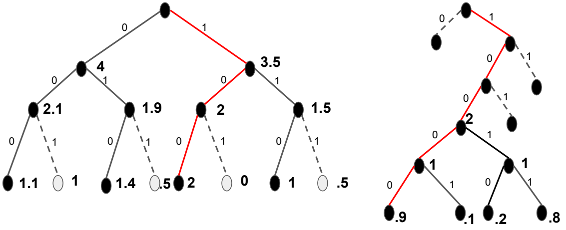
$$P(y_1^N|\tilde{u}_1^N) \quad > \quad P(y_1^N|\hat{u}_1^N)$$

- Let j be such that $\hat{u}_j \neq \tilde{u}_j$ and $\hat{u}_1^{j-1} = \tilde{u}_1^{j-1}$

$$\begin{aligned} c_j &\geq \max_{u_1^N \in \mathbb{F}_2^N : u_1^{j-1} = \hat{u}_1^{j-1}} P(y_1^N|u_1^N) \\ &\geq P(y_1^N|\tilde{u}_1^N) \\ &> P(y_1^N|\hat{u}_1^N) = c \quad \text{(contradiction)} \end{aligned}$$

- Can be extended to precoded polar codes with dynamic frozen symbols.
- Future Directions
  - Dynamics of this update process to be studied
  - Extension of the SC-BT algorithm to shortened, punctured polar codes

Successive Cancellation Look Ahead (SC-LA) Decoding



Look Ahead Depth $D = 2$.

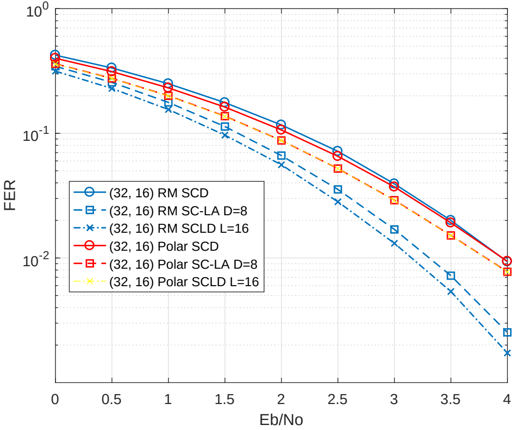
- When a frozen symbol is realized at depth D, the metrics computed at the 2^D leaf nodes is used to make decision on the D information bits.
- Can be extended to list decoding and dynamic frozen symbols.
- Complexity is tractable.

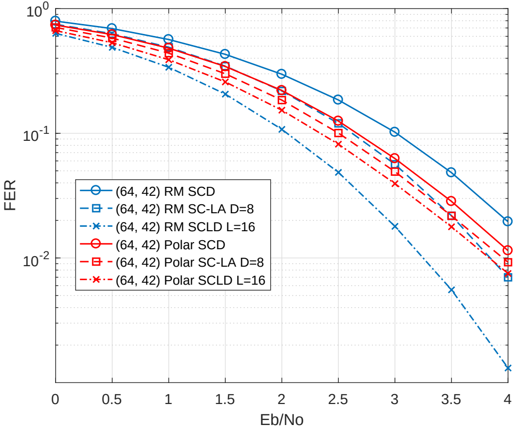
Complexity of traversing 2^D paths

$$
\begin{aligned}
C(i) &= \begin{cases} f(i) + 2C(i+1) & i \in \mathcal{F}^c \\ f(i) + C(i+1) & i \in F \end{cases} \\
C(j) &= \sum_{i=0}^{D-1} f(i+j)2^i + 2^D \sum_{i=D}^{D+f-1} f(i+j) \\
&\simeq O(2^D \log N)
\end{aligned}
$$

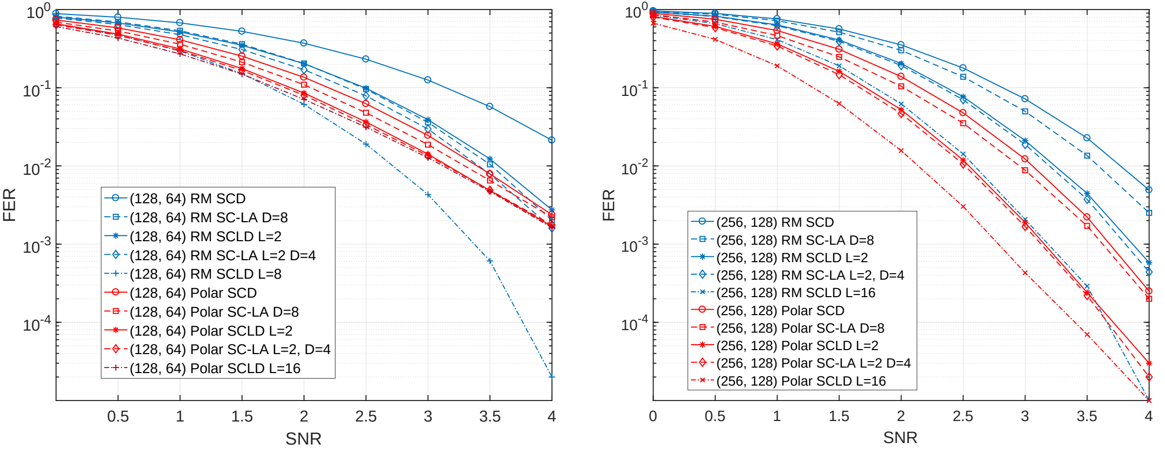
- Total complexity: $O(\frac{N}{D} 2^D \log N)$
- Future Work
  - Adapt the depth to the density of frozen bits

Evaluation of SC-LA Decoding





Evaluation of SC-LA Decoding



Thanks!