A new linearly unstable mode in the core-annular flow of two immiscible fluids

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The linear stability characteristics of pressure-driven core-annular pipe flow of two immiscible fluids are considered to investigate the effects of the density and viscosity ratios, the Reynolds number, the interface location and the interfacial tension. Both liquid-liquid and gas-liquid systems are examined. A new type of interfacial mode associated with the axisymmetric and corkscrew perturbations is discovered for certain ranges of the viscosity and density ratios in the immiscible liquid-liquid system. Two distinct unstable regions in the long and short wavelengths are observed. The long-wavelengths unstable region forms a close loop indicating that it is not a Tollmien-Schlichting mode. The new interfacial mode observed in the present study is similar to that discovered by Mohammadi & Smits (2017) in two-layer Couette flow for low viscosity ratios. In contrast to the two distinct unstable regions found in the immiscible configuration, the corresponding miscible system contains only one unstable mode. It is found that in the liquid-liquid systems, the corkscrew (axisymmetric) perturbation is most dominant when the annular fluid is less (more) viscous than the core fluid. On the other hand, the axisymmetric perturbation is always the dominant one in the gas-liquid system. In gas-liquid systems, the interfacial tension stabilises the short-wave and destabilises the long-wave perturbations, while increasing the interface radius stabilises the flow due to the presence of a plug region in the pipe.

1. Introduction

Two-layer flows are commonly observed in many natural phenomena, such as magma flows, glaciers, Earth's outer core, ocean and atmosphere (Govindarajan & Sahu 2014), and industrial applications, such as crude oil transport in pipelines (Saffman & Taylor 1958; Joseph *et al.* 1997; Cao *et al.* 2003), coating technology (Weinstein & Ruschak 2004), displacement flow (Redapangu *et al.* 2012), de-icing aircraft wings (Yih 1990), to name a few. The pioneering work by Yih (1967) demonstrated the existence of an interfacial unstable mode associated with an infinitesimal small long-wave perturbation at any Reynolds number in plane Couette and plane Poiseuille flows of two immiscible fluids with different viscosities separated by a sharp interface. Since then, several researchers have studied the interfacial instability in the long-wave limit (Hooper 1985), short-wave limit (Hooper & Boyd 1983) and also via full linear stability analysis in two-layer plane Poiseuille (Yiantsios & Higgins 1988*a*,*b*; Sahu *et al.* 2007; Valluri *et al.* 2010; Sahu & Matar 2010), Couette (Mohammadi & Smits 2017), three-layer channel (Sahu *et al.* 2007; Malik & Hooper 2005; Redapangu *et al.* 2012) and core-annular cylindrical pipe (Hickox 1971; Joseph *et al.* 1984; Usha & Sahu 2019; Salin & Talon 2019) flows. The mechanism of the short-wave interfacial instability was provided by Hinch (1984). By conducting an

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energy budget analysis for the linear perturbations, Boomkamp & Miesen (1996) showed that the interfacial instability is driven by the work done at the interface due to the jump in viscosity across the interface.

Instability in the core-annular cylindrical pipe flow of two miscible (Scoffoni *et al.* 2001; Selvam *et al.* 2007, 2009; Sahu 2016, 2019) and immiscible fluids (Hickox 1971; Joseph *et al.* 1984; Usha & Sahu 2019; Salin & Talon 2019) has received increasing attention recently not only because of its importance in many practical applications but also due to its fundamental interest. The single fluid Hagen-Poiseuille flow is known to be linearly stable for all values of Reynolds number (Schmid & Henningson 2001), which is not true in two-fluid flows in a cylindrical pipe. The focus of all the above investigations in core-annular cylindrical pipe flows was to demonstrate the behaviour of the linearly unstable axisymmetric and corkscrew perturbations associated with viscosity stratification (in miscible configuration) or viscosity contrast (in immiscible configuration) between the fluids in density matched systems. The main findings of the earlier studies in the stability of core-annular pipe flows of miscible and immiscible fluids are summarised below. (i) While the immiscible configuration is unstable if the core fluid is more viscous than the annular fluid (Joseph *et al.* 1997), the miscible configuration is found to be unstable beyond a critical viscosity ratio (Selvam *et al.* 2007). (ii) In the miscible configurations, the axisymmetric perturbation is dominant when the core fluid is more viscous than the annular fluid, but when the core fluid is less viscous, the corkscrew perturbation is most dangerous (Selvam *et al.* 2007). This result is in contrast with that of the immiscible core-annular pipe flow in which the axisymmetric perturbation was found to be most unstable when the core fluid is less viscous than the annular fluid (Usha & Sahu 2019). It is also worth mentioning that in the Hagen-Poiseuille single fluid flow (albeit stable for any Reynolds number), the corkscrew perturbation is always the least stable one (Schmid & Henningson 2001). (iii) Selvam *et al.* (2009); Salin & Talon (2019) demonstrated the transition from the convective instability to the absolute instability in miscible and immiscible core-annular flows of two immiscible fluids of different viscosities but of the same density.

In the present work, it is demonstrated the appearance of a new mode of instability distinct from the Tollmien-Schlichting (TS) mode and Yih's interfacial mode in core-annular cylindrical pipe flow of two immiscible fluids. In this context, it is important to discuss the earlier studies on the stability of two-layer plane Poiseuille flow of two miscible fluids (Ranganathan & Govindarajan 2001; Govindarajan 2004; Malik & Hooper 2005; Sahu & Govindarajan 2016) and two-layer Couette flow of two immiscible fluids (Ern *et al.* 2003; Mohammadi & Smits 2017). Ern *et al.* (2003) demonstrated that the most unstable mode in the miscible configuration is similar to that observed in immiscible flows without surface tension for low diffusivity and in the limit of zero thickness of the mixed region. However, in the miscible configuration, they also found that for a certain range of diffusivity and interfacial thickness, the growth rate of the perturbations is higher than the corresponding interfacial mode. In a three-layer channel flow, Govindarajan (2004) reported the existence of a linearly unstable regime distinct from the Tollmien-Schlichting (TS) mode and showed that these unstable regions merged when the mixed layer overlaps with the critical layer (i.e., the location at which the perturbation phase velocity is equal to the mean streamwise velocity). By comprising the instability behaviour of two miscible and immiscible fluids in a two-layer plane Poiseuille flow, Malik & Hooper (2005) showed that when the thickness of the mixed region is comparable to the thickness of the critical layer, the most unstable mode resembles the interfacial mode (Yih 1967). In a miscible channel flow, Talon & Meiburg (2011) observed four different types of modes depending on the location of the mixed region in the Stokes flow regime. By analyzing

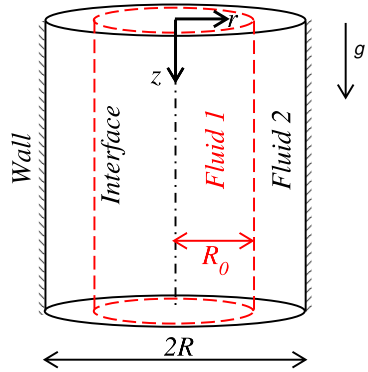

Figure 1: (Colour online) Schematic of the pressure-driven core-annular flow of two immiscible fluids in a vertical pipe of radius R. The acceleration due to gravity, g acts in the positive $z-$direction. The fluids are separated by a sharp interface located at $r = R_0$.

the concentration perturbation, they showed that the most unstable mode is similar to that of Ern *et al.* (2003).

Most relevant to the present study is the discovery of a new type of interfacial instability distinct from the Yih's mode (Yih 1967) in a two-layer Couette flow for low viscosity ratio by Mohammadi & Smits (2017). They also investigated the influence of the density ratio and interfacial tension. In the present study, besides demonstrating the distinct interfacial mode of instability in a core-annular cylindrical pipe flow, the effect of the viscosity and density ratios is also studied, with an emphasis on investigating the linear instability in the core-annular gas-liquid system that has not yet been studied to the best of the author's knowledge. The new interfacial instability mode is compared with the most unstable mode observed in the corresponding miscible core-annular flow. Two configurations have been considered in the gas-liquid system, namely when the annular fluid is a liquid and the core fluid is a gas, and vice-versa. The effects of interface location and interfacial tension have also been investigated. The rest of this paper is organised as follows. The problem is formulated and the governing linear stability equations for immiscible and miscible core-annular configuration are derived in §2. The associated boundary conditions are also presented in this section. The linear stability results are discussed in §3. Concluding remarks are provided in §4.

2. Formulation

2.1. *Immiscible core-annular configuration*

Linear instability characteristics of a steady and fully-developed core-annular pressure-driven flow of two Newtonian and immiscible, incompressible fluids in a vertical cylindrical pipe of radius R are considered. This schematic diagram is shown in Fig. 1. A cylindrical coordinate system (r, θ, z) is used, where r, θ and z denote the radial, azimuthal and axial coordinates, respectively. The density and viscosity of the core fluid (fluid '1' in $0 \leqslant r \leqslant R_0$) and annular fluid (fluid '2' in $R_0 \leqslant r \leqslant R$) are denoted by (ρ_1, μ_1) and (ρ_2, μ_2), respectively. The interfacial tension acting at the interface separating the immiscible fluids is denoted by σ. Both liquid-liquid and gas-liquid systems are considered in this study. The flow dynamics is governed by the continuity and the Navier-Stokes equations in the cylindrical coordinate system, which are non-dimensionalized using the radius of the pipe (R) and average velocity $\left(V = Q/\pi R^2\right)$ as the length and velocity scales, respectively. Here, Q is the volumetric flow rate. In the non-dimensionalization, the properties of fluid '1' are used as the reference scales. The reduced dimensionless pressure p_k in fluid k ($k = 1, 2$) is related to the corresponding total dimensional pressure, $p_{d,k}$ as

$$p_k = (p_{d,k} + \rho_k g z)/\rho_k V^2. \tag{2.1}$$

The various dimensionless numbers are the Reynolds number, ($Re = \rho_1 V R/\mu_1$), the viscosity ratio ($\mu_r = \mu_2/\mu_1$), the density ratio ($\rho_r = \rho_2/\rho_1$), the dimensionless radius of the interface, ($R_i = R_0/R$) and the inverse capillary number, ($\Gamma = \sigma/\mu_1 V$).

2.1.1. *Basic state*

The basic state is a steady, parallel, fully-developed unidirectional flow in the axial direction, $U_{z,k}$ in the core ($k = 1$) and annular ($k = 2$) regions of the pipe. The basic state velocity profiles in the core ($r \in [0, R_i]$) and annular ($r \in [R_i, 1]$) regions are given by

$$U_{z,1} = -\frac{dP}{dz}\frac{Re}{4}({R_i}^2 - r^2) - \frac{dP}{dz}\frac{Re}{4\mu_r}({R_i}^2 - 1), \tag{2.2}$$

$$U_{z,2} = -\frac{dP}{dz}\frac{Re}{4\mu_r}(1 - r^2), \tag{2.3}$$

respectively. Eqs. (2.2) and (2.3) are obtained using the following boundary conditions: (i) the no-slip boundary condition at the pipe wall ($r = 1$), (ii) the velocity maximum condition ($U'_{z,1} = 0$) at the centerline of the pipe ($r = 0$), and (iii) the continuity of velocities ($U_{z,1} = U_{z,2}$) and the shear stresses ($dU_{z,1}/dr = \mu_r dU_{z,2}/dr$) at the interface ($r = R_i$). The pressure gradient, dP/dz (whose value is negative for the flow in the positive z direction) is calculated by maintaining the constant volumetric flow condition, such that dimensionless average velocity, $V = Q/2\pi \int_0^1 U_z r dr = 1$.

2.1.2. *Linear stability equations*

The temporal linear stability equations for the basic flow (Eqs (2.2)-(2.3)) subjected to infinitesimal perturbations are discussed in this section. A normal mode analysis is used to express each flow variable as a sum of the basic state and a time-dependent perturbation (designated by hat):

$$(\mathbf{u}_{r,k}, \mathbf{u}_{\theta,k}, \mathbf{u}_{z,k}, p_i)(r, \theta, z, t) = (0, 0, U_{z,k}(r), P(z)) + (\mathrm{i}\hat{u}_{r,k}, \hat{u}_{\theta,k}, \hat{u}_{z,k}, \hat{p}_k)(r, \theta, z, t), \tag{2.4}$$

where

$$(\mathrm{i}\hat{u}_{r,k}, \hat{u}_{\theta,k}, \hat{u}_{z,k}, \hat{p}_k) = (\mathrm{i}u_{r,k}, u_{\theta,k}, u_{z,k}, p_k)(r) e^{\mathrm{i}(\alpha z + \beta\theta - \alpha c t)}.$$

Similarly, the perturbed interface can be represented as $R_i + r_i e^{\mathrm{i}(\alpha z + \beta\theta - \alpha c t)}$. Here, $\mathrm{i} \equiv \sqrt{-1}$, α, β and $c (\equiv c_r + \mathrm{i} c_i)$ are the wavenumbers in the axial and azimuthal directions (real), and the phase speed (complex) of the perturbation, respectively. The real and imaginary parts of c are denoted by c_r and c_i, respectively. Thus a given mode is temporally unstable if $c_i > 0$, stable if $c_i < 0$ and neutrally stable if $c_i = 0$. The governing temporal linear stability equations are derived using the standard approach (Schmid & Henningson 2001; Usha & Sahu 2019), i.e. by substituting the perturbations in the dimensionless continuity and Navier-Stokes equations and then subtracting the corresponding unperturbed equations followed by linearising the resulting equations. After suppressing the hat notations, the temporal linear stability equations for both the layers, $k = (1, 2)$ are given by

$$u'_{r,k} + \frac{u_{r,k}}{r} + \frac{\beta u_{\theta,k}}{r} + \alpha u_{z,k} = 0, \tag{2.5}$$

$$\rho_k \left(-\alpha c u_{r,k} + \alpha u_{r,k} U_{z,k}\right) = p'_k - \frac{\mathrm{i}\mu_k}{Re}\left[u''_{r,k} + \frac{u'_{r,k}}{r} - \left(\frac{\beta^2+1}{r^2} + \alpha^2\right) u_{r,k} - \frac{2\beta}{r^2} u_{\theta,k}\right], \tag{2.6}$$

$$\rho_k \left(-\alpha c u_{\theta,k} + \alpha u_{\theta,k} U_{z,k}\right) = -\frac{\beta p_k}{r} - \frac{\mathrm{i}\mu_k}{Re}\left[u''_{\theta,k} + \frac{u'_{\theta,k}}{r} - \left(\frac{\beta^2+1}{r^2} + \alpha^2\right) u_{\theta,k} - \frac{2\beta}{r^2} u_{r,k}\right], \tag{2.7}$$

$$\rho_k \left(-\alpha c u_{z,k} + U'_{z,k} u_{r,k} + \alpha U_{z,k} u_{z,k}\right) = -\alpha p_k - \frac{\mathrm{i}\mu_k}{Re}\left[u''_{z,k} + \frac{u'_{z,k}}{r} - \left(\frac{\beta^2}{r^2} + \alpha^2\right) u_{z,k}\right], \tag{2.8}$$

where the prime denotes differentiation with respect to r, $\rho_k = (1, \rho_r)$ and $\mu_k = (1, \mu_r)$. Here, $k = 1, 2$ indicates the flow region. The stability equations in each layer are the same as those given in Schmid & Henningson (2001). The boundary conditions for the perturbation variables are discussed below.

At the centerline of the pipe ($r = 0$), the boundary conditions are

$$u_{r,1} = 0, \;\; u_{\theta,1} = 0, \;\; u'_{z,1} = 0, \;\; p'_1 = 0 \;\text{ for }\; \beta = 0, \tag{2.9}$$

$$u_{r,1} + u_{\theta,1} = 0, \;\; 2u'_{r,1} + u'_{\theta,1} = 0, \;\; u_{z,1} = 0, \;\; p_1 = 0, \;\text{ for }\; \beta = 1, \tag{2.10}$$

$$u_{r,1} = 0, \;\; u_{\theta,1} = 0, \;\; u_{z,1} = 0, \;\; p_1 = 0, \;\text{ for }\; \beta \geqslant 2. \tag{2.11}$$

At the pipe wall ($r = 1$), the boundary conditions are

$$u_{r,2} = 0, \;\; u_{\theta,2} = 0, \;\; u_{z,2} = 0, \tag{2.12}$$

for all values of β.

The tangential stress balance equations for the perturbation at $r = R_i$ in the azimuthal and axial directions are given by

$$\mu_r \left[-\beta u_{r,2} + R_i {u'}_{\theta,2} - u_{\theta,2}\right] + \left[\beta u_{r,1} - R_i {u'}_{\theta,1} + u_{\theta,1}\right] = 0, \text{ and} \tag{2.13}$$

$$\mu_r \left[-\alpha u_{r,2} + r_i {U''}_{z,2} + {u'}_{z,2}\right] = -\alpha u_{r,1} + r_i {U''}_{z,1} + {u'}_{z,1}, \tag{2.14}$$

respectively. The normal stress balance boundary condition at $r = R_i$ are given by

$$Re(p_1 - p_2) + 2\mathrm{i}\left[\mu_r {u'}_{r,2} - {u'}_{r,1}\right] = -\frac{\mathit{\Gamma} r_i}{R_i^2}\left[1 - \beta^2 - \alpha^2 R_i^2\right]. \tag{2.15}$$

The velocity components are also continuous at the interface ($r = R_i$), i.e.

$$u_{r,1} = u_{r,2}, \;\; u_{\theta,1} = u_{\theta,2}, \;\; u_{z,1} = u_{z,2}. \tag{2.16}$$

The kinematic boundary condition for perturbation is given by

$$r_i = \frac{u_{r,1}}{\alpha(U_{z,1} - c)} = \frac{u_{r,2}}{\alpha(U_{z,2} - c)}. \tag{2.17}$$

Eqs. (2.5) $-$ (2.8), along with the boundary conditions (2.9)-(2.17) constitute an eigenvalue problem with the eigenvalue as the frequency of the perturbation ($\omega = \alpha c$) and eigenvectors $\left[u_{r,k}, u_{\theta,k}, u_{z,k}, p_i\right]^T$. The domains $[0, R_i]$ and $[R_i, 1]$ are discretized using the Chebyshev spectral collocation method (Canuto *et al.* 1987), and the eigenvalue problem

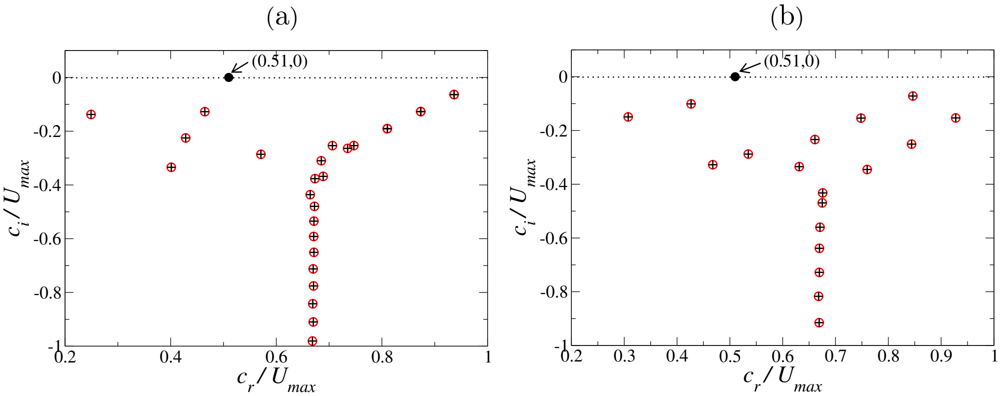

Figure 2: (Colour online) Comparison of the eigenvalue spectrums obtained from the present simulations for the immiscible configuration with $\Gamma = 0$ ('○') and those given in Schmid & Henningson (2001) for the single fluid flow through a pipe ('+'). (a) $\alpha = 1$, $\beta = 0$ and (b) $\alpha = 0.5$, $\beta = 1$. The rest of the parameters are $Re = 1000$, $R_i = 0.7$, $\mu_r = 1$ and $\rho_r = 1$. The additional mode ('●') obtained in the immiscible configuration (mode 'I') is a neutrally stable ($c_i = 0$) mode with $c_r = U_z|_{R_i} = 0.51U_{max}$.

is solved using a public domain software, LAPACK, such that for domain $[0, R_i]$:

$$r_j = \frac{R_i}{2}\left[1 - cos\left(\frac{\pi(j-1)}{N-1}\right)\right], \tag{2.18}$$

and for domain $[R_i, 1]$

$$r_j = \frac{1}{2}\left[1 + cos\left(\frac{\pi(j-1)}{N-1}\right)\right] + \frac{R_i}{2}\left[1 - cos\left(\frac{\pi(j-1)}{N-1}\right)\right], \tag{2.19}$$

where r_j are the locations of the Chebyshev collocation points, and N is the number of collocation points in each layer. The governing equations for the corresponding miscible core-annular configuration are described in the Appendix A.

2.2. *Validation*

The linear stability solver developed for the immiscible configuration, as discussed in §2.1, has been validated against several known results for the single fluid flow (Schmid & Henningson 2001) and also plane Poiseuille flow configuration of two immiscible fluids (Sahu *et al.* 2007). Table 1 shows a comparison between the most unstable eigenmode obtained using the current solver with Hu & Joseph (1989) for two different set of parameters. Several authors have used the results of Hu & Joseph (1989) to validate their solvers (e.g., Orazzo *et al.* (2014)). In Table 1, it can be seen that the present results are in good agreement with Hu & Joseph (1989). In addition, a grid convergence test is also performed (see Fig. A3 in Appendix B) to ensure that the number of grids used in the present study is adequate to generate the eigenvalues accurately at least up to five decimal places. It is found that using more than 41 grids in each layer in the case of immiscible configuration and 81 grids in the miscible configuration is sufficient to achieve the desired accuracy. Figs. 2(a) and (b) show the comparisons of the eigenvalue spectrums obtained from present simulations for the immiscible configuration (§2.1) with $\Gamma = 0$ and those presented in Schmid & Henningson (2001) in the case of single fluid flow through a pipe for ($\alpha = 1$, $\beta = 0$) and ($\alpha = 0.5$, $\beta = 1$), respectively. The rest of the parameters are $Re = 1000$, $R_i = 0.7$, $\mu_r = 1$ and $\rho_r = 1$. The real and imaginary parts

Table 1: Comparison of the most unstable phase speed (c) with Hu & Joseph (1989). Here, $Re_c = \rho_1 V_c R/\mu_1$ is the Reynolds number used by Hu & Joseph (1989), where V_c is the centerline velocity.

Parameters	Hu & Joseph (1989)	Present
Set 1: $Re_c = 499.5$, $R_i = 0.9$, $\Gamma = 2$, $\mu_r = 0.05$, $\rho_r = 1$, $\alpha = 5.0$, $\beta = 0$	0.38425 + 0.02075i	0.38405 + 0.02081i
Set 2: $Re_c = 37.82$, $R_i = 0.7$, $\Gamma = 0$, $\mu_r = 0.5$, $\rho_r = 1$, $\alpha = 10$, $\beta = 0$	0.66929 + 0.00413i	0.66916 + 0.00405i

of the phase speed are normalised with the maximum velocity (U_{max}) of the basic state. It can be seen that all the eigenvalues are overlapped, except for one additional mode in the case of the immiscible configuration (shown by '●'). This is a neutrally stable ($c_i = 0$) mode with $c_r = U_z|_{R_i}$, which is a solution of the kinematic boundary condition (Eq. 2.17). Hereafter, this mode is termed as mode 'I'. It is observed (also can be seen below in several figures) that mode 'I' is always present in the immiscible core-annular flow configuration (Fig. 1). In the following, the mode 'I' will be used to distinguish the other unstable modes, namely mode '1' and mode '2', in the interfacial core-annular flow.

3. Results and discussion

3.1. *Liquid-liquid system*

In order to identify the most dominant mode of the perturbation in liquid-liquid systems ($\rho_r = \mathcal{O}(1)$), the variations of the normalised growth rate, $\alpha c_{i,max}/U_{max}$ associated with the most unstable axisymmetric ($\beta = 0$) and corkscrew ($\beta = 1$) perturbations (excluding the 'I' mode) are plotted for different viscosity ratios for $\rho_r = 1.1$ and density ratios for $\mu_r = 1.2$ in Figs. 3(a) and (b), respectively. The rest of the parameters are $Re = 1000$, $R_i = 0.7$, $\Gamma = 0.1$, and $c_{i,max}$ is the phase speed of the perturbation corresponding to the most unstable wavenumber (α) for each set of parameters. It is found (not shown) that the higher modes ($\beta \geqslant 2$) are stable for the range of parameters considered in this study. It can be seen in Fig. 3(a) that for $\rho_r = 1.1$, while the corkscrew ($\beta = 1$) perturbation is most dominant for $\mu_r < 1$ (i.e. when the annular fluid is less viscous than the core fluid), the axisymmetric ($\beta = 0$) perturbation is more unstable for $\mu_r > 1$ (i.e., when the annular fluid is more viscous than the core fluid). This is also true for other values of the density ratios, albeit for $\rho_r = \mathcal{O}(1)$, i.e., in liquid-liquid systems, as shown in Fig. 3(b). Usha & Sahu (2019) also found that the axisymmetric perturbation is the most dominant one in the core-annular flow of two immiscible fluids for $\mu_r > 1$ and $\rho_r = 1$. However, this behaviour contrasts with the miscible configuration of two isodense fluids, in which the axisymmetric perturbation is more dominant for $\mu_r < 1$, but the corkscrew perturbation is more unstable for $\mu_r > 1$ (Selvam *et al.* 2007; Sahu & Govindarajan 2016).

3.1.1. *Axisymmetric perturbation: $\beta = 0$*

In this section, the linear instability behaviour of the axisymmetric perturbation ($\beta = 0$) in the immiscible core-annular flow is discussed. The unstable mode in the immiscible configuration is also compared with that of the corresponding miscible configuration. Figs. 4(a) and (b) show the neutral stability curves for the most unstable mode (excluding the 'I' mode) in the $(Re, \alpha)-$plane and the variations of the normalised real part of

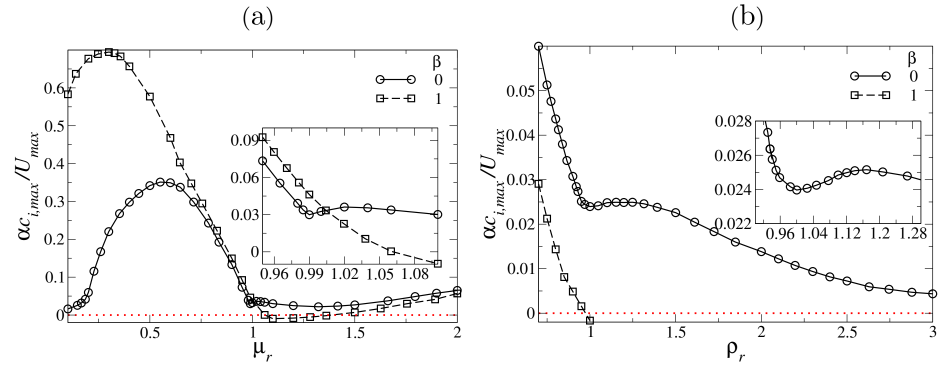

Figure 3: Effect of β on the variations of $\alpha c_{i,max}/U_{max}$ (excluding the neutrally stable interfacial 'I' mode) with (a) the viscosity ratio, μ_r for $\rho_r = 1.1$, and (b) the density ratio, ρ_r for $\mu_r = 1.2$ in the immiscible core-annular configuration. The rest of the parameters are $Re = 1000$, $R_i = 0.7$ and $\Gamma = 0.1$. The insets in panels (a) and (b) are the magnified views near $\mu_r = 1$ and $\rho_r = 1$, respectively.

the phase speed (c_r/U_{max}) along the neutral stability boundaries, respectively. The parameters used for the immiscible configuration are $R_i = 0.7$, $\Gamma = 0$, $\rho_r = 1$, $\mu_r = 0.1$ and $\beta = 0$. It can be seen in Fig. 4(a) that there are two distinct unstable regions corresponding to the long wavelengths (small α) and short wavelengths (large α). The normalised phase speeds of these modes are about 0.5 and 0.9 (see, Fig. 4b), which are termed as 'mode 1' and 'mode 2', respectively. It can also be seen that only one mode is unstable in the corresponding miscible core-annular configuration with $Sc = 1$, $q = 0.02$ (shown by the dashed line in Figs. 4(a) and (b)). The value of c_r/U_{max} in the miscible configuration lies in between the two unstable modes in the immiscible configuration. It is also verified that the result qualitatively remains the same even when the value of q is 10% of the pipe radius and for a range of finite Sc values (see, Fig. A2 in Appendix A). It is found that mode '1' in the low α region becomes stable at sufficiently high values of Re. This indicates that this mode is inviscidly stable and thus, can not be a Tollmien-Schlichting (TS) mode. Mohammadi & Smits (2017) found a similar unstable mode in two-layer Couette flow for low viscosity ratios. This point is discussed further in §3.1.2.

The eigenvalue spectrums associated with the immiscible configuration for $Re = 500$, $\alpha = 1$ (one typical set of parameters in the long wavelengths unstable region) and $Re = 100$, $\alpha = 5$ (one typical set of parameters in the short wavelengths unstable region) are shown in Figs. 4(c) and (d), respectively. It can be seen that for $Re = 500$, $\alpha = 1$, mode '1' is unstable ($c_{r1}, c_{i1} = 0.49100, 0.0067$) and mode '2' is stable ($c_{r2}, c_{i2} = 0.93783, -0.02550$). On the other hand, for $Re = 100$, $\alpha = 5$, mode '1' becomes stable ($c_{r1}, c_{i1} = 0.53618, -0.10152$) and mode '2' is unstable ($c_{r2}, c_{i2} = 0.91531, 0.07105$). It is also observed that, for $\mu_r < 1$, while in the long wavelengths region the real part of the phase speed of the most unstable mode is smaller than that of the 'I' mode, in the short wavelengths region it is higher or of the same order.

To further understand the behaviour of the two distinct unstable modes in the immiscible configuration, the neutral stability curves are plotted for different values of the viscosity ratio in Figs. 5(a-f). The rest of the parameters are the same as those used to generate Fig. 4(a). It can be seen that as we increase the value of μ_r (while remaining less than one), the neutral stability boundaries associated with mode '1' and mode '2'

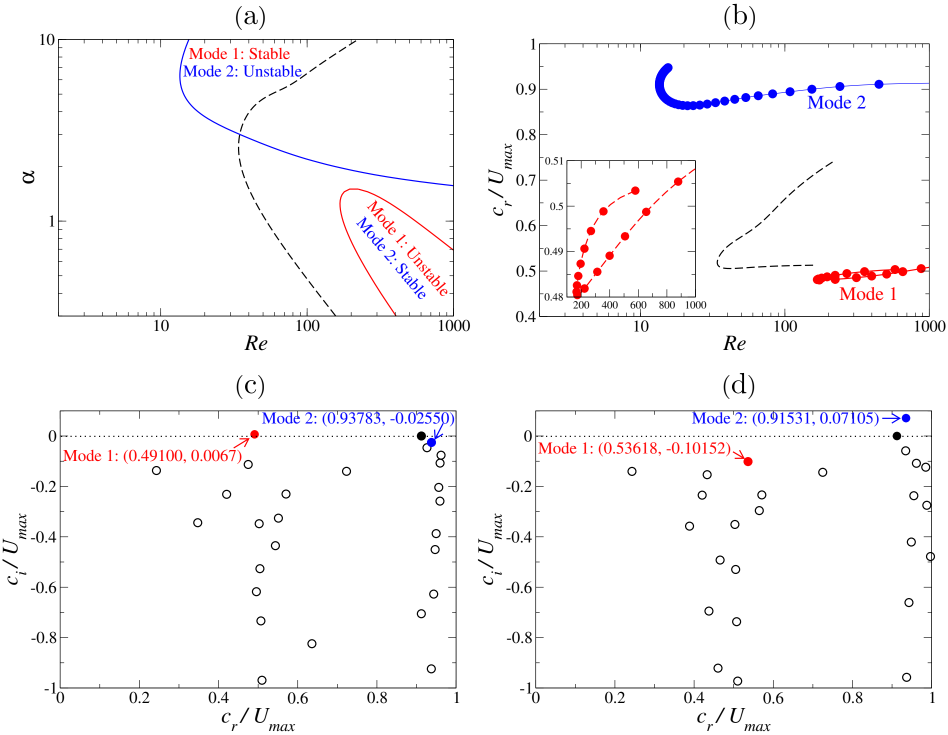

Figure 4: (Colour online) (a) The neutral stability curves corresponding to the most unstable axisymmetric ($\beta = 0$) perturbation (excluding the 'I' mode). (b) Variation of the real part of the phase speed of the most unstable mode as a function of Re. The inset in panel (b) corresponds to mode '1' in the linear scale. The result associated with the corresponding miscible configuration (with $Sc = 1$, $q = 0.02$) is shown by the black dashed line in panels (a) and (b). The eigenvalue spectrums in the immiscible configuration for (c) $Re = 500$, $\alpha = 1$ and (d) $Re = 100$, $\alpha = 5$. The mode 'I' with $c_r = 0.91234$ is shown by symbol (●) in panels (c) and (d). The rest of the parameters are $R_i = 0.7$, $\Gamma = 0$, $\rho_r = 1$ and $\mu_r = 0.1$.

come closer and merge at $\mu_r \approx 0.12$. The overlap region (where both the modes are unstable) grows while the unstable region associated with mode 2 shrinks as we further increase the viscosity ratio, and it disappears for $\mu_r = 0.95$. Close inspection of Figs. 5(a-f) also reveals that the smallest Reynolds number for which either mode '1' or mode '2' is unstable increases with an increase in the viscosity ratio. In Figs. 6(a) and (b), the dispersion curves (c_i/U_{max} versus α) for different viscosity ratios are plotted for $Re = 100$ and $Re = 1500$, respectively. The rest of the parameters are the same as those used to generate Figs. 5(a-f). In Figs. 6(a) and (b), it can be seen that $c_i/U_{max} > 0$ over a finite band of wavenumbers, indicating the presence of linear instability. Fig. 6(a) shows that only mode '2' (in the large α region) is unstable for $Re = 100$, which can also be seen in the neutral stability curve plotted in Fig. 5. For $Re = 100$, it can be observed that the "most-dominant" mode that corresponds to the value of α for which c_i/U_{max} is maximum, decreases with increasing the viscosity ratio (μ_r). In contrast, depending on the value of the viscosity ratio (μ_r), mode '1' and/or mode '2' become the most dominant mode for $Re = 1500$. It is also observed that the value of α associated with the most dominant mode has a non-monotonic variation with μ_r. In particular, inspection of Fig.

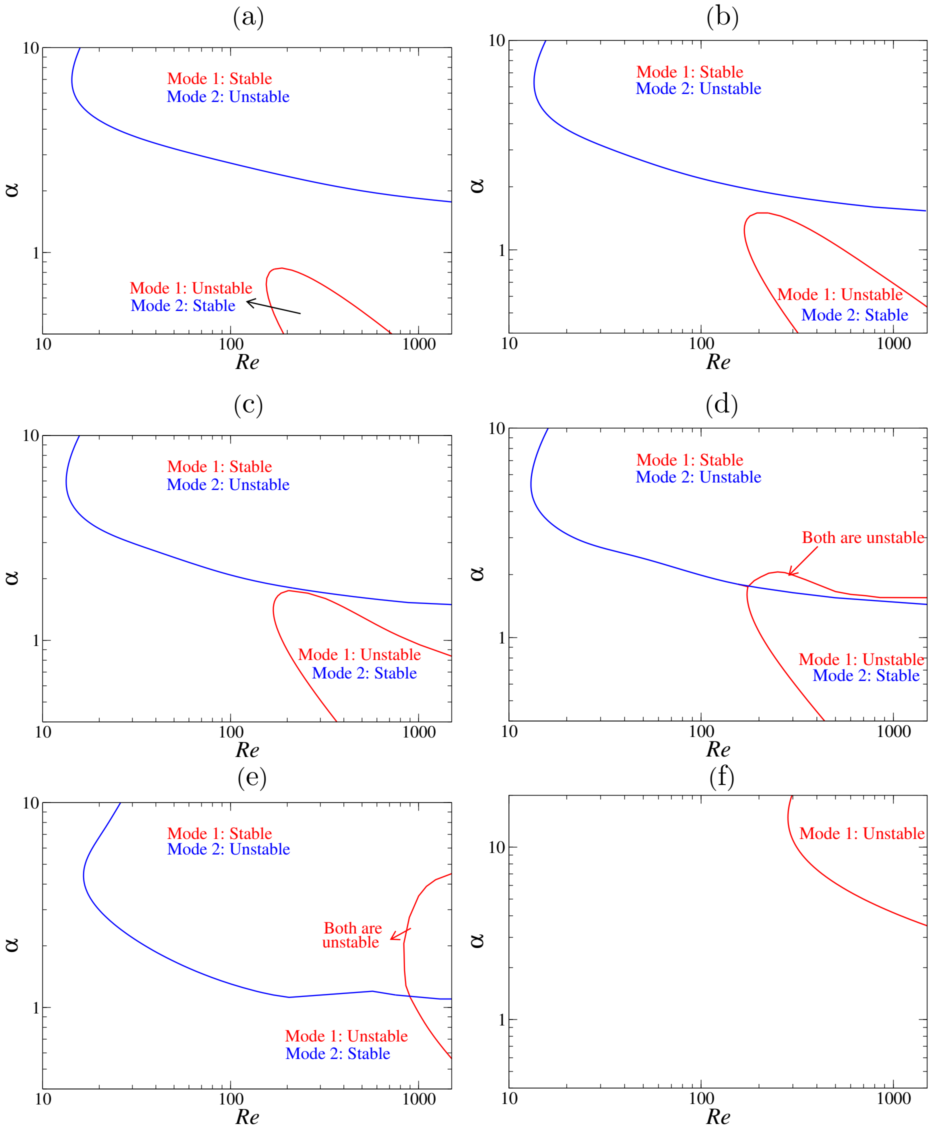

Figure 5: (Colour online) The neutral stability curves associated with the most unstable axisymmetric ($\beta = 0$) perturbation (excluding the mode 'I') for different viscosity ratios in the immiscible configuration. (a) $\mu_r = 0.05$, (b) $\mu_r = 0.1$, (c) $\mu_r = 0.12$, (d) $\mu_r = 0.15$, (e) $\mu_r = 0.5$ and (f) $\mu_r = 0.95$. The rest of the parameters are $R_i = 0.7$, $\Gamma = 0$ and $\rho_r = 1$.

6(b) reveals the following: (i) only mode '2' (with large α value) is unstable for $\mu_r = 0.05$, (ii) for $\mu_r = 0.15$, mode '1' (with small α value) is the most dominant mode as the value of $c_{i,max}/U_{max}$ is higher than that of mode '2', and (iii) mode '2' is the most dominant mode for $\mu_r = 0.5$.

Then, the effect of the density ratio on the neutral stability curves is investigated for the axisymmetric perturbation ($\beta = 0$) in the immiscible configuration with $\mu_r = 0.1$ in Fig. 7. The rest of the parameters are $\Gamma = 0$ and $R_i = 0.7$. It can be seen that the distinct modes merge and becomes a single unstable mode for $\rho_r = 0.1$ and 10. In order

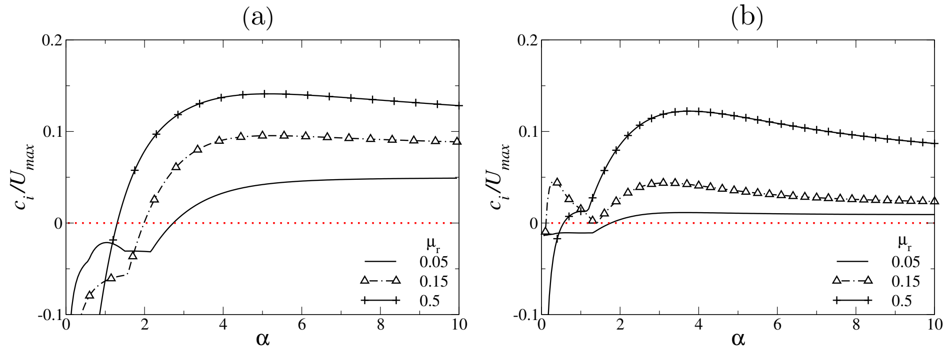

Figure 6: (Colour online) Effect of the viscosity ratio on the variation of c_i/U_{max} versus α for (a) $Re = 100$ and (b) $Re = 1500$. The rest of the parameters are $\beta = 0$, $\rho_r = 1$, $\Gamma = 0$ and $R_i = 0.7$.

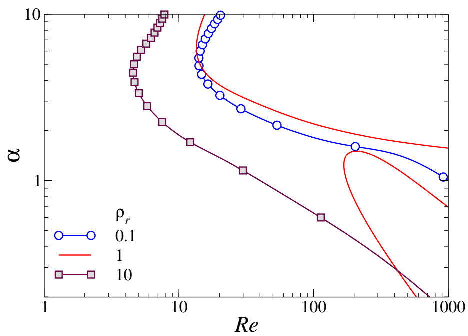

Figure 7: (Colour online) Effect of the density ratio on the neutral stability curve for $\mu_r = 0.1$ in the immiscible core-annular configuration. The rest of the parameters are $\beta = 0$, $\Gamma = 0$ and $R_i = 0.7$.

to find the range of the density ratios for which the two distinct modes are present, the variations of c_i/U_{max} and c_r/U_{max} with α are plotted for $\mu_r = 0.1$, $Re = 1000$, $\Gamma = 0$ and $R_i = 0.7$ in Figs. 8(a) and (b), respectively. It is observed that two unstable modes associated with low and high α values appear in the range $0.7 \leqslant \rho_r \leqslant 5$ for the set of parameters considered. Inspection of Figs. 8(a) and (b) reveals that for $\rho_r = 0.5$ and $\rho_r = 7$ there is only one unstable mode ($c_i/U_{max} > 0$).

3.1.2. *Corkscrew perturbation:* $\beta = 1$

After establishing the new mode of instability associated with the axisymmetric perturbation ($\beta = 0$), the linear stability behaviour of the corkscrew perturbation ($\beta = 1$) is investigated in this section. Fig. 9(a) depicts the neutral stability curves associated with the most unstable mode (excluding the mode 'I' with its real part, $c_{rI} = U_z|_{R_i}$) in the $(Re, \alpha)-$plane. The rest of the parameters used to generate these results are $R_i = 0.7$, $\Gamma = 0$, $\rho_r = 1$ and $\mu_r = 1.2$. It can be seen that the corkscrew perturbation ($\beta = 1$) too exhibits two distinct unstable regions corresponding to the long (small α) and short wavelengths (large α) for the set of parameters considered. This is also clearly evident in Fig. 9(b) which presents the variations of the normalised real part of the phase speed (c_r/U_{max}) along the neutral stability boundaries. However, close inspection of Figs. 9(a) and (b) reveals that, in contrast to the axisymmetric perturbation ($\beta = 0$), the long and

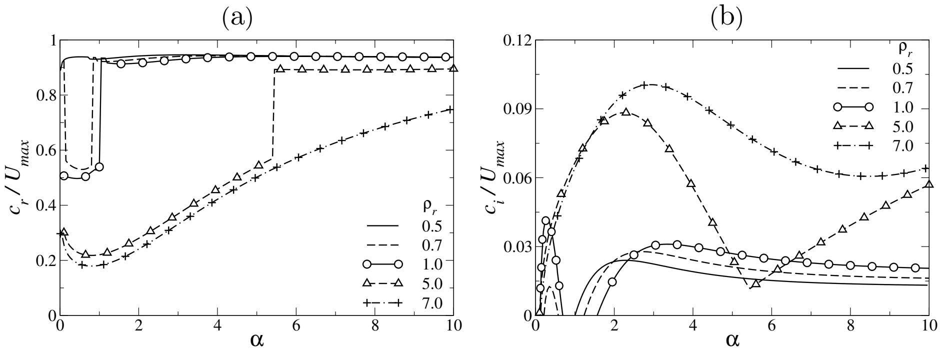

Figure 8: Variations of the normalised real (c_r/U_{max}) and imaginary (c_i/U_{max}) parts of the phase speed of the most unstable axisymmetric perturbation ($\beta = 0$). (a) c_r/U_{max} versus α and (b) c_i/U_{max} versus α. The rest of the parameters are $\mu_r = 0.1$, $Re = 1000$, $\mathit{\Gamma} = 0$ and $R_i = 0.7$.

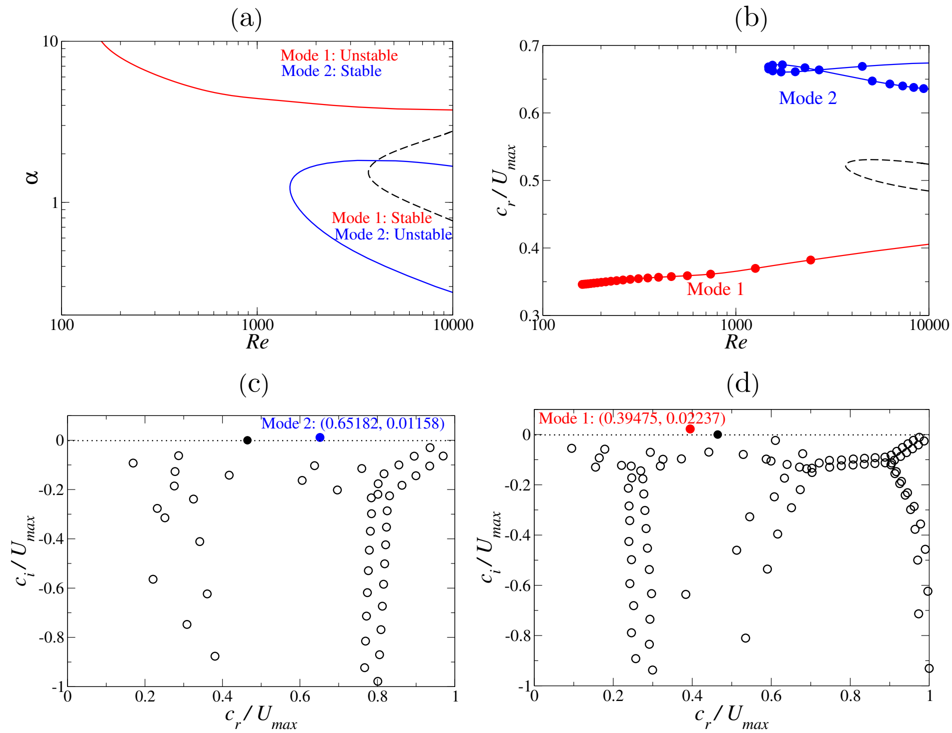

Figure 9: (Colour online) (a) The neutral stability curves for the most unstable corkscrew ($\beta = 1$) perturbation. (b) Variation of the real part of the phase speed of the most unstable mode as a function of Re. The result associated with a miscible configuration (with $Sc = 1$, $q = 0.02$) is shown by black dashed line in panels (a) and (b). The eigenvalue spectrums for (c) $Re = 3000$, $\alpha = 1$ and (d) $Re = 3000$, $\alpha = 6$ in the immiscible configuration. The mode 'I' with $c_r = U_z|_{R_i} = 0.46448$ is shown by symbol (●) in panels (c) and (d). The rest of the parameters are $R_i = 0.7$, $\mathit{\Gamma} = 0$, $\rho_r = 1$ and $\mu_r = 1.2$.

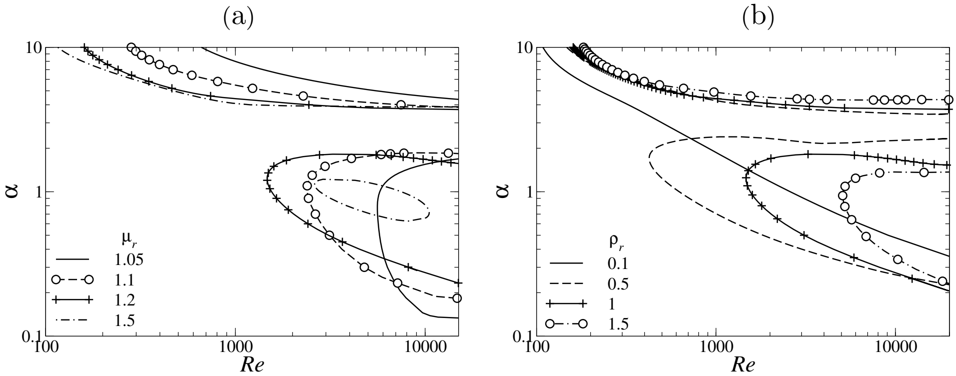

Figure 10: The neutral stability curves associated with the most unstable corkscrew ($\beta = 1$) perturbation (excluding the mode 'I') for different (a) viscosity ratio for $\rho_r = 1$ and (b) density ratio for $\mu_r = 1.2$ in the immiscible configuration. The rest of the parameters are $\Gamma = 0$ and $R_i = 0.7$.

short wavelengths unstable regions are associated with mode '2' (whose $c_r > c_{rI}$) and mode '1' (whose $c_r < c_{rI}$) in the case of corkscrew perturbation ($\beta = 1$). The results for the corresponding miscible configuration (with $Sc = 1$, $q = 0.02$) is shown by black dashed line in Figs. 9(a) and (b). It can be seen in Fig. 9(a) that the neutral stability curves for mode '2' (in the small α regime) in the immiscible configuration look similar to that of the single unstable Tollmien-Schlichting (TS) mode observed in the miscible case. However, the critical Reynolds number associated with mode '2' in the immiscible configuration is much smaller than that of the TS mode in the miscible configuration.

The characteristics of modes '1' and '2' in the immiscible configuration for $\beta = 1$ are illustrated in Figs. 9(c) and (d), which show the eigenvalue spectrums for two typical sets of parameters, namely ($Re = 3000$, $\alpha = 1$) and ($Re = 3000$, $\alpha = 6$) in the unstable long and short wavelength regions, respectively. In Fig. 9(c), it can be seen that the real part of the phase speed of the most unstable mode, c_r is greater than c_{rI}; thus, it is mode '2' by definition. On the other hand, in Fig. 9(d), the phase speed of the most unstable mode, c_r is less than c_{rI}; thus, it is mode '1' by definition.

The effect of the viscosity ratio for $\rho_r = 1$ and the density ratio for $\mu_r = 1.2$ on the neutral stability curve associated with the most unstable corkscrew ($\beta = 1$) perturbation (excluding the mode 'I') in the immiscible configuration is investigated in Figs. 10(a) and (b), respectively. The rest of the parameters are fixed at $\Gamma = 0$ and $R_i = 0.7$. The two distinct unstable regions in the long and short wavelengths perturbations are apparent Fig. 10(a) for different viscosity ratios. Another important point to be noted here that for $\mu_r = 1.5$, the neutral stability boundary in the long wavelengths region form a closed loop. It can be seen in Fig. 10(b) that while only mode '1' is unstable for low density ratios (see, for instance, $\rho_r = 0.1$), the two distinct unstable regions are present for $\rho_r \geqslant 0.5$. However, increasing the density ratio shifts the neutral boundary towards high Reynolds number for the set of parameters considered in Fig. 10(b). It is found (not shown) that all the neutral stability boundaries for mode '2' in Figs. 10(a) and (b) form close loops (albeit it happens at large values of Re). This behaviour indicates that the unstable mode '2' in the interfacial configuration is not a Tollmien-Schlichting (TS) mode. The new type of interfacial mode observed in this study for the immiscible core-annular configuration is similar to that found in two-layer Couette flow for low viscosity ratio (Mohammadi & Smits 2017).

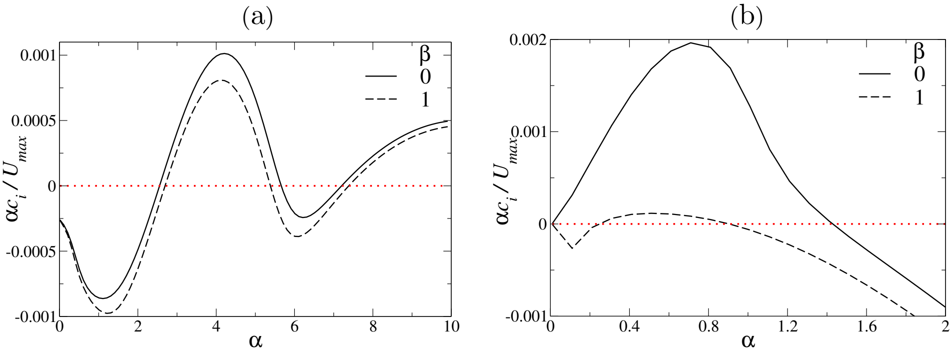

Figure 11: (Colour online) Effect of β on the dispersion curves associated with the most unstable mode (excluding the mode 'I'). (a) Core (gas) - annular (liquid) configuration ($\rho_r = 10^3$, $\mu_r = 10^2$). (b) Core (liquid) - annular (gas) configuration ($\rho_r = 10^{-3}$, $\mu_r = 10^{-2}$). The rest of the parameters are $Re = 10^4$, $R_i = 0.7$ and $\Gamma = 0.1$.

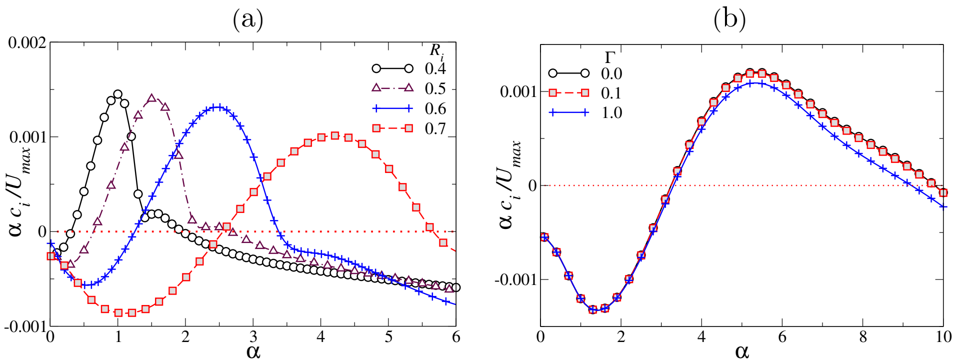

Figure 12: (Colour online) Dispersion curves associated with the most unstable mode (excluding the mode 'I') for $\beta = 0$ (axisymmetric perturbation). (a) Effect of R_i for $Re = 10^4$, $\Gamma = 0.1$, and (b) effect of Γ for $Re = 5000$ and $R_i = 0.7$. The rest of the parameters are $\rho_r = 10^3$ and $\mu_r = 10^2$.

3.2. *Gas-liquid systems*

Finally, a parametric study is conducted to study the effect of the location of the interface (R_i) and the inverse capillary number (Γ) on the linear stability behaviour in gas-liquid systems. Two configurations, namely, (i) when the core fluid is a gas and the annular fluid is a liquid (i.e., $\rho_r = 10^3$ and $\mu_r = 10^2$) and (ii) when the core fluid is a liquid and the annular fluid is a gas (i.e., $\rho_r = 10^{-3}$ and $\mu_r = 10^{-2}$) are considered. Figs. 11(a) and (b) depict the dispersion curves ($\alpha c_i/U_{max}$ versus α) associated with the most unstable axisymmetric ($\beta = 0$) and corkscrew ($\beta = 1$) perturbations (excluding the 'I' mode) in the gas-liquid systems for ($\rho_r = 10^3$, $\mu_r = 10^2$) and ($\rho_r = 10^{-3}$, $\mu_r = 10^{-2}$), respectively. It can be seen that, unlike the liquid-liquid system (discussed in §3.1), the axisymmetric perturbation ($\beta = 0$) is the most dominant mode in gas-liquid system. Thus, only the axisymmetric perturbation ($\beta = 0$) is examined hereafter in this section.

The effect of the interface location (R_i) and the inverse capillary number (Γ) on the dispersion curves for the most unstable mode (excluding the mode 'I') are shown in Figs. 12(a) and (b) for $\beta = 0$, $\rho_r = 10^3$ and $\mu_r = 10^2$ (when the core fluid is a gas and the annular fluid is a liquid). The rest of the parameters in Fig. 12(a) are $Re = 10^4$ and

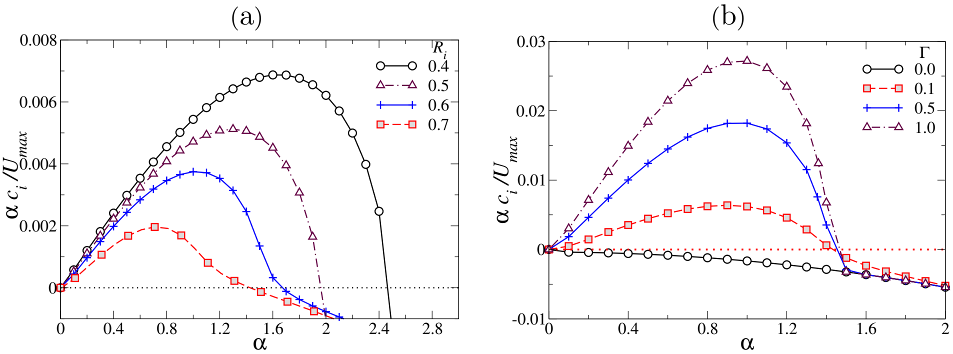

Figure 13: (Colour online) Dispersion curves associated with the most unstable mode (excluding the mode 'I') for $\beta = 0$ (axisymmetric perturbation). (a) Effect of R_i for $Re = 10^4$, $\Gamma = 0.1$, and (b) effect of Γ for $Re = 2000$ and $R_i = 0.7$. The rest of the parameters are $\rho_r = 10^{-3}$ and $\mu_r = 10^{-2}$.

$\Gamma = 0.1$, and in Fig. 12(b) are $Re = 5000$ and $R_i = 0.7$. Increasing R_i decreases the gradient of velocity in the annular region ($U'_{z,2}$) and decreases the centreline velocity to maintain the constant volumetric flow rate condition. Thus, increasing R_i makes the flow in the annular region like a plug, which in turn stabilises the flow as seen in Fig. 12(a). It can also be seen in Fig. 12(a) that the wavelength of the perturbation ($2\pi/\alpha$) associated with the highest growth rate decreases as the value of R_i increases. It can be seen in Fig. 12(b) that increasing Γ, which corresponds to increasing the surface tension, stabilises the short-wave (high α) perturbation.

Figs. 13(a) and (b) depict the effect of the interface location (R_i) for $Re = 10^4$, $\Gamma = 0.1$ and the inverse capillary number (Γ) for $Re = 2000$, $R_i = 0.7$ on the growth rate of the perturbation for $\rho_r = 10^{-3}$ and $\mu_r = 10^{-2}$. In this case, as the core is a liquid (highly viscous as compared to gas), the plug flow region appears in the core layer, which stabilises the flow. In other words, increasing R_i decreases the maximum growth rate of the perturbation (Fig. 13a). In the case of visco-plastic fluid flow in a channel, Frigaard (2001) also found that the presence of unyielded (plug) region highly stabilises the flow. It is found that increasing Γ destabilises the long-wave perturbation (Fig. 12b). Thus, it can be concluded that increasing Γ, which corresponds to increasing the surface tension, stabilises the short-wave (high α) but destabilises the long-wave perturbations (low α) via the Rayleigh-Plateau instability. It is also observed (not shown) that the effect of Γ on the stability characteristic is similar in the liquid-liquid configuration described in §3.1.

4. Conclusions

The linear stability behaviour of the axisymmetric ($\beta = 0$) and corkscrew ($\beta = 1$) perturbations in the core-annular pressure-driven flow of two immiscible fluids in a cylindrical pipe is examined and compared to that observed in the corresponding configuration of two miscible fluids. The effects of the viscosity ratio (μ_r), the density ratio (ρ_r), the Reynolds number (Re), the dimensionless interface location (R_i) and the inverse capillary number (Γ) have been investigated. Both liquid-liquid and gas-liquid systems are considered. A new mode of instability distinct from the Tollmien-Schlichting (TS) mode and Yih's interface mode (Yih 1967) is discovered for a certain range of viscosity and density ratios in the immiscible liquid-liquid system ($\rho_r = \mathcal{O}(1)$). The corkscrew perturbation also exhibits the new mode of instability for a certain range of density ratios. Contrary to the immiscible core-annular configuration, in which two regions of instability are observed for a range of viscosity and density ratios, only one mode is found to be unstable in the miscible core-annular flow. The new interfacial mode observed in the present study is similar to that found in two-layer Couette flow for low viscosity ratios (Mohammadi & Smits 2017). It is also observed that in the liquid-liquid systems ($\rho_r = \mathcal{O}(1)$), while the corkscrew perturbation is most dominant when the annular fluid is less viscous than the core fluid, the axisymmetric perturbation becomes more unstable when the annular fluid is more viscous than the core fluid. In contrast to the liquid-liquid system, the axisymmetric perturbation is always the dominant one in the gas-liquid system. It is found that increasing the interface radius stabilises the flow due to the presence of a plug flow region.

Declaration of Interests: The authors report no conflict of interest.

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Appendix A. Miscible core-annular configuration

In order to compare the linear stability behaviour of the immiscible configuration with the corresponding miscible system, the basic state and the linear stability equations associated with the core-annular flow of two miscible fluids are discussed briefly in this section. The reader is referred to Sahu (2016) for more details. The schematic diagram of the pressure-driven core-annular miscible flow configuration is shown in Fig. A1. In this configuration, the fluids are miscible and separated by a mixed region of thickness q_0 in region $R_0 - q_0/2 \leqslant r \leqslant R_0 + q_0/2$ of the pipe. The dimensionless dynamic viscosity is given by:

$$\mu_0 = \exp(s_0 \ln \mu_r), \tag{A 1}$$

where s_0 is given by:

$$\begin{aligned} s_0 &= 0, \qquad 0 \leqslant r \leqslant R_i - q/2, \\ s_0 &= \sum_{i=1}^{6} a_i r^{i-1}, \quad R_i - q/2 \leqslant r \leqslant R_i + q/2, \\ s_0 &= 1, \qquad R_i + q/2 \leqslant r \leqslant 1, \end{aligned} \tag{A 2}$$

where a_i ($i = 1, 6$) are obtained by assuming that the scalar is continuous up to the second derivative at $r = R_i - q/2$ and $r = R_i + q/2$ (Govindarajan 2004; Sahu & Govindarajan 2011), wherein $q = q_0/R$. The other variables are nondimensionalised in the same manner as described in §2.1 for the immiscible configuration. When the Péclet number $Pe(\equiv ReSc)$ is large, s_0 could be approximated by an error function that depends on the combination $(r - R_i)\sqrt{Pe/z}$. In the stability calculation, the dependence of s_0 on z is neglected. This "quasi-steady" approximation to represent basic concentration profile in miscible flows is justified if the wavelength $2\pi/\alpha$ of the disturbance is much shorter than the length-scale over which s_0 varies with z, namely $q^2 Pe$. It is also to be

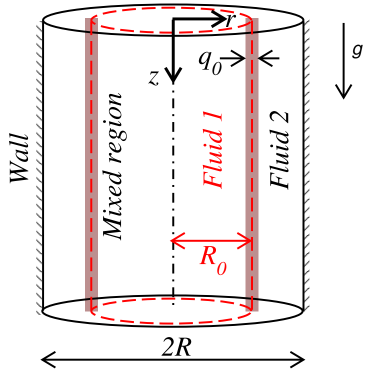

Figure A1: (Colour online) Schematic of the pressure-driven core-annular flows of two miscible fluids in a vertical pipe of radius R. The acceleration due to gravity, g acts in the positive $z-$direction. The fluids are separated by a mixed region of thickness q_0 occupying the region $R_0 - q_0/2 \leqslant r \leqslant R_0 + q_0/2$.

noted here that, after Tan & Homsy (1986), several authors have used the quasi-steady approximation to represent basic concentration profile in miscible flows in the form of a hyperbolic tangent (Ern *et al.* 2003), an error function (Selvam *et al.* 2007; Talon & Meiburg 2011), and a fifth-order polynomial (Ranganathan & Govindarajan 2001; Sahu 2016).

In the case of the miscible core-annular flow, the basic state is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu_0\frac{\partial U_z}{\partial r}\right] = \frac{dP}{dz}Re, \tag{A 3}$$

which is solved using the no-slip boundary condition at the pipe wall and the symmetric boundary condition at the centerline of the pipe. Unlike the immiscible configuration, the interfacial boundary conditions are not needed in this case as the dynamics is characterised by the diffusion coefficient ($\mathcal{D}$) of the scalar and not the interfacial tension (no sharp interface). The dimensionless pressure gradient, dP/dz is obtained using constant volumetric flow condition.

In the derivation of the linear stability equations for the miscible configuration, the scalar variable can be expressed as $s_0(r) + s(r)e^{\mathrm{i}(\alpha z+\beta\theta-\alpha ct)}$, such that the amplitude of the perturbation viscosity, $\mu = (\partial\mu_0/\partial s_0)\, s$. The normal mode analysis used in this case is also similar to that given in §2.1.2. The linear stability equations for the miscible configuration are given by

$$u_r' + \frac{u_r}{r} + \frac{\beta u_\theta}{r} + \alpha u_z = 0, \tag{A 4}$$

$$\rho(-\alpha c u_r + \alpha u_r U_z) = p' - \frac{\mathrm{i}}{Re}\Big[\mu_0\left\{{u_r}'' + \frac{{u_r}'}{r} - \left(\frac{\beta^2+1}{r^2} + \alpha^2\right)u_r - \frac{2\beta}{r^2}u_\theta\right\} + 2\mu_0' {u_r}' + \alpha U_z' \mu\Big], \tag{A 5}$$

$$\rho(-\alpha c u_\theta + \alpha u_\theta U_z) = -\frac{\beta p}{r} - \frac{\mathrm{i}\mu_0}{Re}\left\{{u_\theta}'' + \frac{{u_\theta}'}{r} - \left(\frac{\beta^2+1}{r^2} + \alpha^2\right)u_\theta - \frac{2\beta}{r^2}u_r\right\} - \frac{\mathrm{i}{\mu_0}'}{Re}\left[{u_\theta}' - \frac{u_\theta}{r} - \frac{\beta u_r}{r}\right], \tag{A 6}$$

$$\rho(-\alpha c u_z + U_z{}' u_r + \alpha U_z u_z) = -\alpha p - \frac{\mathrm{i}\mu_0}{Re}\left\{u_z{}'' + \frac{u_z{}'}{r} - \left(\frac{\beta^2}{r^2} + \alpha^2\right) u_z\right\} -$$
$$\frac{\mathrm{i}\mu_0'}{Re}\left[v_z' - \alpha v_r\right] - \frac{\mathrm{i}U_z'}{Re}\mu' - \frac{\mathrm{i}\mu}{Re}\left[U_z{}'' + \frac{U_z'}{r}\right], \quad \text{(A 7)}$$

$$-\alpha c s + s_0{}' u_r + \alpha U_z s = -\frac{\mathrm{i}}{ReSc}\left\{s'' + \frac{s'}{r} - \left(\frac{\beta^2}{r^2} + \alpha^2\right) s\right\}, \quad \text{(A 8)}$$

Here $\rho = s_0\rho_r + (1 - s_0)$ and $Sc(\equiv \mu_1/\rho\mathcal{D})$ is the Schmidt number. It is noted here that Eqs. (A 4)-(A 7) are similar to the stability equations for each layer in the case of immiscible core-annular flow configuration (§2.1.2). The boundary conditions for the perturbation velocity field (v_r, v_θ, v_z) at the centerline and wall of the pipe are the same as Eqs. (2.9) and (2.12). The boundary conditions for the scalar variable, s are $s = 0$ and $s' = 0$ at the centerline and wall of the pipe, respectively. For more details and validation of the stability analysis of the miscible configuration presented in this section, the reader is referred to our previous studies Sahu & Govindarajan (2011); Sahu (2016, 2019).

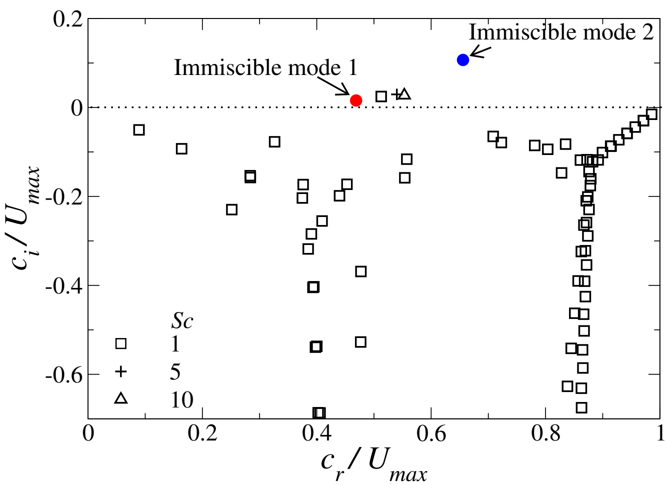

Figure A2: (Colour online) Effect of Sc on the most unstable mode in the miscible core-annular flow configuration for $Re = 1500$, $\alpha = 2.5$, $\beta = 0$, $q = 0.02$, $\mu_r = 0.5$, $R_i = 0.7$ and $\rho_r = 1$. The complete eigenvalue spectrum corresponds to $Sc = 1$. The unstable modes '1' and '2' in the corresponding immiscible configuration with $\Gamma = 0$ are shown by ('●') and ('●'), respectively. It can be seen that increasing Sc has a non-monotonic effect on the growth rate of the most unstable mode in the miscible configuration.

Appendix B. Grid convergence test

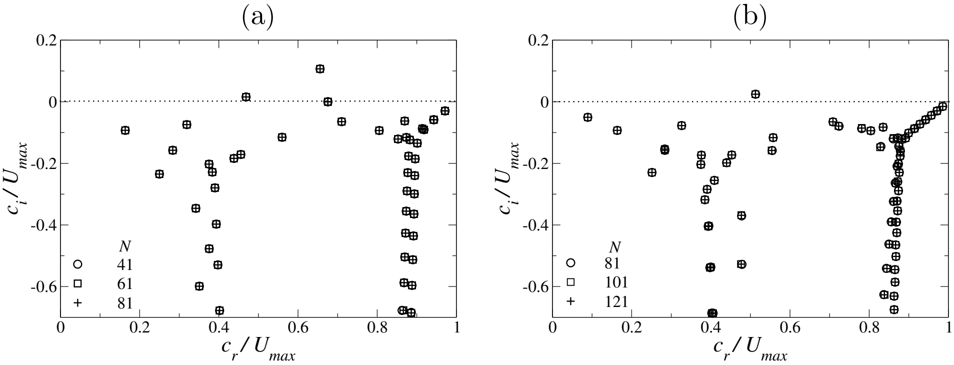

Figure A3: The eigenvalue spectrums obtained using different numbers of grids (N) in the (a) immiscible configuration (Fig. 1) with $\Gamma = 0$ and (b) miscible configuration (Fig. A1) with $Sc = 1$ and $q = 0.02$. The rest of the parameters are $Re = 1500$, $\alpha = 2.5$, $\beta = 0$, $R_i = 0.7$, $\mu_r = 0.5$ and $\rho_r = 1$.

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