Yager’s New Class of Implications $J_f$ and some Classical Tautologies

J. Balasubramaniam

Department of Mathematics and Computer Sciences,
Sri Sathya Sai Institute of Higher Learning,
Prasanthi Nilayam, A.P-515134, India.

Abstract

Recently, Yager (Info Sci, Vol 167, pp.193 - 216, 2004) has introduced a new class of fuzzy implications, denoted $J_f$, called the $f$-generated implications and has discussed some of their desirable properties, such as neutrality, exchange principle, etc. In this work, we discuss the class of $J_f$ implications with respect to three classical logic tautologies, viz., distributivity, law of importation and contrapositive symmetry. Necessary and sufficient conditions under which $J_f$ implications are distributive over $t$-norms and $t$-conorms and satisfy the law of importation with respect to a $t$-norm have been presented. Since the natural negations of $J_f$ implications, given by $N_{J_f}(x) = J_f(x, 0)$, in general, are not strong, we give sufficient conditions under which they become strong and possess contrapositive symmetry with respect to their natural negations. When the natural negations of $J_f$ are not strong, we discuss the contrapositivisation of $J_f$. Along the lines of $J_f$ implications, a new class of implications called $h$-generated implications, $J_h$, has been proposed and the interplay between these two types of implications has been discussed. Notably, it is shown that while the natural negations of $J_f$ are non-filling those of $J_h$ are non-vanishing, properties which determine the compatibility of a contrapositivisation technique.

Key words: Yager’s Implications, $f$-generated Implications, Distributivity of Fuzzy Implications, Law of Importation, Contrapositive Symmetry, Contrapositivisation, $h$-generated Implications.

Email address: jbala@ieee.org (J. Balasubramaniam).

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1 Introduction

Fuzzy implication operators play an important role both in Approximate Reasoning and Fuzzy Control Theory. The most established and well-studied classes of fuzzy implications are $R$-, $S$- and $QL$-implications (see, for example, [15], [16], [19], [26] for their definitions and properties). Recently, Yager [32] has introduced a new class of implications, denoted $J_f$, called the $f$-generated implications - which in general are different from the above categories (see [3], [4]) - and discussed their desirable properties as listed in [19], such as neutrality, exchange principle, etc.

1.1 Motivation for this work

Lately there has been a spate of works that discusses and explores the validity of many classical logic tautologies in fuzzy logic, especially those that involve fuzzy implications. Three such classical logic tautologies involving fuzzy implications that have obtained maximum attention from researchers are those that deal with the distributivity of fuzzy implications over $t$-norms and $t$-conorms, the satisfaction of the law of importation with respect to a $t$-norm and contrapositive symmetry.

Recently there has been a lot of discussion [7], [12], [13], [14], [22], [27] centred around a paper by Combs and Andrews [11] where they attempt to exploit the equivalence

\[(p \land q) \rightarrow r \equiv (p \rightarrow r) \lor (q \rightarrow r)\]  

(1)

towards eliminating combinatorial rule explosion in fuzzy systems. (1) is only one of four such equations as listed in [17], which deals with the distributivity of implication operators with respect to $t$-norms and $t$-conorms, the rest of them being:

\[(p \lor q) \rightarrow r \equiv (p \rightarrow r) \land (q \rightarrow r)\]  

(2)

\[r \rightarrow (s \land t) \equiv (r \rightarrow s) \land (r \rightarrow t)\]  

(3)

\[r \rightarrow (s \lor t) \equiv (r \rightarrow s) \lor (r \rightarrow t)\]  

(4)

In [27], Trillas and Alsina have investigated the conditions under which the following general form of (1), where $p, q, r \in [0, 1]$:

\[J(T(p,q), r) \equiv S(J(p, r), J(q, r))\]  

(5)
holds for the classes of $R$-, $S$- and $QL$-implications, where $T$ and $S$ denote any $t$-norm and $t$-conorm respectively. The generalisations of equations (2) - (4) are as follows:

$$J(S(p, q), r) \equiv T(J(p, r), J(q, r))$$  \hspace{1cm} (6)

$$J(r, T_1(s, t)) \equiv T_2(J(r, s), J(r, t))$$  \hspace{1cm} (7)

$$J(r, S_1(s, t)) \equiv S_2(J(r, s), J(r, t))$$  \hspace{1cm} (8)

where $p, q, r, s, t \in [0, 1]$.

Conditions under which equations (6) - (8) hold for $R$- and $S$-implications have appeared in [7]. Except for the case when $J$ is an $R$-implication obtained from a strict $t$-norm $T$ and $S_1 = S_2$ is a nilpotent $t$-conorm in (8), in all the other cases, if $J$ is an $R$- or an $S$-implication, the $t$-norm $T$ and the $t$-conorm $S$ do get fixed to $T_M(x, y) = \min(x, y)$ and $S_M(x, y) = \max(x, y)$, respectively. Also the equation (7) has been discussed in [1], [2] under the assumptions that $T = T_1 = T_2$ is a strict $t$-norm and the implication $J$ is continuous except at $(0,0)$.

The above equations play an important role in lossless rule reduction in Fuzzy Rule Based Systems [5], [7], [24], [30]. Thus it becomes both interesting and important to discuss the validity of these distributive equations for a given fuzzy implication in the hope of obtaining $t$-norms $T$ and $t$-conorms $S$ other than $T_M$ and $S_M$, respectively. We will see below that the family of $f$-generated implications has more solutions to (7) than an $R$- or an $S$-implication. On the other hand, for Yager’s $f$-generated implication too the solution to (8) is not fully settled. Thus it is worthwhile to study the distributivity of $J_f$ over $t$-norms and $t$-conorms.

The equation $(x \land y) \rightarrow z \equiv (x \rightarrow (y \rightarrow z))$, known as the law of importation, is another desirable tautology in classical logic. The general form of the above equivalence is given by

$$J(T(x, y), z) \equiv J(x, J(y, z)), \quad x, y, z \in [0, 1],$$

where $T$ is a $t$-norm and $J$ a fuzzy implication. In $A$-implications defined by Turksen et al. [29], the general form of the law of importation, with $T$ as the product $t$-norm $T_P(x, y) = x \cdot y$, was taken as one of the axioms. Baczyński [1] has studied the law of importation in conjunction with the general form of the following distributive property of fuzzy implications

$$[r \rightarrow (s \land t)] \equiv [(r \rightarrow s) \land (r \rightarrow t)]$$

and has given a characterisation. Bouchon-Meunier and Kreinovich [23] have characterised fuzzy implications that have the law of importation as one of the axioms along with the general form of the above distributive equation.
They have considered the minimum t-norm $T_M$ for $T$ and claim that Mam-dani’s choice of implication “min” is “not so strange after all”. [8] discusses the validity of this tautology for $R$-, $S$- and $QL$-implications and its possible applications in Approximate Reasoning are explored.

Contrapositive symmetry is yet another classical tautology desirable for fuzzy implications. Contrapositive symmetry plays a significant role in classical logic structures wherein “proof by contradiction” is a commonly employed method to validate conjectures. Works in [9], [10], [20] discuss ways of imparting contrapositive symmetry with respect to any arbitrary strong negation $N$. Also contrapositive symmetry of fuzzy implications has been studied in a functional framework in [2] along with the law of importation and the general form of the above distributive equation.

This work, where we discuss the class of $J_f$ implications with respect to three classical logic tautologies, viz., distributivity over $t$-norms and $t$-conorms, law of importation and contrapositive symmetry, can be seen as part of the above efforts.

Yager in [32] has done an extensive analysis of the impact of this new class of implications in Approximate Reasoning by introducing concepts like strictness of implications and sharpness of inference, among others. This work can also be seen as a continuation of the above study on the classical tautologies satisfied by Yager’s $f$-generated implications that have an influence in Approximate Reasoning. For more recent works on the role of fuzzy logic operators in Computing with words see [28],[33].

1.2 Outline of this work

Firstly, by discussing the four different general forms of distributive equations we show that the Yager’s class of $f$-generated implications does have more solutions for one of them than that possessed by $R$-, $S$- or $QL$-implications. We also give necessary and sufficient conditions under which the class of $J_f$ implications satisfies the law of importation. Following this, we give sufficient conditions under which the natural negations of $J_f$ implications are strong and the implications $J_f$ possess contrapositive symmetry with respect to their natural negations.

Since the natural negations of $J_f$ implications, in general, are not strong we discuss the contrapositivisation of $J_f$ implications using the techniques proposed in [10]. We have shown that, in general, only the upper contrapositivisation is $N$-Compatible with $J_f$.

Finally, taking cue from the $f$-generated implications $J_f$, we also present a
new class of implications, $J_h$, called $h$-generated implications and show that they have some very useful properties, viz., their natural negations are non-vanishing, and hence the lower contrapositivisation is $N$-Compatible with $J_h$.

The paper is organised as follows. In Section 2 we recall the class of $f$-generated fuzzy implications $J_f$ proposed by Yager in [32] and also some relevant results on $t$-norms and $t$-conorms. In Section 3 we investigate the distributivity of $J_f$ over $t$-norms and $t$-conorms, while in Section 4 the law of importation with respect to a $t$-norm $T$ is explored and in Section 5 the contrapositive symmetry of $J_f$ implications is discussed. In Section 6 we present a new class of implications, $J_h$, called $h$-generated implications and discuss their properties vis-à-vis contrapositivisation. Section 7 gives some concluding remarks.

2 Preliminaries

To make this work self-contained, we briefly mention some of the concepts and results employed in the rest of the work.

2.1 Negations

Definition 1 ([19] Definition 1.1 Pg 3) A negation $N$ is a function from $[0,1]$ to $[0,1]$ such that:

- $N(0) = 1; \ N(1) = 0$;
- $N$ is non-increasing.

Definition 2 A negation $N$ is said to be

- non-vanishing if $N(x) \neq 0$ for any $x \in [0,1)$, i.e., $N(x) = 0$ iff $x = 1$;
- non-filling if $N(x) \neq 1$ for any $x \in (0,1]$, i.e., $N(x) = 1$ iff $x = 0$.

A negation $N$ that is not non-filling (non-vanishing) will be called filling (vanishing).

Definition 3 ([19] Definition 1.2 Pg 3) A negation $N$ is called strict if in addition $N$ is strictly decreasing and continuous.

Note that if a negation $N$ is strict it is both non-vanishing and non-filling, but the converse is not true.

Definition 4 ([19] Definition 1.2 Pg 3) A strong negation $N$ is a strict negation $N$ that is also involutive, i.e., $N(N(x)) = x, \ \forall x \in [0,1]$.  
2.2 T-norms and T-conorms

Definition 5 ([18] Definition 1.1 Pg 4) A function $T$ from $[0,1]^2 \rightarrow [0,1]$ is called a triangular norm (shortly t-norm) if, for all $x, y, z \in [0,1]$,

\[
T(x,y) = T(y,x), \quad (T1)
\]
\[
T(x, T(y,z)) = T(T(x,y), z), \quad (T2)
\]
\[
T(x,y) \leq T(x,z) \quad \text{whenever } y \leq z, \quad (T3)
\]
\[
T(x,1) = x. \quad (T4)
\]

Definition 6 ([18] Definition 1.13 Pg 11) A function $S : [0,1]^2 \rightarrow [0,1]$ is called a triangular conorm (shortly t-conorm) if, for all $x, y, z \in [0,1]$, it satisfies

\[
S(x,y) = S(y,x), \quad (S1)
\]
\[
S(x, S(y,z)) = S(S(x,y), z), \quad (S2)
\]
\[
S(x,y) \leq S(x,z) \quad \text{whenever } y \leq z, \quad (S3)
\]
\[
S(x,0) = x. \quad (S4)
\]

Definition 7 ([18] Definitions 2.9 Pg 26 & 2.13 Pg 28) A t-norm $T$ (t-conorm $S$ resp.) is said to be

- Continuous if it is continuous in both the arguments;
- Idempotent if $T(x,x) = x$ ($S(x,x) = x$) for all $x \in [0,1]$;
- Archimedean if $T$ (S resp.) is such that for every $x, y \in (0,1]$ ($x, y \in [0,1)$ resp.) there is an $n \in \mathbb{N}$ with $x_T^{(n)} < y$ ($x_S^{(n)} > y$);
- Strict if $T$ (S resp.) is continuous and strictly monotone, i.e., $T(x,y) < T(x,z)$ ($S(x,y) < S(x,z)$) whenever $x > 0$ ($x < 1$ resp.) and $y < z$;
- Nilpotent if $T$ (S resp.) is continuous and if each $x \in (0,1)$ is such that $x_T^{(n)} = 0$ ($x_S^{(n)} = 1$) for some $n \in \mathbb{N}$.

Tables 1 lists the basic t-norms and t-conorms along with their properties.

Theorem 1 ([18] Theorem 5.1 Pg 122) $T$ is a continuous Archimedean t-norm iff $T$ has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $f : [0,1] \rightarrow [0, \infty]$ with $f(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that for all $x, y \in [0,1]$ 

\[
T(x,y) = f^{(-1)}(f(x) + f(y)), \quad (9)
\]

where $f^{(-1)}$ is the pseudo-inverse of $f$ and is defined as:

\[
f^{(-1)}(x) = \begin{cases} 
  f^{-1}(x), & \text{if } x \in [0, f(0)] \\
  0, & \text{if } x \in [f(0), \infty]
\end{cases} \quad (10)
\]
Table 1
Examples of t-norms and t-conorms and their properties

<table>
<thead>
<tr>
<th>t-norm $T$</th>
<th>t-conorm $S$</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_M : \min(x, y)$</td>
<td>$S_M : \max(x, y)$</td>
<td>continuous, idempotent</td>
</tr>
<tr>
<td>$T_P : x \cdot y$</td>
<td>$S_P : x + y - x \cdot y$</td>
<td>strict</td>
</tr>
<tr>
<td>$T_{LK} : \max(x + y - 1, 0)$</td>
<td>$S_{LK} : \min(x + y, 1)$</td>
<td>nilpotent</td>
</tr>
<tr>
<td>$T_D : \begin{cases} y, &amp; \text{if } x = 1 \ x, &amp; \text{if } y = 1 \ 0, &amp; \text{otherwise} \end{cases}$</td>
<td>$S_D : \begin{cases} x, &amp; \text{if } y = 0 \ y, &amp; \text{if } x = 0 \ 1, &amp; \text{otherwise} \end{cases}$</td>
<td>Archimedean, not continuous</td>
</tr>
</tbody>
</table>

Note that if $f(0) = \infty$ then $T$ is strict and if $f(0) < \infty$ then $T$ is nilpotent.

**Definition 8 ([18] Definition 3.39 Pg 79)** A multiplicative generator of a t-conorm $S$ is a strictly decreasing function $\phi : [0, 1] \to [0, 1]$, which is left-continuous in $1$ and satisfies $\phi(0) = 1$, such that for all $x, y \in [0, 1]$ we have

$$S(x, y) = \phi^{-1}(\phi(x) \cdot \phi(y))$$

where $\phi^{-1}$ is the pseudo-inverse of $\phi$.

For more details on the pseudo-inverses of monotone functions see, for example, [18], Section 3.1.

### 2.3 Yager’s Class of Implication Operators

**Definition 9 ([19] Definition 1.15 Pg 22)** A function $J$ from $[0, 1]^2$ to $[0, 1]$ is called a fuzzy implication if for all $x, y, z \in [0, 1]$ it has the following properties:

\[ J(x, z) \geq J(y, z) \text{ if } x \leq y, \]  
\[ J(x, y) \leq J(x, z) \text{ if } y \leq z, \]  
\[ J(0, y) = 1, \]  
\[ J(x, 1) = 1, \]  
\[ J(1, 0) = 0. \]

**Definition 10 (Cf. [26])** A fuzzy implication $J$ is said to have

- the neutrality property or is said to be neutral if

\[ J(1, y) = y, \quad y \in [0, 1]; \]
$$\lambda$$-neutrality: $J_f(1, x) = f^{-1}(1 \cdot f(x)) = f^{-1}(f(x)) = x.$

**Exchange Principle:**

$$J_f(x, J_f(y, z)) = f^{-1}(x \cdot f(J_f(y, z))) = f^{-1}(x \cdot f(f^{-1}(y \cdot f(z)))) = f^{-1}(x \cdot y \cdot f(z)) = J_f(y, J_f(x, z)).$$

Table 2 gives a few examples from the above class $J_f$ (see [32] pp. 198 - 200).
3 On the Distributivity of $f$-generated Implications $J_f$ over $t$-norms and $t$-conorms

In this section, we study the distributivity of the $f$-generated implications $J_f$ over $t$-norms and $t$-conorms, by studying the conditions under which $J_f$ implications satisfy equations (5) - (8).

3.1 On the equations (5) and (6)

Theorem 2 (Cf. [7] Theorems 5 & 6) Any neutral fuzzy implication $J$ that is one-to-one in the first variable, when the second variable is in $(0,1)$, reduces

(i) (5) to (1) and satisfies (1);
(ii) (6) to (2) and satisfies (2).

Proposition 14 ([7] Propositions 3 & 6) Let $J$ be a binary operator on $[0,1]$. Then the following are equivalent:

• $J$ is non-increasing in the first variable;
• $J$ satisfies (1), i.e., (5) with $T = T_M$ and $S = S_M$;
• $J$ satisfies (2), i.e., (6) with $T = T_M$ and $S = S_M$.

Lemma 1 Let $J_f$ be an $f$-generated implication. $J_f$ is one-to-one in the first variable, while the second variable lies in $(0,1)$.

PROOF. Let $y \in (0,1)$ be fixed and let $x_1, x_2 \in [0,1]$ such that $J_f(x_1, y) = J_f(x_2, y)$. Now since $f(y) \in (0, \infty)$ we have

$$J_f(x_1, y) = J_f(x_2, y) \implies f^{-1}(x_1 \cdot f(y)) = f^{-1}(x_2 \cdot f(y)) \implies x_1 \cdot f(y) = x_2 \cdot f(y) \implies x_1 = x_2. \square$$

Theorem 3 Let $J_f$ be an $f$-generated implication. Then $J_f$ satisfies (5) if and only if $S = S_M$ and $T = T_M$.

PROOF. ($\implies$) Let $J_f$ satisfy (5). Since $J_f$ is neutral and one-one in the first variable with the second variable in $(0,1)$, by Theorem 2 (i) we have that $S = S_M$ and $T = T_M$.

($\impliedby$) On the other hand, since $J_f$ is a fuzzy implication it has (J1) and thus by Proposition 14 $J$ satisfies (5) with $S = S_M$ and $T = T_M$. $\square$
**Theorem 4** Let \( J_f \) be an \( f \)-generated implication. Then \( J_f \) satisfies (6) if and only if \( S = S_M \) and \( T = T_M \).

**PROOF.** Again by the one-to-oneness of \( J_f \) in the first variable and Theorem 2 (ii). \(\square\)

### 3.2 On the equations (7) and (8)

To discuss the equations (7) and (8) w.r.to \( J_f \) we consider two cases under each of them, viz., when \( f(0) = \infty \) and \( f(0) < \infty \).

First we note that, since \( J_f \) is neutral, i.e., \( J_f(1, y) = y, \forall y \in [0, 1] \), we have that \( T_1 = T_2 = T \) in (7) and \( S_1 = S_2 = S \) in (8). Hence when \( J_f \) satisfies (7) and (8) they reduce to (11) and (12), respectively:

\[
\begin{align*}
J_f(r, T(s,t)) &\equiv T[J_f(r,s), J_f(r,t)] \quad (11) \\
J_f(r, S(s,t)) &\equiv S[J_f(r,s), J_f(r,t)] \quad (12)
\end{align*}
\]

**Proposition 15 ([7] Propositions 9 & 12)** Let \( J \) be a binary operator on \([0,1]\). Then the following are equivalent:

- \( J \) is non-decreasing in the second variable;
- \( J \) satisfies (3), i.e., (11) with \( T = T_M \);
- \( J \) satisfies (4), i.e., (12) with \( S = S_M \).

#### 3.2.1 On the equations (7) and (8) when \( f(0) = \infty \)

In the following result we show that \( J_f \) implications obtained from \( f \)-generators, such that \( f(0) = \infty \), satisfy (11) for \( t \)-norms other than min.

**Theorem 5** Let \( J_f \) be obtained from an \( f \)-generator where \( f(0) = \infty \). Then \( J_f \) satisfies the distributive law (11) if

\[
\begin{align*}
i) & \quad T = T_M, \text{ or} \\
ii) & \quad T \text{ is the } t \text{-norm obtained using } f \text{ as the additive generator, i.e., } T(x, y) = f^{(-1)}(f(x) + f(y)).
\end{align*}
\]

**PROOF.**

i) Since \( J_f \) is an implication operator, and thus has (J2), from Proposition 15 we see that \( J_f \) satisfies (11) when \( T = T_M \).
ii) Let $T$ be the $t$-norm obtained using $f$ as the additive generator, i.e.,

$$T(x, y) = f^{-1}(f(x) + f(y)).$$

Since $f(0) = \infty$ we know that $f^{-1} = f^{-1}$, $f \circ f^{-1} = id$ and we have

$$J_f(r, T(s, t)) = f^{-1}(r \cdot f(T(s, t))) = f^{-1}(r \cdot f \circ f^{-1}(f(s) + f(t))) = f^{-1}(r \cdot (f(s) + f(t))) = f^{-1}(r \cdot f(s) + r \cdot f(t)) = f^{-1}(f \circ f^{-1}(r \cdot f(s)) + f \circ f^{-1}(r \cdot f(t))) = f^{-1}(f(J_f(r, s)) + f(J_f(r, t))) = T[J_f(r, s), J_f(r, t)]. \quad \square$$

In [7] it was shown that when $J$ is an $R$- or an $S$-implication (7) holds if and only if $T_1 = T_2 = T_M$. Along similar lines the same can be proven when $J$ is a $QL$-implication too. As noted earlier, since (7) has an important role to play in rule reduction, from Theorem 5, we note that when $J$ is an $f$-generated implication, there exist other choices for $T$ than $\min$, unlike in the case of $R$, $S$- and $QL$-implications.

The Example 1 below shows that if the $t$-norm $T$ in (11) is such that its generator is different from the $f$-generator used to obtain $J_f$ then (11) may not be satisfied.

**Example 1** Let $f(x) = -\ln x$, then $f(0) = \infty$, $J_f(x, y) = J_Y(x, y) = y^x$, the Yager’s implication [31]. Considering $f$ as an additive generator we get the product $t$-norm $T_P(x, y) = x \cdot y$. Now, let $T$ be the Lukasiewicz $t$-norm $T_{LK}(x, y) = \max(0, x + y - 1)$.

Now, letting $r = s = t = 0.4$, we have $J_Y(r, s) = J_Y(r, t) = 0.4^{0.4} = 0.693$ while $T_{LK}(s, t) = \max(0, 0.4 + 0.4 - 1) = 0$. Hence $J_Y(r, T_{LK}(s, t)) = 0^{0.4} = 0$ while $T_{LK}[J_Y(r, s), J_Y(r, t)] = T_{LK}(0.693, 0.693) = 0.386$, i.e $J_Y(r, T_{LK}(s, t)) \neq T_{LK}[J_Y(r, s), J_Y(r, t)]$, when $r = s = t = 0.4$. \quad \square

Again by (J2) and Proposition 15 we see that $J_f$ satisfies (12) when $S = S_M$. From the Example 2 below it is reasonable to surmise that a result similar to Theorem 5 may not be possible.

**Example 2** Let $f(x) = -\ln x$, then $f(0) = \infty$, $J_f(x, y) = J_Y(x, y) = y^x$, the Yager’s implication [31].

- Considering $f$ as an additive generator we get the product $t$-norm $T_P$ whose dual $t$-conorm with respect to $1 - x$ is the **strict** Algebraic Sum $t$-conorm $S_P(x, y) = x + y - x \cdot y$. 

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Let us define an increasing continuous function $\phi : [0, 1] \to [0, \infty)$ from $f(x) = -\ln x$ as follows: $\phi(x) = \exp\{-f(x)\} = \exp\{\ln x\} = x$. Now, considering $\phi$ as an additive generator of a $t$-conorm, we get the \textit{nilpotent} Lukasiewicz $t$-conorm $S_{\text{LK}}(x, y) = \min(x + y, 1)$.

From the following Table 3 it is clear that (12) does not hold when $J$ is $J_{\text{Y}}$ and $S$ is either the Lukasiewicz $t$-conorm $S_{\text{LK}}$ or the Algebraic Sum $t$-conorm $S_{\text{S}}$ with $r = 0.3, s = t = 0.1$.

<table>
<thead>
<tr>
<th>S</th>
<th>$S(s, t)$</th>
<th>$J(r, s)$</th>
<th>$J(r, t)$</th>
<th>LHS (12)</th>
<th>RHS (12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{\text{LK}}$</td>
<td>0.2</td>
<td>0.50011</td>
<td>0.6170</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$S_{\text{S}}$</td>
<td>0.19</td>
<td>0.50011</td>
<td>0.607612</td>
<td>0.751186</td>
<td></td>
</tr>
</tbody>
</table>

Table 3
Examples of $t$-conorms (both nilpotent and strict) which do not satisfy (12) with $J_{\text{Y}}$.

Hence, determining $t$-conorms $S$ such that (12) holds when $J = J_f$ is worthy of exploration in view of the importance of (12) in the field of rule reduction.

3.2.2 \textit{On the equations (7) and (8) when $f(0) < \infty$}

Again by the neutrality of $J_f$ it is enough to consider (11) and (12). Towards investigating (11) when $f(0) < \infty$ we need the following modified versions of Theorems 7 and 8 in [7] and the lemma given below.

\textbf{Theorem 6 (Cf. [7] Theorems 7,8)} \textit{Any neutral fuzzy implication $J$ such that $J(., 0)$ is onto reduces}

• (7) to (3) and satisfies (3), i.e., $J$ satisfies (7) only if $T_1 = T_2 = T_M$;

• (8) to (4) and satisfies (4), i.e., $J$ satisfies (8) only if $S_1 = S_2 = S_M$.

\textbf{Lemma 2} \textit{Let $f(0) < \infty$. Then $J_f(., 0)$ is onto.}

\textbf{PROOF.} To show that $J_f(., 0)$ is onto we need to show that for every $y \in [0, 1]$ there exists $x \in [0, 1]$ such that $J_f(x, 0) = y$. Let $y \in [0, 1]$ be arbitrary. Then

$$J_f(x, 0) = y \implies f^{-1}(x \cdot f(0)) = y$$

$$\implies x \cdot f(0) = f(y)$$

$$\implies x = \frac{f(y)}{f(0)}.$$
Now $1 \geq y \geq 0 \implies f(1) \leq f(y) \leq f(0) < \infty \implies 1 \geq x \geq 0$. Thus for any $y \in [0, 1]$ there exists $x = \frac{f(y)}{f(0)}$ such that $J_f(x, 0) = y$ and so $J_f(., 0)$ is onto. □

From Theorem 6 and Lemma 2 we have the following:

**Theorem 7** Let $J_f$ be an $f$-generated implication with $f(0) < \infty$. Then $J_f$ satisfies (7) if and only if $T_1 = T_2 = T_M$.

**Proof.** $(\Rightarrow):$ Let $J_f$ satisfy (7). Then $T_1 = T_2$ by the neutrality of $J_f$. Then by the ontoness of $J_f(., 0)$ and Theorem 6 we have that $T_1 = T_2 = T_M$. $(\Leftarrow)$ On the other hand, if $T_1 = T_2 = T_M$ in (7), since $J_f$ is a fuzzy implication and so has (J2) we have by Proposition 15 that $J_f$ satisfies (11). □

**Theorem 8** Let $J_f$ be an $f$-generated implication with $f(0) < \infty$. Then $J_f$ satisfies (8) if and only if $S_1 = S_2 = S_M$.

A summary of the above results is given in Table 5 in Section 7.

4 On the Law of Importation $J(T(x, y), z) = J(x, J(y, z))$

In this section we consider the following general form of law of importation,

$$J(T(x, y), z) = J(x, J(y, z)), \quad x, y, z \in [0, 1], \quad \text{(LI)}$$

where $T$ is a $t$-norm and $J$ is a fuzzy implication.

**Theorem 9** $J_f$ satisfies the law of importation (LI) if and only if $T = T_P$, the product $t$-norm.

**Proof.** $(\Leftarrow):$ Let $T$ be the product $t$-norm. Then,

$$\text{RHS (LI)} = J_f(x, J_f(y, z))$$

$$= f^{-1}[x \cdot f(J_f(y, z))]$$

$$= f^{-1}[x \cdot f \circ f^{-1}(y \cdot f(z))]$$

$$= f^{-1}[x \cdot (y \cdot f(z))]$$

$$= J_f(x \cdot y, z) = \text{LHS (LI)}$$
(⇒: ) Let $J_f$ obey the law of importation (LI). Let $z \in (0,1)$ then $f(z) \in (0,\infty)$. Now for any $x,y \in [0,1]$, we have

\[
J_f(T(x,y),z) = J_f(x,J_f(y,z))
\]
\[
\implies f^{-1}[T(x,y) \cdot f(z)] = f^{-1}[x \cdot f \circ f^{-1}(y \cdot f(z))]
\]
\[
\implies f \circ f^{-1}[T(x,y) \cdot f(z)] = f \circ f^{-1}[x \cdot y \cdot f(z)]
\]
\[
i.e., \quad [T(x,y) \cdot f(z)] = x \cdot y \cdot f(z)
\]
\[
i.e., \quad T(x,y) = x \cdot y \quad \Box
\]

By the commutativity of a $t$-norm $T$ it is obvious that if a fuzzy implication $J_f$ has (LI) then it has (EP).

5 Contrapositive Symmetry of $f$-generated Implications

In the framework of classical two-valued logic, contrapositivity of a binary implication operator is a tautology, i.e., $\alpha \implies \beta \equiv \neg \beta \implies \neg \alpha$. In fuzzy logic, contrapositive symmetry of a fuzzy implication $J_f$ with respect to strong negation $N$ - CP$(N)$ - plays an important role in the applications of fuzzy implications, viz., Approximate Reasoning, Deductive Systems, Decision Support Systems, Formal Methods of Proof, etc (see also [20], [21]).

Definition 16 A fuzzy implication $J_f$ is said to have contrapositive symmetry with respect to a strong negation $N$, denoted CP$(N)$, if

\[
J_f(x,y) = J_f(N(y),N(x)), \quad x, y \in [0,1]. \quad (CP)
\]

Definition 17 Let $J_f$ be any fuzzy implication. The natural negation of $J_f$, denoted by $N_{J_f}$, is given by $J_f(x,0) = N_{J_f}(x), \forall x \in [0,1]$. Clearly $N_{J_f}(0) = 1$ and $N_{J_f}(1) = 0$.

Usually, the contrapositive symmetry of a fuzzy implication $J_f$ is studied with respect to its natural negation, denoted CP$(N_{J_f})$, provided $N_{J_f}$ is strong. Also in the setting of fuzzy logic, contrapositive symmetry is the characterising property of strong fuzzy implications obtained from a $t$-conorm and a strong negation, which are defined as follows:

Definition 18 ([19] Definition 1.16, Pg 24) An $S$-implication $J_{S,N}$ is obtained from a $t$-conorm $S$ and a strong negation $N$ as follows:

\[
J_{S,N}(x,y) = S(N(x),y), \quad x, y \in [0,1]. \quad (13)
\]
The following theorem characterises $S$-implications:

**Theorem 10** ([19] Theorem 1.13 Pg 24) *A fuzzy implication $J$ is an $S$-implication for an appropriate $t$-conorm $S_J$ and a strong negation $N$ if and only if $J$ has $CP(N)$, the exchange property (EP) and is neutral (NP), where $S_J(x, y) = J(N(x), y)$.*

In general, the natural negation $N_J$ of $J$ need not be strong. Even if $N_J$ is strong $J$ still may not have $CP(N_J)$. For example, consider the fuzzy implication $J_K(x, y) = [1 - x + x \cdot y^2]^{\frac{1}{2}}$. The natural negation of $J_K$ is $N_{J_K}(x) = J_K(x, 0) = [1 - x]^\frac{3}{2}$ which is not a strong negation and hence $J_K$ does not have $CP(N_{J_K})$. On the other hand, though the natural negation of the implication $J_{GG}(x, y) = \min\left\{1, \frac{1 - x}{1 - y}\right\}$, $N_{J_{GG}}(x) = 1 - x$, is a strong negation $J_{GG}$ does not have $CP(1 - x)$.

In this section, we analyse the nature of the natural negations of $J_f$, $N_{J_f}$, under different boundary conditions on the underlying generator $f$ and give a sufficient condition under which $N_{J_f}$ is strong and $J_f$ has $CP(N_{J_f})$.

### 5.1 The family of $J_f$ implications and $CP(N)$

The natural negation of $J_f$, given by $N_{J_f}(x) = J_f(x, 0) = f^{(-1)}(x \cdot f(0))$, is quite evidently a negation. To discuss the nature of $N_{J_f}$ we consider the following two cases:

**Case I:** $f(0) < \infty$
If $f(0) < \infty$ then $N_{J_f}(x) = J_f(x, 0) = f^{(-1)}(x \cdot f(0)), \forall x \in (0, 1)$. Since $f$ and thus $f^{(-1)} = f^{-1}$ are strictly decreasing continuous functions, we have that $N_{J_f}$ is a strict negation. For $N_{J_f}$ to be strong, we need that $N_{J_f}(N_{J_f}(x)) = x, \forall x \in [0, 1]$, which is not the case always (see Example 3 below).

**Example 3** Consider the $f$-generated implication $J_{f_f}(x, y) = 1 - x^{\frac{1}{\lambda}}(1 - y)$ obtained from the Yager’s class of $f$-generators $f(x) = (1 - x)^\lambda$ with $f(0) = 1 < \infty$ (see Table 2). Now, if $\lambda = 0.5$, i.e. $\frac{1}{\lambda} = 2$, then $N_{J_{f_f}}(x) = J_{f_f}(x, 0) = 1 - x^{\frac{1}{2}}$ is a strict negation. That it is not strong can be seen by letting $x = 0.5$ in which case $N_{J_{f_f}}(N_{J_{f_f}}(x)) = 1 - [1 - x^2]^2 = 1 - (1 - 0.25)^2 = 0.4375 \neq 0.5 = x$. On the other hand, if $\lambda = 1$, then $N_{J_{f_f}}(x) = J_{f_f}(x, 0) = 1 - x$, which is a strong negation.

**Case II:** $f(0) = \infty$
In the case when $f(0) = \infty$ it is easy to see that $N_{J_f}$ is not even strict, since $\forall x \in (0, 1]$, we have $J_f(x, 0) = N_{J_f}(x) = f^{-1}(x \cdot f(0)) = f^{-1}(x \cdot \infty) = \infty.$
 Quite obviously, it is not strong either. In fact, $N_{J_f}$ is a vanishing but a non-filling negation.

Thus, as per Definition 16, $J_f$ does not have contrapositive symmetry with respect to its natural negation. The following result gives a sufficient condition under which this happens.

**Theorem 11** Let the $f$-generator be such that $f(0) = 1$ and $f^{-1} = f$. Then the natural negation of $J_f$, $N_{J_f}$, is a strong negation and $J_f$ has $CP(N_{J_f})$.

**PROOF.** Let the $f$-generator be such that $f(0) = 1$ and $f^{-1} = f$. Then the pseudo-inverse of $f$ from (10) is given by:

$$f^{-1}(x) = \begin{cases} f^{-1}(x), & \text{if } x \in [0 = f(1), f(0) = 1] \\ 0, & \text{if } x \in [1, \infty] \end{cases} \quad (14)$$

Now, the natural negation of $J_f$ is given by $J_f(x, 0) = N_{J_f}(x) = f^{-1}(x \cdot f(0)) = f^{-1}(x \cdot 1) = f^{-1}(x)$ for any $x \in [0, 1]$. Since $f$ is strictly decreasing so is $N_{J_f}$. Also, $N_{J_f}(N_{J_f}(x)) = f^{-1} \circ f^{-1}(x) = f \circ f^{-1}(x) = x$, since $f^{-1} = f$. Hence $N_{J_f}$ is a strong negation.

From the following string of equalities we note that $J_f$ has $CP(N_{J_f})$.

$$J_f(N_{J_f}(y), N_{J_f}(x)) = f^{-1}[N_{J_f}(y) \cdot f(N_{J_f}(x))]$$
$$= f^{-1}[f^{-1}(y) \cdot f \circ f^{-1}(x)]$$
$$= f^{-1}[f^{-1}(y) \cdot x]$$
$$= f^{-1}[f(y) \cdot x] = J_f(x, y). \quad \Box$$

**Corollary 19** If $N$ is any strong negation then $J_N$ has $CP(N)$.

Any strong $N$ can be thought of as a decreasing bijection $\phi$ on the unit interval $[0, 1]$ with $\phi = \phi^{-1}$ and hence is a multiplicative generator of a strict $t$-conorm. Also note that for a strong $N$, $J_N$ has $CP(N)$, (EP) and (NP). Thus by Theorem 10 $J_N$ can be represented as an $S$-implication. Now, the $t$-conorm $S_{J_N}$ obtained from $J_N$ according to Theorem 10 is $S_{J_N}(x, y) = J_N(N(x), y) = N[N(x) \cdot N(y)] = \phi^{-1}[\phi(x) \cdot \phi(y)]$, which by Definition 8 is nothing but the $t$-conorm obtained using $\phi$ as the multiplicative generator. Hence, in the case $f(0) = 1$ and $f^{-1} = f$ we do not obtain any new fuzzy implications but only $S$-implications from a strict $t$-conorm $S$ and the strong negation $N_S$ which is also the multiplicative generator $\phi$ of $S$.  

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5.2 $J_f$ and Contrapositivisation

From the discussions in Section 5.1 we observe that the natural negations of $J_f$, in general, are not strong and thus, as per Definition 16, do not have contrapositive symmetry with respect to their natural negation. In fact, the natural negation $N_{J_f}$ of $J_f$ is, in general, only a strict negation if $f(0) < \infty$, while it is a vanishing and a non-filling negation if $f(0) = \infty$.

Towards imparting contrapositive symmetry to such fuzzy implications $J$ with respect to a strong negation $N$ the following two contrapositivisation techniques - upper and lower contrapositivisation - have been proposed by Bandler and Kohout in [10], whose definitions we give below.

**Definition 20** Let $J$ be any fuzzy implication and $N$ a strong negation. The upper and lower contrapositivisations of $J$ with respect to $N$, denoted herein as $\mathcal{U}^N$ and $\mathcal{L}^N$, respectively, are defined as follows:

$$x \mathcal{U}^N y = \max\{J(x, y), J(N(y), N(x))\}$$ (15)

$$x \mathcal{L}^N y = \min\{J(x, y), J(N(y), N(x))\}$$ (16)

for any $x, y \in [0, 1]$.

As can be seen, $\mathcal{U}^N$ and $\mathcal{L}^N$ are both fuzzy implications, as per Definition 9, and always have the contrapositive symmetry with respect to the strong negation $N$ employed in their definitions.

**Definition 21** Let $J$ be a fuzzy implication and $N$ a strong negation. A contrapositivisation technique $\star^N : \longrightarrow$ is said to be $N$-Compatible if the contrapositivisation of $J$ with respect to $N$, denoted as $J^*(x, y) = x \star^N y$ for all $x, y \in [0, 1]$, is such that the natural negation of $J^*$, given by $N_{J^*}(.) = J^*(., 0)$, is equal to the strong negation $N$ employed.

Definition 21, in essence, is asking for $J^*$ to have CP($N_{J^*}$).

The following result has been proven in [9]:

**Proposition 22** Let $J$ be a neutral fuzzy implication with natural negation $J(x, 0) = N_J(x)$ and $N$ a strong negation.

i) The upper contrapositivisation of $J$ with respect to $N$ is $N$-Compatible if and only if $N(x) \geq N_J(x)$, for all $x \in [0, 1]$.

ii) The lower contrapositivisation of $J$ with respect to $N$ is $N$-Compatible if and only if $N(x) \leq N_J(x)$, for all $x \in [0, 1]$.
PROOF. We give the proof of part i) as that of part ii) is similar.

i) Let \( x \in [0,1] \). By definition of \( \mathcal{U}:N \) we have

\[
\begin{align*}
\mathcal{U}:N & \text{ is N-Compatible iff } N(x) = x \mathcal{U}:N 0 \\
& \text{iff } N(x) = \max\{J(x, 0), J(1, N(x))\} \\
& \text{iff } N(x) = \max(N_J(x), N(x)) \\
& \text{iff } N(x) \geq N_J(x), \text{for all } x \in [0, 1]. \quad \Box
\end{align*}
\]

If the upper contrapositivisation of \( J \) with respect to a strong \( N \) is N-Compatible, then from Proposition 22 we know \( N \geq N_J \). Since \( N \) is strong \( N(x) = 1 \) if and only if \( x = 0 \) and we have that for all \( x \in (0,1] \), \( 1 \geq N(x) \geq N_J(x) \) and \( N_J \) is a non-filling negation. In other words, if the natural negation of the fuzzy implication \( J \) is a filling negation we cannot find any strong \( N \) with which the upper contrapositivisation of \( J \) becomes N-Compatible. Similarly, if the natural negation of the fuzzy implication \( J \) is a vanishing negation we cannot find any strong \( N \) with which the lower contrapositivisation of \( J \) becomes N-Compatible.

Now, in the case \( f(0) < \infty \), we have that the natural negation of \( J_f \) is at least strict and so both upper and lower contrapositivisation techniques are N-Compatible, with respect to strong negations \( N \), depending on whether \( N \geq N_J \) or \( N \leq N_J \), respectively. On the other hand, when \( f(0) = \infty \), \( N_J \) is a non-filling but a vanishing negation and thus we cannot have any strong negation \( N \leq N_J \). Therefore, only the upper contrapositivisation technique is N-Compatible with respect to strong negations \( N \geq N_J \).

A summary of results in this section is given in Table 6 in Section 7.

In the following section, taking cue from the Yager’s \( f \)-generated implications, we propose a new class of \( h \)-generated implications, denoted \( J_h \), where \( h \) is defined on \([0,1]\) to \([0,1]\), unlike \( f \) which is from \([0,1]\) to \([0,\infty]\) and study its properties. We also show that one can obtain natural negations \( N_{J_h} \) of \( h \)-generated implications that are non-vanishing and hence the lower contrapositivisation technique is N-Compatible with respect to strong negations \( N \leq N_{J_h} \).

6 A New class of Implications: \( h \)-generated Implications - \( J_h \)

Definition 23 \( An h \)-generator is a function \( h : [0,1] \rightarrow [0,1] \), that is strictly decreasing and continuous such that \( h(0) = 1 \). Let \( h\) be its pseudo-inverse
For any \( \text{Neutrality (NP)} \) : \( J_h(1, x) = h^{-1}(1 \cdot h(x)) = x \), since \( h^{-1} \circ h = id \).

Also, \( J_h \) has the following desirable properties:

\[
\begin{array}{|c|c|c|c|} 
\hline
\text{Name} & h(x) & h(1) & J_h(x, y) \\
\hline
\text{Schweizer-Sklar} & 1 - x^p; p \neq 0 & 0 & [1 - x + x \cdot y^p]^{\frac{1}{p}} \\
\hline
\text{Yager's} & (1 - x)^\lambda; \lambda > 0 & 0 & 1 - x^\frac{1}{\lambda}(1 - y) \\
\hline
- & 1 - \frac{x^n}{n}; n \geq 1 & 1 - \frac{1}{n} & \min\{[n - nx + x \cdot y^n]^{\frac{1}{n}}, 1\} \\
\hline
\end{array}
\]

Table 4
Examples of some \( J_h \) implications with their \( h \)-generators.

given by:

\[
h^{-1}(x) = \begin{cases} 
  h^{-1}(x), & \text{if } x \in [h(1), 1] \\
  1, & \text{if } x \in [0, h(1)] 
\end{cases} \quad (17)
\]

Lemma 3 Let the function \( J_h \) from \([0, 1] \times [0, 1]\) to \([0, 1]\) be defined as

\[
J_h(x, y) = \text{def} h^{-1}(x \cdot h(y)), \quad x, y \in [0, 1]. \quad (18)
\]

\( J_h \) is a fuzzy implication and called the \( h \)-generated implication.

PROOF. That \( J_h \) is a fuzzy implication can be seen from the following:

- \( J_h(1, 0) = h^{-1}(1 \cdot h(0)) = h^{-1}(1 \cdot 1) = 0 \).
- \( J_h(0, 1) = h^{-1}(0 \cdot h(1)) = h^{-1}(0) = 1 = J_h(0, 0) \).
- \( J_h(1, 1) = h^{-1}(1 \cdot h(1)) = h^{-1}(h(1)) = 1 \), since \( h^{-1} \circ h = id \).
- For any \( x, x', y \in [0, 1] \) we have \( x \leq x' \implies x \cdot h(y) \leq x' \cdot h(y) \implies h^{-1}(x \cdot h(y)) \geq h^{-1}(x' \cdot h(y)) \implies J_h(x, y) \geq J_h(x', y) \). Thus \( J_h \) is non-increasing in the first variable.
- For any \( x, y, y' \in [0, 1] \) we have \( y \leq y' \implies x \cdot h(y) \geq x \cdot h(y') \implies h^{-1}(x \cdot h(y)) \leq h^{-1}(x \cdot h(y')) \implies J_h(x, y) \leq J_h(x, y') \). Thus \( J_h \) is non-decreasing in the second variable.
- Since \( 0 \leq x \cdot h(1) \leq h(1) \), \( \forall x \in [0, 1] \), we have \( J_h(x, 1) = h^{-1}(x \cdot h(1)) = 1 \), by definition of \( h^{-1} \).
- \( J_h(0, y) = h^{-1}(0 \cdot h(y)) = h^{-1}(0) = 1 \), for all \( y \in [0, 1] \). \( \Box \)

Without explicitly using the pseudo-inverse (18) can be written in the following form:

\[
J_h(x, y) = h^{-1} \left( \max (x \cdot h(y), h(1)) \right), \quad x, y \in [0, 1]. \quad (19)
\]
• **Exchange Principle (EP):** For every \( h \)-generator \( h \) and \( x, y, z \in [0, 1] \) we get

\[
    J_h(x, J_h(y, z)) = h^{(-1)}(x \cdot h(J_h(y, z))) \\
    = h^{-1}\left(\max\left(x \cdot h(h^{-1}(\max(y \cdot h(z), h(1))), h(1))\right)\right) \\
    = h^{-1}(\max(x \cdot \max(y \cdot h(z), h(1)), h(1))) \\
    = h^{-1}(\max(x \cdot y \cdot h(z), x \cdot h(1), h(1))) \\
    = h^{-1}(\max(x \cdot y \cdot h(z), h(1))) ,
\]

since \( x \cdot h(1) \leq h(1) \). Similarly we get that

\[
    J_h(y, J_h(x, z)) = h^{-1}(\max(y \cdot x \cdot h(z), h(1))) .
\]

Thus \( J_h \) satisfies the exchange principle.

Table 4 gives a few examples from the above class \( J_h \).

### 6.1 \( J_h \) and its natural negation \( N_{J_h} \)

The natural negation of \( J_h \), \( N_{J_h}(x) = J_h(x, 0) = h^{(-1)}(x \cdot h(0)) = h^{(-1)}(x) \), for all \( x \in [0, 1] \) is, in general, only a negation. But,

- \( N_{J_h} \) is a strict negation if \( h(1) = 0 \);
- \( N_{J_h} \) is a strong negation iff \( h = h^{-1} \), in which case \( N_{J_h} = h^{(-1)} = h \).

When \( h = h^{-1} \) from Corollary 19 we see that \( J_h \) has CPS(\( N_{J_h} \)). Let \( h \neq h^{-1} \). Then, if \( h(1) = 0 \) we have that the natural negation \( N_{J_h} \) is strict and hence is both a non-vanishing and non-filling negation. When \( h(1) > 0 \) then \( N_{J_h} \) is a non-vanishing but a filling negation.

### 6.2 \( J_h \) and Contrapositivisation

Let \( h \neq h^{-1} \). Then, if \( h(1) = 0 \) we have that the natural negation \( N_{J_h} \) is strict and hence there exist strong negations \( N \) such that both the upper and lower contrapositivisation of \( J_h \) are N-Compatible, depending on whether \( N \geq N_{J_h} \) or \( N \leq N_{J_h} \), respectively. On the other hand, if \( h(1) > 0 \) then \( N_{J_h} \) is a non-vanishing but a filling negation and only the lower contrapositivisation technique is N-Compatible with respect to strong negations \( N \leq N_{J_h} \).

Figure 1 shows plots of the fuzzy implication \( J_{y, \lambda}(x, y) = 1 - x^{\frac{1}{\lambda}} + x^{\frac{1}{\lambda}} \cdot y \) obtained from the Yager’s class of \( h \)-generators for \( \lambda = 0.5 \) or \( \frac{1}{\lambda} = 2 \) (See
(a) Yager’s implication \( J_{Y2}(x, y) = 1 - x^2 + x^2 \cdot y \) with \( \frac{1}{\lambda} = 2 \)

(b) Lower Contrapositivisation of \( J_{Y2} \) with \( N_1(x) = (1 - \sqrt{x})^2 \)

(c) Upper Contrapositivisation of \( J_{Y2} \) with \( N_2(x) = \sqrt{1 - x^2} \)

(d) Plots of negations \( N_1 (-) \), \( N_2 (-) \), \( N_{J_{Y2}} (...) \)

Fig. 1. Fuzzy implication \( J_{Y2}(x, y) = 1 - x^2 + x^2 \cdot y \) with \( \frac{1}{\lambda} = 2 \) whose natural negation is the strict negation \( N_{J_{Y2}}(x) = 1 - x^2 \) with Lower and Upper Contrapositivisations. Table 4) along with its natural negation \( N_{J_{Y2}}(x) = 1 - x^2 \), the lower and upper contrapositivised implications with respect to negations \( N_1(x) = (1 - \sqrt{x})^2 \) and \( N_2(x) = \sqrt{1 - x^2} \), respectively.

For more details on Contrapositivisation and significance of N-Compatibility see [9], [10], [20], [21].

6.3 Relation between \( f \)- and \( h \)-generators

Let \( f \) be an \( f \)-generator. Then let us define an \( \hat{h} : [0, 1] \to [0, 1] \) as follows

\[
\hat{h}(x) =_{df} \exp\{-f(1 - x)\}
\]  

(20)
Then \( \hat{h} \) is a strictly decreasing function on the unit interval \([0, 1]\), such that 
\[ \hat{h}(0) = \exp\{-f(1 - 0)\} = \exp\{-f(1)\} = 1 \] 
since \( f(1) = 0 \). Now, if \( f(0) = \infty \) then \( \hat{h}(1) = 0 \) while if \( f(0) < \infty \) then \( \hat{h}(1) > 0 \). In either case, the \( \hat{h} \) obtained as in (20) can act as an \( h \)-generator.

Similarly, from an \( h \)-generator one can obtain an \( \hat{f} \)-generator as follows:

\[
\hat{f}(x) = \text{def} - \ln h(1 - x) \tag{21}
\]

While an \( f \)-generator can be seen as the additive generator of some continuous Archimedean \( t \)-norm \( T \), an \( h \)-generator can be seen as the multiplicative generator of some continuous Archimedean \( t \)-conorm \( S \). Thus (20) and (21) are how one obtains the multiplicative generator of the \( N \)-dual \( t \)-conorm \( S \) from the additive generator of the \( t \)-norm \( T \) and viceversa (see [18], pp. 80–81), where \( N \) is the classical negation \( N(x) = 1 - x \).

Also note that if the range of the \( f \)-generator is \([0, 1]\), i.e., \( f(0) = 1 \), then \( f \) itself can act as the \( h \)-generator and \( J_h = J_f \) and \( h(1) = 0 \). This equivalence can be readily seen in the case of Yager’s class of generators from both the Tables 2 and 4. On the other hand, we can still obtain the \( h \)-generator from \( f \) as in (20) (see Example 4 below).

**Example 4** Consider the \( f \)-generator given by \( f(x) = 1 - x \). Then \( f(1) = 0 \) and also \( f(0) = 1 \) and thus letting \( h = f \) we get that \( J_f = J_h(x, y) = 1 - x + x \cdot y \). On the other hand, by employing (20) we obtain the following:

\[
\hat{h}(x) = \exp\{-f(1 - x)\} = \exp\{-x\}
\]

\[
\hat{h}^{(-1)}(x) = \begin{cases} 
-\ln x, & \text{if } x \in [\hat{h}(1) = \frac{1}{e}, \hat{h}(0) = 1] \\
1, & \text{if } x \in [0, \hat{h}(1)]
\end{cases}
\]

\[J_h(x, y) = \hat{h}^{(-1)}(x \cdot \exp\{-y\}) = \min\{- \ln(x \cdot \exp\{-y\}), 1\}
\]

\[N_{J_h}(x) = \min\{- \ln x, 1\}
\]

whereas \( J_f(x, y) = 1 - x + x \cdot y \) and \( N_{J_f}(x) = 1 - x \).

When \( h(1) \neq 0 \) or \( f(0) \neq 1 \), one cannot take \( f = h \) and by appyling (21) and (20) one gets different \( f \) and \( \hat{h} \), respectively (see Example 5).

**Example 5** Consider the \( h \)-generator \( h(x) = 1 - \frac{x^2}{2} \). Then clearly \( h(1) = 0.5 \neq 0 \) and thus is not suitable to be employed as an \( f \)-generator directly. Also \( J_h(x, y) = \min\{1, \sqrt{2 - 2 \cdot x + x \cdot y^2}\} \) as can be seen from Table 4.

On the other hand, using the transformation (21) we have the following:
\[ f(x) = -\ln h(1-x) = -\ln \left[ 1 - \frac{x^2 + 2x}{2} \right] = -\ln \frac{1}{2} \left[ 1 - x^2 + 2x \right] \]

\[ \hat{f}(1) = 0 \quad \hat{f}(0) = \ln 2 \]

\[
\hat{f}^{-1}(x) = \begin{cases} 
1 - \left[ 2 \left(1 - e^{-x} \right) \right]^\frac{1}{2}, & x \in [0, \ln 2] \\
0, & x \in [\ln 2, \infty]
\end{cases} = \max \left\{ 0, 1 - \left[ 2 \left(1 - e^{-x} \right) \right]^\frac{1}{2} \right\}
\]

\[ J_f(x, y) = \max \left\{ 0, 1 - \left[ 2 - 2 \left[\frac{1}{2} \left(1 - y^2 + 2y \right) \right] x \right]^\frac{1}{2} \right\} \]

\[ N_J_f(x) = \max \left\{ 0, 1 - \left[ 2 - 2 \left[\frac{1}{2} \right]^x \right]^\frac{1}{2} \right\} \]

Similarly, the \( f \)-generator \( f(x) = -\ln x \) is such that \( f(1) = 0 \) and \( f(0) \neq 1 \) and thus \( h \neq f \) directly. But by using the transformation (20) we have \( \hat{h}(x) = 1 - x \) and \( J_{f}(x, y) = 1 - x + x \cdot y \) while \( J_f(x, y) = J_{Y}(x, y) = y^x \).

A more detailed study of \( f \)- and \( h \)-generated implications has been carried out in [3], [4].

7 Concluding Remarks

In this work we have studied the newly proposed Yager’s class of \( f \)-generated fuzzy implications with respect to three classical tautologies, viz., distributivity over \( t \)-norms and \( t \)-conorms, law of importation and contrapositive symmetry. The results of the above investigation are given in Tables 5 and 6 for ready reference.

<table>
<thead>
<tr>
<th>J_f</th>
<th>satisfies (5)</th>
<th>satisfies (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(0) = \infty )</td>
<td>( S = S_M, \quad T = T_M )</td>
<td>( S = S_M, \quad T = T_M )</td>
</tr>
<tr>
<td>( f(0) &lt; \infty )</td>
<td>( S = S_M, \quad T = T_M )</td>
<td>( S = S_M, \quad T = T_M )</td>
</tr>
</tbody>
</table>

Table 5
Summary of results in Section 3

We have also suggested some sufficient conditions under which \( J_f \) implications possess contrapositive symmetry with respect to their natural negation. Since the natural negations of \( J_f \), in general, are not strong we resorted to the
well-established contrapositivisation techniques, viz., upper and lower contrapositivisation [10]. We have shown that, in general, only the upper contrapositivisation is N- Compatible with \( J_f \) and hence we have proposed a new class of fuzzy implications called \( h \)-generated implications, denoted \( J_h \), along the lines of \( J_f \), for which class the lower contrapositivisation is N- Compatible.

In this work both necessary and sufficient conditions have been proposed for \( J_f \) to satisfy the considered tautologies (except in the case of (7), (8) and \( \text{CP}(N) \) when \( f(0) = \infty \), where it is only a sufficient condition). Thus determining the necessary conditions so that \( J_f \) satisfies these tautologies when \( f(0) = \infty \) is likely to be both interesting and important.

Yager in [32] has done an extensive analysis of the impact of this new class of implications in Approximate Reasoning by introducing concepts like strictness of implications and sharpness of inference, among others. For more recent works on the role of fuzzy logic operators in Computing with words see [28],[33]. This work can be seen as a continuation of the above study on the classical tautologies satisfied by Yager’s \( f \)-generated implications that have an influence in Approximate Reasoning.

### References


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