T-subnorms with strong associated negation: Some Properties

Balasubramaniam Jayaram
Department of Mathematics,
Indian Institute of Technology Hyderabad,
Yeddumallaram - 502 205, INDIA
jbala@iith.ac.in

Summary

In this work we investigate t-subnorms \( M \) that have strong associated negation. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeaness, conditional cancellativity, left-continuity, nilpotent elements, etc. In particular, we show that under this setting many of these properties are equivalent.

Keywords: T-norms, t-subnorms, Archimedeaness, conditional cancellativity, left-continuity, residual implications.

1 Introduction

The theory of triangular norms and triangular subnorms have been well studied and their applications well-established. Many algebraic and analytical properties of these operations, viz., Archimedeaness, conditional cancellativity, left-continuity, etc., have been studied and their inter-relationships shown (see for instance, Klement et al. [4]).

Yet another way of categorizing t-subnorms is as follows: Given a t-subnorm \( M \), one can obtain its associated negation \( n_M \) (see Definitions 2.2 and 2.4 below). Note that \( n_M \) is usually not a fuzzy negation, i.e., \( n_M(1) \geq 0 \). However, we can broadly consider two sub-classes of t-subnorms based on whether their associated negation \( n_M \) is strong or not.

In this work, we study the class of t-subnorms whose associated negation \( n_M \) is strong. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate some particular classes of these and study the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeaness, conditional cancellativity, left-continuity, etc. In particular, we show that under this setting many of these properties are equivalent.

2 Preliminaries

Definition 2.1. A fuzzy negation is a function \( N: [0, 1] \to [0, 1] \) that is non-increasing and such that \( N(1) = 0 \) and \( N(0) = 1 \). Further, it is said to be strong or involutive, if \( N \circ N = \text{id}_{[0, 1]} \).

Definition 2.2. A t-subnorm is a function \( M: [0, 1]^2 \to [0, 1] \) such that it is monotonic non-decreasing, associative, commutative and \( M(x, y) \leq \min(x, y) \) for all \( x, y \in [0, 1] \), i.e., 1 need not be the neutral element.

Definition 2.3. Let \( M \) be a t-subnorm.

(i) If 1 is the neutral element of \( M \), then it becomes a t-norm. We denote a t-norm by \( T \) in the sequel.

(ii) \( M \) is said to satisfy the Conditional Cancellation Law if, for any \( x, y, z \in (0, 1) \),

\[
M(x, y) = M(x, z) > 0 \text{ implies } y = z. \quad (CCL)
\]

Alternately, (CCL) implies that on the positive domain of \( M \), i.e., on the set \( \{(x, y) \in (0, 1)^2 \mid M(x, y) > 0\} \), \( M \) is strictly increasing.

(iii) \( M \) is said to be Archimedean, if for all \( x, y \in (0, 1) \) there exists an \( n \in \mathbb{N} \) such that \( x^{[n]}_M < y \);

(iv) An element \( x \in (0, 1) \) is a nilpotent element of \( M \) if there exists an \( n \in \mathbb{N} \) such that \( x^{[n]}_M = 0 \);

(v) A t-norm \( T \) is said to be nilpotent, if it is continuous and if each \( x \in (0, 1) \) is a nilpotent element of \( T \).

Definition 2.4. Let \( M \) be any t-subnorm and \( x, y \in [0, 1] \).
• The residual implication $I_M$ of $M$ is given by
  
  \[ I(x, y) = \sup \{ t \in [0, 1] \mid M(x, t) \leq y \} \]  
  \[ (1) \]

• The associated negation $n_M$ of $M$ is given by
  
  \[ n_M(x) = \sup \{ t \in [0, 1] \mid M(x, t) = 0 \} \]  
  \[ (2) \]

Clearly, $n_M$ is a non-increasing function. Note that though $n_M(0) = 1$, it need not be a fuzzy negation, since $n_M(1)$ can be greater than 0. Hence, only in the case $n_M$ is a fuzzy negation we call $n_M$ the natural negation of $M$ in this work. However, many results hold even if $n_M(1) > 0$ and hence to preserve this generality in such situations we term $n_M$ as the associated negation.

For instance, the following result is true even when $n_M(1) > 0$.

**Lemma 2.5** (cf. [1], Proposition 2.3.4). Let $M$ be any $t$-subnorm and $n_M$ its associated negation. Then we have the following:

(i) $M(x, y) = 0 \implies y \leq n_M(x)$.

(ii) $y < n_M(x) \implies M(x, y) = 0$.

(iii) If $M$ is left-continuous then $y = n_M(x) \implies M(x, y) = 0$, i.e., the reverse implication of (i) also holds.

**Lemma 2.6.** Let $M$ be any $t$-subnorm with $n_M$ being a natural negation with $e$ as its fixed point, i.e., $n_M(e) = e$. Then

(i) Every $x \in (0, e)$ is a nilpotent element; in fact, $x^{[2]}_M = 0$ for all $x \in (0, e)$.

(ii) In addition, if $M$ is either conditionally cancellative or left-continuous, then $e$ is also a nilpotent element.

**Proof.** (i) By definition,

\[ n_M(e) = \sup \{ t \in [0, 1] \mid M(e, t) = 0 \} = e, \]

implies that $M(e, e^-) = 0$, from whence we get $M(x, x) \leq M(e, e^-) = 0$ for all $x \in [0, e)$. In other words, $x^{[2]}_M = 0$ for all $x \in (0, e)$.

(ii) If $M$ is conditionally cancellative, then $M(e, e) = x < e$ and from (ii) above we have $M(x, x) = 0$.

Now,

\[ e^{[4]}_M = M(M(e, e), M(e, e)) = M(x, x) = 0. \]

If $M$ is left-continuous, then $n_M(e) = \max \{ t \in [0, 1] \mid M(e, t) = 0 \} = e$, i.e., $e \in \{ t \in [0, 1] \mid M(e, t) = 0 \}$ and hence $M(e, e) = 0$, i.e., $e$ is also a nilpotent element.

**Remark 2.7.** (i) In the case $n_M$ is a strong natural negation we can show that if $M$ is conditionally cancellative then every $x \in (0, 1)$ is also a nilpotent element, see Remark 5.8(ii).

(ii) Note that without any further assumptions, the set of nilpotent elements need not be the whole of $(0, 1)$. For instance, for the nilpotent minimum $t$-norm

\[ T_{nM}(x, y) = \begin{cases} \text{0, if } x + y \leq 1, \\ \text{min}(x, y), \text{ otherwise}, \end{cases}, \quad x, y \in [0, 1], \]

which is left-continuous but not conditionally cancellative, its set of nilpotent elements is $(0, 0.5)$, while its set of zero divisors is $(0, 1)$.

However, Theorem 6.1 gives an equivalence condition for the whole of $(0, 1)$ to be the set of nilpotent elements under a suitable condition on $n_M$.

3 T-subnorms with strong associated negation = $T$-norms

There are works showing that some classes of $t$-subnorms $M$ whose associated negations $n_M$ are involutive do become $t$-norms. Jenei [3] showed it for the class of left-continuous $M$, while Jayaram [2] did the same for conditionally cancellative $M$. The main result of this section shows that the above results are true in general, i.e., any $t$-subnorm with a strong natural negation is a $t$-norm.

The following result was firstly proven by Jenei in [3]. However, we give a very simple proof of this result without resorting to the rotation-invariance property.

**Theorem 3.1** (Jenei, [3], Theorem 3). If $M$ is a left-continuous $t$-subnorm with $n_M$ being strong, then $M$ is a $t$-norm.

**Proof.** Firstly, note that if $M$ is a left-continuous $t$-subnorm, then its residual satisfies the exchange principle, i.e.,

\[ I_M(x, I_M(y, z)) = I_M(y, I_M(x, z)). \]

It follows from the fact that the neutral element of $M$ does not play any role in the proof, see, for instance the proof given for Theorem 2.5.7 in [1].

If $n_M$ is strong, then for every $y \in [0, 1]$ there exists $y^\prime \in [0, 1]$ such that $n_M(y) = y^\prime$. Now,

\[ I_M(1, y^\prime) = I_M(1, I_M(y, 0)) = I_M(y, I_M(1, 0)) = I_M(y, 0) = y^\prime. \]

Thus, for all $y^\prime \in [0, 1]$,

\[ I_M(1, y^\prime) = \max \{ t \mid M(1, t) \leq y^\prime \} = y^\prime \implies M(1, y^\prime) = y^\prime. \]
Theorem 3.2 (Jayaram [2], Theorem 4). Let $M$ be any conditionally cancellative $t$-subnorm. If $n_M$ is a strong natural negation then $M$ is a $t$-norm.

Now, we prove the main result of this section which shows that the above results are true in general.

Theorem 3.3. Let $M$ be any $t$-subnorm with $n_M$ being a strong natural negation. $M$ is a $t$-norm.

Proof. Note, firstly, that since $n_M(x) = \sup\{t \in [0,1] | M(x,t) = 0\}$, is a strong negation, we have that $n_M(z) = 1 \iff z = 0$ and $n_M(z) = 0 \iff z = 1$. Equivalently, $M(1,z) = 0 \iff z = 0$.

On the contrary, let us assume that $M(1,x) = x' \leq x$ for some $x \in (0,1]$. Since $n_M$ is strong, the following are true:

(i) $n_M(x') > n_M(x)$

(ii) if $p > n_M(x)$ then $M(x,p) > 0$,

(iii) there exists a $y \in (0,1)$ such that $n_M(x') > y > n_M(x)$ and $M(y,x) = q > 0$ while $M(y,x') = 0$.

Now, by associativity we have

\[
\begin{align*}
M(y,M(x,1)) &= M(y,x') = 0 \\
M(M(y,x),1) &= M(q,1) = 0,
\end{align*}
\]

a contradiction. Thus $M(1,x) = x$ for all $x \in [0,1]$ and hence we have the result. \hfill \Box

In the following sections, we deal with $t$-subnorms whose associated negations are strong, or equivalently $t$-norms whose associated negations are strong. We discuss the inter-relationships between the different algebraic and analytical properties for this subclass of $t$-norms; in particular, Archimedeaness, Conditional Cancellativity, (Left-)continuity and Nilpotence that are relevant to our context. We begin with listing out some established results and go on to present some new ones.

4 Continuity and Nilpotence

Let $T$ be a $t$-norm and $n_T$ a strong negation. The following result, whose proof is straight-forward, shows the equivalence between continuity and nilpotence:

Theorem 4.1 (Klement et al. [4]). Let $T$ be a $t$-norm with $n_T$ being strong. Then the following are equivalent:

(i) $T$ is continuous.

(ii) $T$ is a nilpotent $t$-norm.

Further, we know that every nilpotent $t$-norm is both Archimedean and Conditionally cancellative, since every nilpotent $t$-norm is isomorphic to the Łukasiewicz $t$-norm and Archimedeaness and Conditionally cancellativity are preserved under isomorphism, see [4], Examples 2.14(iv) and 2.15(v). Trivially, every nilpotent $t$-norm is also left-continuous.

5 Conditional Cancellativity, Left Continuity and Nilpotence

Recently, in Jayaram [2], the following problem of U.Höhle, given in Klement et al. [5] has been solved. Further it was shown that it characterizes the set of all conditionally cancellative $t$-subnorms.

(U.Höhle, [5], Problem 11) Characterize all left-continuous $t$-norms $T$ which satisfy

\[ I_T(x,T(x,y)) = \max(n_T(x),y), \quad x, y \in [0,1]. \tag{3} \]

where $I_T, n_T$ are as given in (1) and (2) with $M = T$.

Theorem 5.1 (Jayaram [2], Theorem 1). Let $M$ be any $t$-subnorm, not necessarily left-continuous. Then the following are equivalent:

(i) The adjoint pair $(I, M)$ satisfies (3).

(ii) $M$ is a Conditionally Cancellative $t$-subnorm.

Remark 5.2. The following statements follow from Theorem 5.1 with $M = T$, a $t$-norm:

(i) If a (right) continuous $T$ satisfies (3) along with its residual then $T$ is necessarily Archimedean, see [4], Proposition 2.15(ii).

(ii) However, if a left-continuous $T$ satisfies (3) along with its residual then $T$ need not be Archimedean and hence not continuous. An example is Hajek’s $t$-norm or the following $t$-norm $T_{OY}$ of Ouyang et al. [7], Example 3.4, which is a (CCL) $t$-norm (and hence a $t$-subnorm too) that is left-continuous but not continuous at $(0.5,0.5)$ and hence is not Archimedean (see Figure 1(a)):

\[
T_{OY}(x,y) = \begin{cases} 
2(x - 0.5)(y - 0.5) + 0.5, & \text{if } (x, y) \in (0.5, 1)^2 \\
2y(x - 0.5), & \text{if } (x, y) \in (0.5, 1) \times [0, 0.5) \\
2x(y - 0.5), & \text{if } (x, y) \in [0, 0.5] \times (0.5, 1] \\
0, & \text{otherwise}
\end{cases}
\]

Theorem 5.3 (Jenei, [3], Theorem 2). Let $T$ be a left-continuous $t$-norm with $n_T$ being strong. Then the following are equivalent:

(i) $T$ is a left-continuous $t$-norm.

(ii) $T$ is Archimedean.

(iii) $T$ is Conditionally Cancellative.

(iv) $T$ is nilpotent.

(v) $T$ is strong.
(i) \( T \) is a conditionally cancellative t-norm.

(ii) \( T \) is a nilpotent t-norm.

In fact, for a conditionally cancellative t-subnorm \( M \) we can give an equivalent condition for it to be left-continuous.

**Theorem 5.4.** Let \( M \) be a (CCL) t-subnorm. Then the following are equivalent:

(i) \( M(x, n_M(x)) = 0, \quad x \in [0, 1] \).

(ii) \( M \) is left-continuous.

**Proof.** (i) \( \Rightarrow \) (ii): Let \( M(x, n_M(x)) = 0 \), for all \( x \in [0, 1] \). On the contrary, let us assume that \( M \) is non-left-continuous. Then there exist \( x_0 \in (0, 1] \), \( y_0 \in (0, 1) \) and an increasing sequence \( (x_n)_{n \in \mathbb{N}} \) where \( x_n \in [0, 1) \), such that \( \lim_{n \to \infty} x_n = x_0 \), but

\[
\lim_{n \to \infty} M(x_n, y_0) = M(x_0, y_0) = z' < z_0 = M(x_0, y_0).
\]

Observe that

\[
I_M(y_0, z') = \sup \{ t \in [0, 1] \mid M(y_0, t) \leq z' \} = x_0,
\]

since from the monotonicity of \( M \) we have \( M(y_0, x_n) \leq z' \) for every \( n \in \mathbb{N} \) and \( M(y_0, x_0) = z_0 > z' \). Since \( M \) is (CCL), we have

\[
I_M(y_0, z') = I_M(y_0, M(y_0, x_0^-)) = \max(n(y_0), x_0^-).
\]

Now, we have two cases. On the one hand, if \( I_M(y_0, z') = x_0 < x_0 \), then it is a contradiction to (4). On the other hand, if \( I_M(y_0, z') = n(y_0) \), then this implies that \( n(y_0) = x_0 \) from (4) and hence

\[
M(x_0, y_0) = M(y_0, n(y_0)) = z_0 = 0,
\]

by the hypothesis and hence there does not exist any \( z' < z_0 \) and hence \( M \) is left-continuous.

(ii) \( \Rightarrow \) (i): Follows from Lemma 2.5(iii).

**Remark 5.5.** In Theorem 5.4 we do not need \( n_M \) to be a negation, i.e., \( n_M(1) \geq 0 \). Consider the following t-subnorm \( M_{P_t} \) (cf. Example 3.15 of [4], see Figure 1(b)) which is a left-continuous (CCL) but \( n_{M_{P_t}} \) is not a negation since \( n_{M_{P_t}}(1) = 0.2 \)

\[
M_{P_t} = \begin{cases} 
0.2 + \frac{3(x - 0.2)(y - 0.2)}{4}, & \text{if } (x, y) \in (0.2, 1]^2 \\
0, & \text{otherwise}
\end{cases}
\]

**Theorem 5.6.** Let \( M \) be a (CCL) t-subnorm whose \( n_M \) is strong. Then \( M \) is left-continuous.

\[
\begin{align*}
&\text{Proof. } \text{If possible, let } M(x_0, n(x_0)) = p > 0 \text{ for some } x_0 \in (0, 1). \text{ Since } M \text{ is (CCL), we have } M(1^-, x_0) < x_0 \text{ and hence by associativity we have} \\
&M(1^-, M(x_0, n(x_0))) = M(1^-, p) \\
&M(M(1^-, x_0), n(x_0)) = 0
\end{align*}
\]

from whence it follows \( M(1^-, p) = 0 \), i.e., \( n(p) = 1 \), a contradiction to the fact that \( n_M \) is strong. Thus \( p = 0 \) and the result follows from Theorem 5.4. \( \square \)

**Theorem 5.7.** Let \( M \) be a t-subnorm such that \( n_M \) is strong. Then the following are equivalent:

(i) \( M \) is conditionally cancellative.

(ii) \( M \) is a nilpotent t-norm.

**Proof.** If \( M \) satisfies (CCL) then \( M \) is left-continuous, from Theorem 5.6 and now, using Theorem 5.3 we have the result. \( \square \)

**Remark 5.8.** (i) The nilpotent minimum t-norm \( T_{nM} \) is an example of a t-subnorm \( M \) whose \( n_M \) is involutive and \( M \) satisfies (LEM) with \( n_M \) but is not conditionally cancellative and hence is not a nilpotent t-norm.

(ii) In the case \( n_M \) is a strong natural negation, from Theorem 5.6 we see that conditionally cancellativity is equivalent to left-continuity and from Theorem 5.7 that every \( x \in (0, 1) \) is a nilpotent element.

6 Archimedeaness, Left Continuity and Nilpotence

We begin with a result that shows that if \( n_M \) is strong, then the Archimedeaness is equivalent to every element \( x \in (0, 1) \) being nilpotent. However, unless \( M \) is also left-continuous, \( M \) is not a nilpotent t-norm.

**Theorem 6.1.** Let \( M \) be any t-subnorm such that \( n_M \) is not completely vanishing, i.e., there exists \( z \in (0, 1) \) such that \( n_M(z) > 0 \). The following are equivalent:

(i) Every \( x \in (0, 1) \) is a nilpotent element.

(ii) \( M \) is Archimedean.

**Proof.** (i) \( \Rightarrow \) (ii): Follows from Proposition 2.15 (iv) in Klement et al. [4].

(ii) \( \Rightarrow \) (i): Let \( M \) be any Archimedean t-subnorm such that \( n_M \) is not completely vanishing, i.e., there exist \( z \in (0, 1) \) such that \( n_M(z) > 0 \). By Lemma 2.5(ii) we see that for any \( 0 < z' < n_M(z) \), we have \( M(z', z) = 0 \).
For any $x \in [0,1)$, by the Archimedeaness of $M$, there exists an $n, p \in \mathbb{N}$ such that $x_M^n < z'$ and $x_M^p < z$ from whence we have that

$$x_M^{n+p} = M \left(x_M^n, x_M^p\right) \leq M(z', z) = 0.$$ 

\[\square\]

Corollary 6.2. Let $M$ be any t-norm such that $n_M$ is a strong negation. Then the following are equivalent:

(i) Every $x \in (0,1)$ is a nilpotent element.

(ii) $M$ is Archimedean.

The following result is due to Kolesárová [6]:

Theorem 6.3. Let $T$ be any Archimedean t-norm. Then the following are equivalent:

(i) $T$ is left-continuous.

(ii) $T$ is continuous.

Corollary 6.4. A left-continuous Archimedean t-subnorm $M$ whose $n_M$ is involutive is a nilpotent t-norm.

Proof. From Theorem 3.1 we see that $M$ is a left-continuous t-norm. From Theorem 6.3, since $M$ is Archimedean it is continuous. Also by Theorem 6.1, we have that every $x \in (0,1)$ is a nilpotent element. Thus $T$ is nilpotent, i.e., isomorphic to $T_L$. \[\square\]

Remark 6.5. (i) Note that there exist left-continuous Archimedean t-subnorms $M$ that are not continuous and hence their $n_M$ is not involutive. For instance, consider the t-subnorm

$$M(x, y) = \begin{cases} 
    x + y - 1, & \text{if } x + y > \frac{3}{2}, \\
    0, & \text{otherwise},
\end{cases} x, y \in [0,1].$$

(ii) The nilpotent minimum t-norm $T_{n\text{M}}$ is an example of a left-continuous t-subnorm $M$ whose $n_M$ is involutive but is not Archimedean and hence is not a nilpotent t-norm.

(iii) However, it is not clear whether there exists any non-nilpotent Archimedean t-subnorm $M$ whose $n_M$ is involutive. Clearly such t-(sub)norms are not left-continuous.

Problem 1. Does there exist any non-nilpotent Archimedean t-subnorm $M$ whose $n_M$ is involutive. In other words, is an Archimedean t-subnorm $M$ whose $n_M$ is involutive necessarily left-continuous?

7 Archimedeanness and Conditional Cancellativity

In general, there does not exist any inter-relationships between Archimedeanness and conditional cancellativity, as the following examples show.

Example 7.1. (i) The Ouyang t-norm $T_OY$ is an example of a t-(sub)norm which is not Archimedean but is both left-continuous and conditionally cancellative.

(ii) The following t-norm is neither Archimedean nor left-continuous but is conditionally cancellative:

$$T(x, y) = \begin{cases} 
    0, & \text{if } xy \leq \frac{1}{2} \& \text{max}(x, y) < 1 \\
    xy, & \text{if } xy > \frac{1}{2} \\
    \min(x, y), & \text{otherwise}
\end{cases}.$$
(iii) The following t-subnorm is Archimedean and continuous, but not conditionally cancellative:

\[ M(x, y) = \max(0, \min(x + y - 1, x - a, y - a, 1 - 2a)) \]

where \( a \in (0, 0.5) \). For instance, with \( a = 0.25 \) we have \( M(0.75, 0.75) = M(0.75, 0.8) = 0.5 \).

(iv) The nilpotent minimum \( T_{nM} \), whose \( n_M \) is strong, is neither Archimedean nor conditionally cancellative, but is left-continuous.

In the case, when \( n_M \) is strong we have the following partial implication.

**Lemma 7.2.** Let \( M \) be any t-subnorm whose \( n_M \) is strong. If \( M \) is conditionally cancellative then \( M \) is Archimedean.

**Proof.** From Theorem 5.7, we have that if \( M \) is conditionally cancellative then \( M \) is a nilpotent t-norm from whence it follows that \( M \) is Archimedean. \( \square \)

**Problem 2.** Does there exist any Archimedean t-subnorm \( M \) whose \( n_M \) is involutive but is not conditionally cancellative? In other words, is an Archimedean t-subnorm \( M \) whose \( n_M \) is involutive necessarily conditionally cancellative?

8 Concluding Remarks

In this work, we have shown that t-subnorms whose associated negations are strong are necessarily t-norms. Further, we have studied the inter-relationships between some algebraic and analytical properties of such t-(sub)norms. Figure 2 gives a pictorial summary of the results that exist so far.

References


