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On the characterizations of (S, N)-implications☆

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Abstract

The characterization of S-implications generated from strong negations presented firstly by Trillas and Valverde in 1985 is well-known in the literature. In this paper we show that some assumptions are needless and present two characterizations of S-implications with mutually independent requirements. We also present characterizations of (S, N)-implications obtained from continuous fuzzy negations or strict negations. Besides these main results some new facts concerning fuzzy implications, fuzzy negations and laws of contraposition are proved.

Keywords: (S, N)-Implication; S-Implication; Fuzzy implication; Fuzzy negation; Laws of contraposition; Triangular conorm

1. Introduction and motivation

Characterizations and representations of fuzzy logical connectives are one of the most important and interesting mathematical problems in the fuzzy logic. In recent scientific literature we can find many such results (see [21,8] for strong and strict negations; [19,12] for triangular norms and conorms; [15,20,9,4,5,18] for fuzzy implications; [10,11] for all the above operators). We recall firstly some definitions that we will specially use.

Definition 1.1 (*see Fodor and Roubens [10, p. 3], Klement et al. [12, Definition 11.3], Gottwald [11, Definition 5.2.1], Bustince et al. [5, p. 210]*). A decreasing function $N\colon [0, 1] \to [0, 1]$ is called a fuzzy negation if $N(0) = 1$, $N(1) = 0$. A fuzzy negation N is called

(i) strict if it is strictly decreasing and continuous;
(ii) strong if it is an involution, i.e., $N(N(x)) = x$, for all $x \in [0, 1]$.

Definition 1.2 (*see Schweizer and Sklar [19, Chapter 5], Klement et al. [12, Definition 1.13]*). A function $S\colon [0, 1]^2 \to [0, 1]$ is called a triangular conorm (t-conorm) if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

$$S(x, y) = S(y, x), \tag{S1}$$

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$$S(x, S(y, z)) = S(S(x, y), z), \tag{S2}$$

$$S(x, y) \leqslant S(x, z) \quad \text{whenever } y \leqslant z, \tag{S3}$$

$$S(x, 0) = x. \tag{S4}$$

In the literature we can find several diverse definitions of fuzzy implications (cf. [10,5]). In our article this family is defined as below.

Definition 1.3. A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:

$$I \text{ is decreasing in the first variable}, \tag{I1}$$

$$I \text{ is increasing in the second variable}, \tag{I2}$$

$$I(0,0) = 1, \quad I(1,1) = 1, \quad I(1,0) = 0. \tag{I3}$$

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

From the above definition we can deduce, that for each fuzzy implication $I(0,x) = I(x,1) = 1$ for $x \in [0,1]$. Therefore, Definition 1.3 is equivalent to the definition proposed by Fodor and Roubens [10, Definition 1.15]. Moreover, I satisfies also the normality condition $I(0,1) = 1$ and, consequently, every fuzzy implication restricted to the set $\{0,1\}^2$ coincides with the classical Boolean implication.

There are many important methods for generating fuzzy implications (see [7,10,11]). This paper deals with one class of fuzzy implications, which can be seen as a generalization of the 'material' implication:

$$p \to q \equiv \neg p \vee q.$$

Definition 1.4 (*cf. Trillas and Valverde [23], Dubois and Prade [7], Fodor and Roubens [10], Klir and Yuan [13], Alsina and Trillas [3]*). A function $I\colon [0,1]^2 \to [0,1]$ is called an (S,N)-implication if there exist a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x), y), \quad x, y \in [0,1]. \tag{1}$$

If N is a strong negation, then I is called a strong implication (S-implication). Moreover, if an (S,N)-implication is generated from S and N, then we will often denote this by $I_{S,N}$.

We would like to underline here that some authors use the name S-implication, even if the negation N is not strong (see [12, Definition 11.5]). Since the name S-implication was firstly introduced in fuzzy logic framework by Trillas and Valverde (see [22,23, Definition 3.2]) with restrictive assumptions (S is a continuous t-conorm and N is a strong negation), we use, in the general case, the name '(S,N)-implication' proposed firstly by Alsina and Trillas [3]. It is important to note that various assumptions on the function N are still considered in many recent works (cf. [12, Definition 11.5, 11, Definition 5.4.1, 5, p. 216, 1, p. 213]). It can be easily verified that every (S,N)-implication is a fuzzy implication.

Example 1.5. Examples of fuzzy negations, t-conorms and basic S-implications can be found in the literature (see [10–13]). Here we present only some examples of (S,N)-implications which will be useful in the sequel.

(i) If S is any t-conorm and N is the Gödel (intuitionistic) negation

$$N_{\mathbf{G1}}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0, \end{cases}$$

then we always obtain the least (S,N)-implication

$$I_{\mathbf{G1}}(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ y & \text{if } x > 0, \end{cases} \qquad x, y \in [0,1].$$

(ii) If S is any t-conorm and N is the dual Gödel negation

$$N_{\mathbf{G2}}(x) = \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{if } x < 1, \end{cases} \qquad x \in [0, 1],$$

then we always obtain the greatest (S, N)-implication

$$I_{\mathbf{G2}}(x, y) = \begin{cases} 1 & \text{if } x < 1, \\ y & \text{if } x = 1, \end{cases} \qquad x, y \in [0, 1].$$

A first characterization of S-implications was presented by Trillas and Valverde [23] and it is often written in the following form.

Theorem 1.6. (*Trillas and Valverde [23, Theorem 3.2], cf. Fodor and Roubens [10, Theorem 1.13], Gottwald [11, Proposition 5.4.5]*). *A fuzzy implication I is an S-implication generated from some* t-*conorm S and some strong negation N if and only if I satisfies the left neutrality property*, *i.e.*,

$$I(1, y) = y, \quad y \in [0, 1], \tag{NP}$$

I satisfies the exchange principle, *i.e.*,

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1] \tag{EP}$$

and I satisfies the law of contraposition (CP) *with respect to N*, *i.e.*,

$$I(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1]. \tag{CP}$$

The above theorem can be easily transformed to the following version.

Theorem 1.7. *For a function $I : [0, 1]^2 \to [0, 1]$ the following statements are equivalent*:

(i) *I is an S-implication*.
(ii) *I satisfies* (I1), (I2), (NP), (EP) *and* (CP) *with respect to a strong negation N*.

A similar theorem but for S-implications defined on partially ordered bounded sets was given by De Baets [6]. In this paper we analyze the above theorem in depth and show that the law of CP can be replaced in Theorem 1.7 by a simpler condition. As first results we present two characterizations of S-implications with mutually independent requirements. We also show (see Corollary 2.5) that the law of CP cannot be used in the characterization of (S, N)-implications generated from non-strong fuzzy negations and thus the above theorem cannot be generalized in a natural way to our general situation. This explains the reconsideration of the problem of the characterization of all (S, N)-implications. Hence in this work we shall prove the characterizations of (S, N)-implications generated from continuous fuzzy negations or strict negations. Some new results concerning fuzzy negations, fuzzy implications and different laws of CPs are also proven besides these main results.

2. Discussion on characterizations of S-implications

Our first goal is to present a new characterization of S-implications. Towards this end, we examine some interdependencies between the classical law of CP and the basic properties of fuzzy implications. To use shorter notation, if a function I satisfies (CP) with respect to a fuzzy negation N, then we say that I satisfies CP(N). The first lemma can be easily proved.

Lemma 2.1. *If a function $I : [0, 1]^2 \to [0, 1]$ satisfies* (I1) *and* (I3), *then the function $N_I : [0, 1] \to [0, 1]$ defined by*

$$N_I(x) := I(x, 0), \quad x \in [0, 1] \tag{2}$$

is a fuzzy negation.

If I is a fuzzy implication, then the function N_I defined by (2) is called the natural negation of I.

Lemma 2.2 (*cf. Bustince et al. [5, Lemma 1(i), (ii) and (v)]*). *Let* $I: [0, 1]^2 \to [0, 1]$ *be any function and* N *a fuzzy negation.*

(i) *If I satisfies* (I1) *and* CP(N), *then I satisfies* (I2).
(ii) *If I satisfies* (I2) *and* CP(N), *then I satisfies* (I1).
(iii) *If I satisfies* (NP) *and* CP(N), *then I satisfies* (I3) *and* $N = N_I$ *is a strong negation.*

Proof. Let $x, y, z \in [0, 1]$.

(i) If $y \leqslant z$, then $N(z) \leqslant N(y)$, so by (I1) we have $I(N(y), N(x)) \leqslant I(N(z), N(x))$, hence, from (CP), we get

$$I(x, y) = I(N(y), N(x)) \leqslant I(N(z), N(x)) = I(x, z),$$

thus I satisfies (I2).

(ii) If $x \leqslant y$, then $N(y) \leqslant N(x)$, so by (I2) we have $I(N(z), N(y)) \leqslant I(N(z), N(x))$, hence, from (CP), we get

$$I(x, z) = I(N(z), N(x)) \geqslant I(N(z), N(y)) = I(y, z),$$

therefore I satisfies (I1).

(iii) Since I satisfies (NP), it is obvious that $I(1, 1) = 1$ and $I(1, 0) = 0$. Moreover, $I(0, 0) = I(1, 1)$, so it satisfies (I3). Next, since I satisfies CP(N) and (NP) we get

$$N_I(x) = I(x, 0) = I(N(0), N(x)) = I(1, N(x)) = N(x), \quad x \in [0, 1].$$

Further, for any $y \in [0, 1]$ we have

$$y = I(1, y) = I(N_I(y), N_I(1)) = I(N_I(y), 0) = N_I(N_I(y)),$$

so N_I is a strong negation. □

Immediately, from the last point we get

Corollary 2.3. *Let* $I: [0, 1]^2 \to [0, 1]$ *be a function that satisfies* (NP). *If* N_I *is not a strong negation, then I does not satisfy* (CP) *with any fuzzy negation N.*

On the other hand from Lemma 1(vi) and (ix) in [5] and Lemma 2.2(iii) above we have

Lemma 2.4. *Let* $I: [0, 1]^2 \to [0, 1]$ *be any function and* N_I *a strong negation.*

(i) *If I satisfies* CP(N_I), *then I satisfies* (NP).
(ii) *If I satisfies* (EP), *then I satisfies* (I3), (NP) *and* (CP) *only with respect to* N_I.

From the above facts the following corollary can be easily obtained.

Corollary 2.5. *If* $I \in \mathcal{FI}$ *satisfies* (NP) *and* (EP), *then the following statements are equivalent*:

(i) N_I *is a strong negation.*
(ii) *I satisfies* (CP) *with respect to* N_I.

As a result we obtain a new characterization of *S*-implications.

Theorem 2.6 (*cf. Theorem 1.7*). *For a function* $I: [0, 1]^2 \to [0, 1]$ *the following statements are equivalent*:

(i) *I is an S-implication.*
(ii) *I satisfies* (I1), (EP) *and* N_I *is a strong negation.*

Moreover, the representation of S-implication (1) *is unique in this case.*

Table 1
The mutual exclusivity of properties in Theorem 2.6

Function I	(I1)	(EP)	N_I is a strong negation
$I_{\mathbf{RS}}(x,y)=\begin{cases}1 & \text{if } x\leqslant y\\ 0 & \text{if } x>y\end{cases}$	✓	×	×
$I(x,y)=xy$	×	✓	×
$I(x,y)=\begin{cases}0 & \text{if } x<0.5 \text{ and } y>0\\ 1 & \text{if } x\geqslant 0.5 \text{ and } y>0\\ 1-x & \text{if } y=0\end{cases}$	×	×	✓
$I_{\mathbf{YG}}(x,y)=\begin{cases}1 & \text{if } x=y=0\\ y^x & \text{otherwise}\end{cases}$	✓	✓	×
$I(x,y)=\begin{cases}1-x & \text{if } y=0\\ y & \text{if } x=1\\ 1 & \text{otherwise}\end{cases}$	✓	×	✓
$I(x,y)=\begin{cases}1-x & \text{if } y=0\\ y & \text{if } x=1\\ 0 & \text{otherwise}\end{cases}$	×	✓	✓

Proof. (i) $\Longrightarrow$ (ii). It can be easily verified that every S-implication satisfies (I1), (EP) and N_I is a strong negation.

(ii) $\Longrightarrow$ (i). Let us assume that I satisfies (I1), (EP) and N_I is a strong negation. From Lemma 2.4(ii) it follows that I satisfies (I3), (NP) and CP(N_I). By Lemma 2.2(i) the function I satisfies (I2), so $I \in \mathcal{FI}$. By virtue of Theorem 1.7 we get that I is an S-implication generated from some t-conorm S and some strong negation N. Assume that there exist two fuzzy negations N_1, N_2 and two t-conorms S_1, S_2 such that $I(x,y) = S_1(N_1(x), y) = S_2(N_2(x), y)$ for all $x, y \in [0,1]$. Putting $y = 0$ we get $N_1(x) = N_2(x)$ for $x \in [0,1]$, i.e., $N_1 = N_2$. Now, let us fix arbitrarily $x_0, y_0 \in [0,1]$. Since N_1 is a strong negation there exists $x_1 \in [0,1]$ such that $N_1(x_1) = x_0$. Thus $S_1(x_0, y_0) = S_1(N_1(x_1), y_0) = S_2(N_1(x_1), y_0) = S_2(x_0, y_0)$, i.e., $S_1 = S_2$. Therefore N and S are uniquely determined. It can be proven (see the proof of Theorem 1.7 and Remark 4.6(ii)) that in this case $N = N_I$ and $S(x,y) = I(N_I(x), y)$ for every $x, y \in [0,1]$. □

By virtue of Lemma 2.2 we can substitute above the requirement (I1) by (I2).

Example 2.7. In Table 1 we show that the properties in above theorem are independent from each other. It should be noted, that $I_{\mathbf{RS}}$ is the fuzzy implication introduced by Rescher [16], while $I_{\mathbf{YG}}$ is the fuzzy implication introduced by Yager [24].

If we want to have the law of CP in the characterization of S-implications, like in Theorem 1.7, then the alternative characterization with mutually independent assumptions is the following.

Theorem 2.8. *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent*:

(i) *I is an S-implication.*
(ii) *I satisfies* (I1) (*or* (I2)), (NP), (EP) *and* (CP) *with respect to a strong negation N.*

It is interesting that the left neutrality (NP) cannot be deleted from above assumptions, like in Theorem 2.6 (and next characterizations, see Theorems 5.1 and 5.2). As a counterexample consider the constant function $I(x,y) = 1$. It satisfies (I1), (I2), (EP) and (CP) with every strong negation, but it is not an S-implication.

3. Laws of left and right contraposition

We see, by Corollary 2.5, that the law of CP cannot be used in the characterization of the family of (S, N)-implications. We will use a different approach. Since classical negation satisfies the law of double negation, the following laws are also tautologies in the classical logic:

$$\neg p \to q \equiv \neg q \to p, \quad p \to \neg q \equiv q \to \neg p.$$

But not all fuzzy negations are strong, i.e., involutive, so we can consider various laws of CP in fuzzy logic.

Definition 3.1. Let $I : [0,1]^2 \to [0,1]$ be any function and N a fuzzy negation.

(i) We say that I satisfies the law of right CP with respect to N, if

$$I(x, N(y)) = I(y, N(x)), \quad x, y \in [0,1]. \tag{R-CP}$$

(ii) We say that I satisfies the law of left CP with respect to N, if

$$I(N(x), y) = I(N(y), x), \quad x, y \in [0,1]. \tag{L-CP}$$

If I satisfies the law of right (left) CP with respect to N, then we also denote this by R-CP(N) (respectively, by L-CP(N)).

We can easily observe that all the three laws of CP are equivalent, when N is a strong negation. But if a negation N is not strong, then this may not be true. In the rest of this section we discuss the above laws of CP with other properties of fuzzy implications. We start with the law of right CP.

Lemma 3.2. *Let $I : [0,1]^2 \to [0,1]$ be any function and N a continuous fuzzy negation.*

(i) *If I satisfies* (I1) *and* R-CP(N), *then I satisfies* (I2).
(ii) *If I satisfies* (I2) *and* R-CP(N), *then I satisfies* (I1).

Proof. Let $x, y, z \in [0,1]$ be fixed.

(i) If $y \leqslant z$, then there exists $y' \geqslant z'$ such that $N(y') = y$ and $N(z') = z$. Therefore from (R-CP) and (I1) we get

$$\begin{aligned} I(x, y) &= I(x, N(y')) = I(y', N(x)) \\ &\leqslant I(z', N(x)) = I(x, N(z')) = I(x, z). \end{aligned}$$

(ii) If $x \leqslant y$, then $N(x) \geqslant N(y)$. Since N is continuous, there exists $z' \in [0,1]$ such that $N(z') = z$, so by (R-CP) and (I2) we have

$$\begin{aligned} I(x, z) &= I(x, N(z')) = I(z', N(x)) \\ &\geqslant I(z', N(y)) = I(y, N(z')) = I(y, z). \quad \square \end{aligned}$$

Lemma 3.3. *If a function $I : [0,1]^2 \to [0,1]$ satisfies* (NP) *and* (R-CP) *with a fuzzy negation N, then I satisfies* (I3) *and $N = N_I$.*

Proof. It is obvious, from (NP), that $I(1,0) = 0$ and $I(1,1) = 1$, thus $I(0,0) = I(0, N(1)) = I(1, N(0)) = I(1,1) = 1$, so I satisfies (I3). Further, for any $x \in [0,1]$ we have $N_I(x) = I(x,0) = I(x, N(1)) = I(1, N(x)) = N(x)$. $\square$

Lemma 3.4. *If a function $I : [0,1]^2 \to [0,1]$ satisfies* (EP) *and N_I is a fuzzy negation, then I satisfies* R-CP(N_I).

Proof. For any $x, y \in [0,1]$ we get

$$I(x, N_I(y)) = I(x, I(y, 0)) = I(y, I(x, 0)) = I(y, N_I(x)). \quad \square$$

Lemma 3.5. *Let $I : [0,1]^2 \to [0,1]$ be any function and N_I a continuous fuzzy negation.*

(i) *If I satisfies* R-CP(N_I), *then I satisfies* (NP).
(ii) *If I satisfies* (EP), *then I satisfies* (I3), (NP) *and* (R-CP) *only with respect to N_I.*

Proof.

(i) Let us fix $y \in [0, 1]$. N_I is a continuous negation, so there exists $y' \in [0, 1]$ such that $N_I(y') = y$. Thus

$$I(1, y) = I(1, N_I(y')) = I(y', N_I(1)) = I(y', 0) = N_I(y') = y.$$

(ii) The property R-CP(N_I) follows from Lemma 3.4. From the previous point we deduce that it satisfies (NP) and thus, by Lemma 3.3, it satisfies (I3) and N_I is the only negation for *I* in (R-CP). □

Corollary 3.6. *If* $I \in \mathcal{FI}$ *satisfies* (NP) *and* (EP), *then I satisfies* (R-CP) *only with respect to* N_I.

Now we discuss the law of left CP.

Lemma 3.7. *Let* $I: [0, 1]^2 \to [0, 1]$ *be any function and N a continuous fuzzy negation.*

(i) *If I satisfies* (I1) *and* L-CP(*N*), *then I satisfies* (I2).
(ii) *If I satisfies* (I2) *and* L-CP(*N*), *then I satisfies* (I1).

Proof. Let $x, y, z \in [0, 1]$ be fixed.

(i) If $y \leqslant z$, then $N(z) \leqslant N(y)$. Since *N* is continuous, there exists $x' \in [0, 1]$ such that $N(x') = x$. Now from (L-CP) and (I1) we get

$$\begin{aligned} I(x, y) &= I(N(x'), y) = I(N(y), x') \\ &\leqslant I(N(z), x') = I(N(x'), z) = I(x, z). \end{aligned}$$

(ii) If $x \leqslant y$, then there exists $x' \geqslant y'$ such that $N(x') = x$ and $N(y') = y$. Therefore from (L-CP) we have

$$\begin{aligned} I(x, z) &= I(N(x'), z) = I(N(z), x') \\ &\geqslant I(N(z), y') = I(N(y'), z) = I(y, z). \end{aligned}$$ □

Proposition 3.8. *If* N_1, N_2 *are two fuzzy negations such that* $N_1 \circ N_2 = \mathrm{id}_{[0,1]}$, *then*

(i) N_1 *is a continuous fuzzy negation*,
(ii) N_2 *is a strictly decreasing fuzzy negation*,
(iii) N_2 *is a strict negation if and only if* N_1 *is a strict negation. In both cases* $N_1 = N_2^{-1}$.

Proof.

(i) It is obvious, that N_1 is onto [0, 1]. Additionally, since N_1 is decreasing, it is continuous.
(ii) Let us assume that $N_2(x_1) = N_2(x_2)$ for some $x_1, x_2 \in [0, 1]$. Given that $N_1 \circ N_2(x) = x$, for all $x \in [0, 1]$ we get $x_1 = N_1(N_2(x_1)) = N_1(N_2(x_2)) = x_2$, so N_2 is one-to-one, and, consequently, a strictly decreasing fuzzy negation.
(iii) Let us assume that N_2 is a continuous fuzzy negation. From (ii) above we know that N_2 is one-to-one, so N_2 is a decreasing bijection on [0, 1], i.e., it is a strict negation. Hence N_2^{-1} is also a strict negation and $N_1 = N_2^{-1}$. In particular, N_1 is a strict negation.
Conversely, if we assume that N_1 is a strictly decreasing negation, then from (i) above we know that N_1 is continuous, so N_1 is a strict negation. Hence N_1^{-1} is also a strict negation and $N_2 = N_1^{-1}$. In particular, N_2 is a strict negation. □

Lemma 3.9. *If a function* $I: [0, 1]^2 \to [0, 1]$ satisfies (NP), (L-CP) *with some fuzzy negation N and* N_I *is a fuzzy negation, then*

(i) *if I satisfies, in addition,* (I1) *and* (I2), *then* $I \in \mathcal{FI}$,
(ii) $N_I \circ N = \mathrm{id}_{[0,1]}$,
(iii) N_I *is a continuous fuzzy negation*,

(iv) *N is a strictly decreasing fuzzy negation,*
(v) *N is a continuous fuzzy negation if and only if N_I is a strictly decreasing fuzzy negation. In both cases $N = N_I^{-1}$.*

Proof.
(i) Since I satisfies (NP), it is obvious that $I(1,1) = 1$ and $I(1,0) = 0$. Moreover, $I(0,0) = I(N(1),0) = I(N(0),1) = I(1,1)$, so it satisfies (I3) and consequently $I \in \mathcal{FI}$.
(ii) Let us take a fixed $x \in [0,1]$. Then

$$x = I(1,x) = I(N(0),x) = I(N(x),0) = N_I(N(x)).$$

The other three points follow immediately from Proposition 3.8. □

As a result we obtain

Corollary 3.10. *Let $I: [0,1]^2 \to [0,1]$ be a function that satisfies* (NP). *If N_I is not a continuous negation, then I does not satisfy* (L-CP) *with any fuzzy negation N.*

Conversely we have

Lemma 3.11. *If a function $I: [0,1]^2 \to [0,1]$ satisfies* (EP), *N_I is a continuous fuzzy negation and N is a strictly decreasing fuzzy negation such that $N_I \circ N = \mathrm{id}_{[0,1]}$, then I satisfies* (L-CP) *with respect to N.*

Proof. By our assumptions we have, for all $x, y \in [0,1]$

$$\begin{aligned} I(N(x),y) &= I(N(x), N_I \circ N(y)) = I(N(x), I(N(y),0)) \\ &= I(N(y), I(N(x),0)) = I(N(y), N_I \circ N(x)) \\ &= I(N(y),x). \quad \square \end{aligned}$$

When N_I is a strict negation we get the following result.

Lemma 3.12. *Let $I: [0,1]^2 \to [0,1]$ be any function and N_I be a strict negation.*

(i) *If I satisfies* L-CP(N_I^{-1}), *then I satisfies* (NP).
(ii) *If I satisfies* (EP), *then I satisfies* (NP) *and* (L-CP) *only with respect to N_I^{-1}.*

Proof.
(i) For every $y \in [0,1]$ we get

$$I(1,y) = I(N_I^{-1}(0),y) = I(N_I^{-1}(y),0) = N_I(N_I^{-1}(y)) = y.$$

(ii) Because of Lemma 3.11 we know that I satisfies the law of left CP with respect to N_I^{-1}. That I satisfies (NP) follows from (i) above. By virtue of Lemma 3.9(ii) we deduce that N_I^{-1} is the only negation with which I has (L-CP). □

If $I: [0,1]^2 \to [0,1]$ satisfies (NP), (EP) and N_I is a continuous but not strict negation, then the situation is a little more complicated. It is well-known that if $[a,b]$ and $[c,d]$ are two closed subintervals of $[-\infty,+\infty]$ and $f: [a,b] \to [c,d]$ is a monotone function, then the set of discontinuous points of f is a countable subset of $[a,b]$ (see [17]). In this case we will use the pseudo-inverse $f^{(-1)}: [c,d] \to [a,b]$ of f defined by

$$f^{(-1)}(y) := \sup\{x \in [a,b] \mid (f(x)-y)(f(b)-f(a)) < 0\}, \quad y \in [c,d].$$

For a decreasing and non-constant function $f: [a,b] \to [c,d]$, the pseudo-inverse of f can be defined by a simpler formula (see [12, Section 3.1])

$$f^{(-1)}(y) = \sup\{x \in [a,b] \mid f(x) > y\}, \quad y \in [c,d].$$

Proposition 3.13. *If N is a continuous fuzzy negation, then the function* $\mathfrak{N}\colon [0,1] \to [0,1]$ *defined by*

$$\mathfrak{N}(x) = \begin{cases} N^{(-1)}(x) & \text{if } x \in (0,1], \\ 1 & \text{if } x = 0, \end{cases} \qquad x \in [0,1] \tag{3}$$

is a strictly decreasing fuzzy negation. Moreover

$$\mathfrak{N}^{(-1)} = N, \tag{4}$$

$$N \circ \mathfrak{N} = \mathrm{id}_{[0,1]}, \tag{5}$$

$$\mathfrak{N} \circ N\ |_{\mathrm{Ran}(\mathfrak{N})} = \mathrm{id}_{\mathrm{Ran}(\mathfrak{N})}. \tag{6}$$

Proof. Let N be a continuous fuzzy negation. By virtue of Corollary 3.3(ii) from [12] we get that the pseudo-inverse $N^{(-1)}$ is a decreasing function. Moreover

$$N^{(-1)}(1) = \sup\{x \in [0,1] \mid N(x) > 1\} = \sup \emptyset = 0,$$

but

$$N^{(-1)}(0) = \sup\{x \in [0,1] \mid N(x) > 0\}$$

can be less than 1. Let us define a function $\mathfrak{N}\colon [0,1] \to [0,1]$ by (3). Firstly, see that this function is now a well-defined fuzzy negation. Further, since N is continuous, from the above mentioned corollary we have

$$(N^{(-1)})^{(-1)} = N, \tag{7}$$

which implies (4). Indeed, let us define the following two sets:

$$A(y) = \{x \in [0,1] \mid \mathfrak{N}(x) > y\},$$
$$B(y) = \{x \in [0,1] \mid N^{(-1)}(x) > y\},$$

for every $y \in [0,1]$. We consider the following three cases:

- If $A(y) = \emptyset$, then $y = 1$ since $\mathfrak{N}(0) = 1$. This implies that $\mathfrak{N}^{(-1)}(1) = \sup \emptyset = 0 = N(1)$, i.e., we get (4) for $y = 1$.
- If $A(y) = \{0\}$, then $\mathfrak{N}^{(-1)}(y) = \sup\{0\} = 0$. Moreover, only for $x = 0$ we have $\mathfrak{N}(0) > y$, i.e., for all $x \in (0,1]$ we have $\mathfrak{N}(x) \leqslant y$. Hence, by the definition of $\mathfrak{N}$, we get that $N^{(-1)}(x) \leqslant y$ for all $x \in (0,1]$. Therefore $B(y) = \{0\}$ or $B(y) = \emptyset$. In both cases we get $(N^{(-1)})^{(-1)}(y) = \sup B(y) = 0$, for such y. By (7) we have that (4) is true in this case.
- If $A(y) \neq \emptyset$ and $A(y) \neq \{0\}$, then there exists $x_0 \in (0,1]$ such that $x_0 \in A(y)$. Since for all $x \in [x_0, 1]$ we have $\mathfrak{N}(x) = N^{(-1)}(x)$, again from (7) we get $\sup A(y) = \sup B(y)$ i.e., (4) is also true.

By Remark 3.4(ii) from [12] (for $f = \mathfrak{N}$) the fuzzy negation $\mathfrak{N}$ is strictly monotonic on $N([0,1))$. Continuity of N implies that $(0,1] \subset N([0,1))$, so $\mathfrak{N}$ is strictly decreasing on $[0,1]$.

Finally, by virtue of Remark 3.4(vi) from [12] (for $f = \mathfrak{N}$), since $\mathfrak{N}$ is a strictly decreasing fuzzy negation, we obtain

$$N \circ \mathfrak{N}(x) = \mathfrak{N}^{(-1)} \circ \mathfrak{N}(x) = x, \quad x \in [0,1],$$
$$\mathfrak{N} \circ N(x) = \mathfrak{N} \circ \mathfrak{N}^{(-1)}(x) = x, \quad x \in \mathrm{Ran}(\mathfrak{N}).$$

Therefore (5) and (6) are true. □

Remark 3.14. It should be noted, that even if N is a continuous negation, the pseudo-inverse $N^{(-1)}$ need not be a fuzzy negation. To see this let us consider the following continuous fuzzy negation:

$$N(x) = \begin{cases} -2x + 1 & \text{if } x \in [0, 0.5], \\ 0 & \text{if } x \in (0.5, 1], \end{cases}$$

for which $N^{(-1)}(x) = -0.5x + 1$ is strictly decreasing, but not a fuzzy negation.

Thus, because of Lemma 3.11, if N_I is a continuous fuzzy negation, then we can consider the modified pseudo-inverse $\mathfrak{R}_I$ given by

$$\mathfrak{R}_I(x) = \begin{cases} N_I^{(-1)}(x) & \text{if } x \in (0,1], \\ 1 & \text{if } x = 0, \end{cases} \qquad x \in [0,1] \tag{8}$$

as the potential candidate for the fuzzy negation in (L-CP).

Corollary 3.15. *Let* $I: [0,1]^2 \to [0,1]$ *be any function and* N_I *be a continuous negation.*

(i) *If I satisfies* L-CP($\mathfrak{R}_I$), *then I satisfies* (NP).
(ii) *If I satisfies* (EP), *then I satisfies* L-CP($\mathfrak{R}_I$) *and* (NP).

Remark 3.16. If a function $I: [0,1]^2 \to [0,1]$ satisfies (EP) and N_I is a continuous but not a one-to-one fuzzy negation, then it is possible that there exist infinitely many strictly decreasing fuzzy negations for which (L-CP) holds. The negation $\mathfrak{R}_I$ is one of them. As an example consider an (S,N)-implication I generated by any t-conorm and the following continuous fuzzy negation:

$$N(x) = \begin{cases} -2x+1 & \text{if } x \in [0, 0.25], \\ 0.5 & \text{if } x \in (0.25, 0.75), \\ -2x+2 & \text{if } x \in [0.75, 1]. \end{cases}$$

From above corollary we have, that I satisfies L-CP($\mathfrak{R}_I$), where

$$\mathfrak{R}_I(x) = \begin{cases} -0.5x+1 & \text{if } x \in [0, 0.5), \\ -0.5x+0.5 & \text{if } x \in [0.5, 1]. \end{cases}$$

But also $N \circ N_1 = \mathrm{id}_{[0,1]}$, where N_1 is defined by

$$N_1(x) = \begin{cases} -0.5x+1 & \text{if } x \in [0, 0.5], \\ -0.5x+0.5 & \text{if } x \in (0.5, 1]. \end{cases}$$

Hence, because of Lemma 3.11, we see that I also satisfies L-CP(N_1).

4. Triangular conorms from fuzzy implications and continuous negations

In this section we will consider the dual situation, i.e., the method of obtaining t-conorms from fuzzy implications and fuzzy negations. But firstly we investigate some properties of (S,N)-implications. It is important to note that all (S,N)-implications are fuzzy implications.

Proposition 4.1 (*cf. Trillas and Valverde [23, Theorem 3.1]*). *If* $I_{S,N}$ *is an* (S,N)-*implication based on some* t-*conorm S and some fuzzy negation N*, *then*

(i) $I_{S,N} \in \mathcal{FI}$ *and* $I_{S,N}$ *satisfies* (NP), (EP),
(ii) $N_{I_{S,N}} = N$,
(iii) $I_{S,N}$ *satisfies* R-CP(*N*),
(iv) *if N is strict*, *then* $I_{S,N}$ *satisfies* L-CP(N^{-1}),
(v) *if N is strong*, *then* $I_{S,N}$ *satisfies* CP(*N*).

Proof.

(i) It is a straightforward verification that $I_{S,N} \in \mathcal{FI}$. Further, $I_{S,N}$ satisfies (NP), since

$$I_{S,N}(1,y) = S(N(1),y) = S(0,y) = y, \quad y \in [0,1].$$

From the associativity and the commutativity of S we also have (EP).

(ii) For any $x \in [0, 1]$ we have

$$N_{I_{S,N}}(x) = I_{S,N}(x, 0) = S(N(x), 0) = N(x).$$

(iii) Since $I_{S,N}$ satisfies (EP), from Corollary 3.6 it satisfies R-CP(*N*).
(iv) If *N* is a strict negation, then because of Lemma 3.12(ii) we can deduce that $I_{S,N}$ satisfies L-CP(N^{-1}).
(iv) If *N* is a strong negation, then because of Lemma 2.4(ii) we can deduce that $I_{S,N}$ satisfies CP(*N*). □

Remark 4.2. It is important to note, that conditions (I1), (I2), (I3), (NP), (EP) and (R-CP) are not enough to obtain an (S, N)-implication, i.e., they are necessary but not sufficient. Let us consider the Yager implication

$$I_{\mathbf{YG}}(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ and } y = 0, \\ y^x & \text{otherwise.} \end{cases}$$

It satisfies all the above axioms ((R-CP) with respect to $N_{I_{\mathbf{YG}}}$). Let us assume that it is an (S, N)-implication for some t-conorm *S* and some fuzzy negation *N*. By Proposition 4.1 we get that $N = N_{I_{\mathbf{YG}}}$, but easy calculations give, that $N_{I_{\mathbf{YG}}} = N_{\mathbf{G1}}$. Hence, from Example 1.5, it follows that $I_{\mathbf{YG}} = I_{\mathbf{G1}}$, a contradiction.

Proposition 4.3. *Let I be a fuzzy implication and N a fuzzy negation. Let us define a function* $S_{I,N}\colon [0, 1]^2 \to [0, 1]$ *as follows*:

$$S_{I,N}(x, y) = I(N(x), y), \quad x, y \in [0, 1]. \tag{9}$$

Then

(i) $S_{I,N}(1, x) = S_{I,N}(x, 1) = 1$, *for all* $x \in [0, 1]$,
(ii) $S_{I,N}$ *is increasing in both variables* (*in particular* $S_{I,N}$ *satisfies* (S3)),
(iii) $S_{I,N}$ *is commutative*, *i.e.*, $S_{I,N}$ *satisfies* (S1) *if and only if I satisfies the law of left* CP (L-CP) *with respect to N*.

In addition, *if I satisfies* L-CP(*N*), *then*

(iv) $S_{I,N}$ *satisfies* (S4) *if and only if I satisfies* (NP),
(v) $S_{I,N}$ *is associative i.e.*, *it satisfies* (S2) *if and only if I satisfies* (EP).

Proof.

(i) Since every fuzzy implication satisfies $I(0, x) = I(x, 1) = 1$ for every $x \in [0, 1]$, we have

$$S_{I,N}(1, x) = I(N(1), x) = I(0, x) = 1,$$
$$S_{I,N}(x, 1) = I(N(x), 1) = 1,$$

for all $x \in [0, 1]$.

(ii) That $S_{I,N}$ is increasing in both variables is a direct consequence of the monotonicity of *I* in the first and second variables and the monotonicity of *N*.

(iii) If *I* satisfies L-CP(*N*), then

$$S_{I,N}(x, y) = I(N(x), y) = I(N(y), x) = S_{I,N}(y, x), \quad x \in [0, 1].$$

On the other side, by the commutativity of $S_{I,N}$ we have

$$I(N(x), y) = S_{I,N}(x, y) = S_{I,N}(y, x) = I(N(y), x), \quad x, y \in [0, 1],$$

i.e., *I* satisfies L-CP(*N*).

(iv) If *I* satisfies (NP), then, by the commutativity,

$$S_{I,N}(x, 0) = S_{I,N}(0, x) = I(N(0), x) = I(1, x) = x, \quad x \in [0, 1].$$

On the other hand,

$$I(1, x) = I(N(0), x) = S_{I,N}(0, x) = x, \quad x \in [0, 1].$$

(v) If I satisfies (EP), then because of L-CP(N),

$$\begin{aligned} S_{I,N}(x, S_{I,N}(y,z)) &= I(N(x), I(N(y),z)) = I(N(x), I(N(z),y)) \\ &= I(N(z), I(N(x),y)) = S_{I,N}(z, S_{I,N}(x,y)) \\ &= S_{I,N}(S_{I,N}(x,y),z), \end{aligned}$$

for any $x, y, z \in [0,1]$.
On the other hand, if $S_{I,N}$ is associative and commutative, then

$$\begin{aligned} I(x, I(y,z)) &= S_{I,N}(N(x), S_{I,N}(N(y),z)) \\ &= S_{I,N}(N(y), S_{I,N}(N(x),z)) = I(y, I(x,z)), \end{aligned}$$

for all $x, y, z \in [0,1]$. □

Remark 4.4. From the above fact and Lemma 3.9(iii) it follows that if we want to obtain a t-conorm by (9), we should consider a fuzzy implication *I* for which N_I is a continuous fuzzy negation. Again, (NP) and (EP) are necessary but not sufficient to define a t-conorm by (9). As an interesting example we take the fuzzy implication $\boldsymbol{I_{G1}}$. By Example 1.5 it is an (S, N)-implication, so it satisfies conditions (NP) and (EP). Since $N_{I_{\mathbf{G1}}} = N_{\mathbf{G1}}$ is not continuous, it follows that we cannot obtain a t-conorm by (9) for $\boldsymbol{I_{G1}}$ and any fuzzy negation.

If a fuzzy implication *I* satisfies (NP), (EP) and N_I is a continuous fuzzy negation, then by virtue of Corollary 3.15 and previous proposition we obtain that (9) can be considered for the modified pseudo-inverse of the natural negation of *I*.

Corollary 4.5. *If $I \in \mathcal{FI}$ satisfies* (NP), (EP) *and N_I is a continuous fuzzy negation, then the function S_I defined by*

$$S_I(x,y) = I(\mathfrak{R}_I(x), y), \quad x, y \in [0,1] \tag{10}$$

is a t-*conorm, where $\mathfrak{R}_I$ is as defined in* (8).

Remark 4.6. (i) If $I \in \mathcal{FI}$ satisfies both (NP), (EP) and N_I is a strict negation, then because of Lemma 3.12(ii) *I* satisfies (L-CP) only with respect to N_I^{-1}. Thus (9) can be considered only for the inverse of the natural negation of *I*. In this case the formula for a t-conorm is unique and the following:

$$S_I(x,y) = I(N_I^{-1}(x), y), \quad x, y \in [0,1]. \tag{11}$$

(ii) In the special case, when N_I is a strong negation, we have that $N_I^{-1} = N_I$, i.e., the formula for a t-conorm is also unique and the following:

$$S_I(x,y) = I(N_I(x), y) = I(I(x,0), y), \quad x, y \in [0,1].$$

5. Main results

From the above discussion, we can state the following results which characterize some subclasses of (S, N)-implications.

Theorem 5.1. *For a function $I: [0,1]^2 \to [0,1]$ the following statements are equivalent*:

(i) *I is an (S, N)-implication generated from some* t-*conorm S and some continuous fuzzy negation N.*
(ii) *I satisfies* (I1), (EP) *and N_I is a continuous fuzzy negation*.

Moreover, the representation of (S, N)-implication (1) *is unique in this case*.

Proof. (i) $\Longrightarrow$ (ii). Assume that *I* is an (S, N)-implication based on a t-conorm *S* and a continuous negation *N*. Then, by Proposition 4.1, it is a fuzzy implication which satisfies (EP). In particular *I* satisfies (I1). Moreover, by the same result and our assumptions $N_I = N$ is continuous.

(ii) $\Longrightarrow$ (i). From Lemma 3.5 it follows that I satisfies (I3), (NP) and (R-CP) only with respect to N_I. Lemma 3.2 implies that I satisfies (I2), so $I \in \mathcal{FI}$. Further, by virtue of Corollary 3.15, I satisfies L-CP($\mathfrak{R}_I$). Because of Corollary 4.5 the function S_I defined by (10) is a t-conorm. We will show that $I_{S_I,N_I} = I$. Fix arbitrarily $x, y \in [0, 1]$. If $x \in \text{Ran}(\mathfrak{R}_I)$, then by (6) we have

$$I_{S_I,N_I}(x, y) = S_I(N_I(x), y) = I(\mathfrak{R}_I \circ N_I(x), y) = I(x, y).$$

If $x \notin \text{Ran}(\mathfrak{R}_I)$, then from the continuity of N_I there exists $x_0 \in \text{Ran}(\mathfrak{R}_I)$ such that $N_I(x) = N_I(x_0)$. Firstly, see that $I(x, y) = I(x_0, y)$ for all $y \in [0, 1]$. Indeed, let us fix arbitrarily $y \in [0, 1]$. From the continuity of N_I there exists $y' \in [0, 1]$ such that $N_I(y') = y$, so

$$\begin{aligned} I(x, y) &= I(x, N_I(y')) = I(y', N_I(x)) = I(y', N_I(x_0)) = I(x_0, N_I(y')) \\ &= I(x_0, y). \end{aligned}$$

From the above fact we get

$$I_{S_I,N_I}(x, y) = S_I(N_I(x), y) = S_I(N_I(x_0), y) = I(x_0, y) = I(x, y),$$

so I is an (S, N)-implication.

Finally, assume that there exist two continuous fuzzy negations N_1, N_2 and two t-conorms S_1, S_2 such that $I(x, y) = S_1(N_1(x), y) = S_2(N_2(x), y)$ for all $x, y \in [0, 1]$. Fix arbitrarily $x_0, y_0 \in [0, 1]$. Firstly, observe that $N_1 = N_2 = N_I$. Now, since N_I is a continuous negation there exists $x_1 \in [0, 1]$ such that $N_I(x_1) = x_0$. Thus $S_1(x_0, y_0) = S_1(N_I(x_1), y_0) = S_2(N_I(x_1), y_0) = S_2(x_0, y_0)$, i.e., $S_1 = S_2$. We showed that N and S are unique determined. In fact $S = S_I$ defined by (10). □

It should be noted that by virtue of Lemma 3.2 we can substitute in the above theorem the requirement (I1) by (I2).

The next result is a special case of Theorem 5.1.

Theorem 5.2. *For a function* $I : [0, 1]^2 \to [0, 1]$ *the following statements are equivalent*:

(i) *I is an (S, N)-implication generated from some* t-*conorm S and some continuous fuzzy negation N.*
(ii) *I satisfies* (I1), (EP) *and N_I is a strict negation.*

Moreover, the representation of (S, N)-implication (1) *is unique in this case.*

Proof. (i) $\Longrightarrow$ (ii). Assume firstly, that I is an (S, N)-implication based on a t-conorm S and a strict negation N. Then, by Proposition 4.1 it is a fuzzy implication which satisfies (NP) and (EP). In particular I satisfies (I1). Moreover, by the same result and our assumptions $N_I = N$ is a strict negation.

(ii) $\Longrightarrow$ (i). To prove the converse implication, see firstly, that because of Lemma 3.12(ii) we know that I satisfies (NP) and (L-CP) with a strict negation N_I^{-1}. Thus I satisfies (I3). Further, from Lemma 3.7(i) it satisfies (I2). Therefore $I \in \mathcal{FI}$. Now, because of Remark 4.6(i) the function S_I defined by (11) is a t-conorm. Finally, for every $x, y \in [0, 1]$ we have

$$I_{S_I,N_I}(x, y) = S_I(N_I(x), y) = I(N_I^{-1} \circ N_I(x), y) = I(x, y),$$

so I is an (S, N)-implication generated from S_I and N_I. Uniqueness of the representation (1) follows immediately from Theorem 5.1. □

Again, by virtue of Lemma 3.7 we can substitute in the above theorem the requirement (I1) by (I2).

Remark 5.3. Using the same examples as in Table 1 we obtain that the properties in Theorems 5.1 and 5.2 are independent from each other.

As an interesting consequence of the above characterizations we get the following result.

Proposition 5.4. *For a function $I: [0,1]^2 \to [0,1]$ the following statements are equivalent*:

(i) *I is a continuous (S, N)-implication.*
(ii) *I is an (S, N)-implication generated from some continuous* t-*conorm S and some continuous fuzzy negation N.*

Proof. (i) $\Longrightarrow$ (ii). Let I be a continuous (S, N)-implication generated from a t-conorm S and a fuzzy negation N. The negation $N = N_I$ is continuous since $I(x, 0) = S(N(x), 0) = N(x)$ is continuous. Now, because of Theorem 5.1, S is uniquely determined, i.e., $S = S_I$, where S_I is given by (10). We show that S is continuous. By [12], Proposition 1.19 it is enough to show the continuity of S with respect to the second variable. Assume that S is not continuous with respect to the second variable in some point $(x_0, y_0) \in [0,1]^2$. Let $x_1 = \mathfrak{R}_I(x_0)$. By (10) we get that I is not continuous with respect to the second variable in the point (x_1, y_0); a contradiction. Therefore S is continuous.

(ii) $\Longrightarrow$ (i). This implication is obvious, since a composition of continuous functions is continuous. $\square$

The last two results in this section show some relationships between (S, N)-implications and their conjugates (cf. [14, p. 156]). By Φ we denote the family of all increasing bijections $\varphi: [0,1] \to [0,1]$. We say that functions $f, g: [0,1] \to [0,1]$ are Φ-conjugate, if there exists $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x) := \varphi^{-1}(f(\varphi(x))), \quad x \in [0,1].$$

Analogously, we say that functions of two variables $F, G: [0,1]^2 \to [0,1]$ are Φ-conjugate, if there exists $\varphi \in \Phi$ such that $G = F_\varphi$, where

$$F_\varphi(x, y) := \varphi^{-1}(F(\varphi(x), \varphi(y))), \quad x, y \in [0,1].$$

Theorem 5.5 (*cf. Baczyński [4, Proposition 21]*). *Let $\varphi \in \Phi$. If $I_{S,N}$ is an (S, N)-implication generated from some* t-*conorm S and some fuzzy negation N, then the Φ-conjugate of $I_{S,N}$ is also an (S, N)-implication generated from the Φ-conjugate* t-*conorm of S and the Φ-conjugate fuzzy negation of N, i.e., $(I_{S,N})_\varphi$ is the (S, N)-implication given by*

$$(I_{S,N})_\varphi(x, y) = I_{S_\varphi, N_\varphi}(x, y), \quad x, y \in [0,1].$$

Proof. Let $I_{S,N}$ be an (S, N)-implication based on the suitable functions and $\varphi \in \Phi$. It can be easily shown that N_φ is also a fuzzy negation. Moreover, because of Proposition 2.28(iv) in [12] S_φ is a t-conorm. Hence I_{S_φ, N_φ} is an (S, N)-implication and for all $x, y \in [0,1]$ we get

$$\begin{aligned}(I_{S,N})_\varphi(x, y) &= \varphi^{-1}(I_{S,N}(\varphi(x), \varphi(y))) = \varphi^{-1}(S(N(\varphi(x)), \varphi(y))) \\ &= \varphi^{-1}(S(\varphi \circ \varphi^{-1}(N(\varphi(x))), \varphi(y))) \\ &= \varphi^{-1}(S(\varphi(N_\varphi(x)), \varphi(y))) = S_\varphi(N_\varphi(x), y) \\ &= I_{S_\varphi, N_\varphi}(x, y). \quad \square\end{aligned}$$

Analogously we can prove

Theorem 5.6. *Let I be a fuzzy implication, N a fuzzy negation and $\varphi \in \Phi$. If $S_{I,N}$ given by* (9) *is a* t-*conorm, then the Φ-conjugate of $S_{I,N}$ is also a* t-*conorm generated from the Φ-conjugate fuzzy implication of I and the Φ-conjugate fuzzy negation of N, i.e., the function $(S_{I,N})_\varphi$ is a* t-*conorm given by*

$$(S_{I,N})_\varphi(x, y) = S_{I_\varphi, N_\varphi}(x, y), \quad x, y \in [0,1].$$

6. Conclusion

In this paper we presented a new characterization of S-implications (Theorem 5.1 with mutually independent requirements). We also proved characterizations of (S, N)-implications generated from continuous negations and strict negations. Obtained characterizations are, in our opinion, interesting, because in all the results (Theorems 2.6, 5.1 and 5.2) we have properties that are mutually independent. We should note here, that the proof of Theorem 5.1 can be different from that presented here and based only on Darboux property (cf. [2]). Our goal was to show the uniqueness of the representation of (S, N)-implications and the formula (10) of a t-conorm from which (S, N)-implication is generated.

We see also that our method cannot be adopted for non-continuous negations (cf. Remark 4.4). Moreover, the representation of (S, N)-implications in this case may not be unique (see Example 1.5). All this discussion leaves us with the following question.

Problem 6.1. *What is the characterization of* (S, N)*-implications generated from non-continuous negations*?

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