Approximation Capability of SISO SBR Fuzzy Systems based on Fuzzy Implications

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Summary
In this work, we show that fuzzy inference systems based on Similarity Based Reasoning (SBR) where the modification function is a fuzzy implication is a universal approximator under suitable conditions on the other components of the fuzzy system.

Keywords: Similarity Based Reasoning, Fuzzy implications, Universal approximation.

1 Approximate Reasoning
The term approximate reasoning (AR) refers to methods and methodologies that enable reasoning with imprecise inputs to obtain meaningful outputs. AR schemes involving fuzzy sets are one of the best known applications of fuzzy logic in the wider sense. Fuzzy Inference Systems (FIS) have many degrees of freedom, viz., the underlying fuzzy partition of the input and output spaces, the fuzzy logic operations employed, the fuzzification and defuzzification mechanism used, etc. This freedom gives rise to a variety of FIS with differing capabilities. One of the important factors considered while employing an FIS is its approximation capability. Many studies have appeared on this topic and due to space constraints, we only refer the readers to the following exceptional review on this topic [10] and the references therein.

In this work, we consider a Similarity Based Reasoning (SBR) where similarity between the inputs and the antecedents is used to subsequently modify the consequents to obtain a final output. Such inference schemes are also known as plausible reasoning [3]. After detailing the inference mechanism in an SBR, we show that when the modification functions are modeled based on fuzzy implications, under suitable conditions on the other components of an SBR, the FIS based on SBR do become a universal approximator, i.e., can approximate a continuous function over a compact set to arbitrary accuracy. Also we deal only with single variable functions, alternately where the rule base consists of Single Input Single Output (SISO) rules.

2 Similarity Based Reasoning (SBR)
Definition 2.1 ([1, 5]). (i) A function \( T : [0, 1]^2 \to [0, 1] \) is called a t-norm, if it is increasing in both variables, commutative, associative and has 1 as the neutral element.

(ii) A function \( I : [0, 1]^2 \to [0, 1] \) is called a fuzzy implication if it is non-increasing in the first variable, increasing in the second variable and \( I(0, 0) = 1, I(1, 1) = 1, I(1, 0) = 0 \). The set of all fuzzy implications will be denoted by \( FI \).

(iii) If \( X \) is a non-empty set then \( F(X) \) is the fuzzy power set of \( X \), i.e., \( F(X) = \{ A | A : X \to [0, 1] \} \).

(iv) For an \( A \in F(X) \), the support of \( A \) is \( \text{Supp} A = \{ x \in X | A(x) > 0 \} \).

(v) A fuzzy IF-THEN rule is of the form

\[
\text{IF } \tilde{x} \text{ is } A \text{ THEN } \tilde{y} \text{ is } B ,
\]

where \( \tilde{x}, \tilde{y} \) are linguistic variables and \( A \in F(X) \), \( B \in F(Y) \) are linguistic expressions/values assumed by the linguistic variables over suitable universes of discourse \( X, Y \).

2.1 Similarity Based Reasoning (SBR)
Consider the fuzzy if-then rule (1). Let the given input be \( \tilde{x} = A' \). Inference in Similarity Based Reasoning (SBR) schemes in AR is based on the calculation of a measure of compatibility or similarity \( M(A, A') \) of the input \( A' \) to the antecedent \( A \) of the rule, and the use of a modification function \( J \) to modify the consequent \( B \), according to the value of \( M(A, A') \).
Some of the well-known examples of SBR are Compatibility Modification Inference (CMI) [4], “Approximate Analogical Reasoning Scheme” (AARS) in [11] and “Consequent Dilation Rule” (CDR) in [7], Smets and Magrez [8], Chen [2], etc. In this section, we detail the typical inferencing mechanism in SBR, but only in the case of SISO fuzzy rule bases.

2.2 Matching function M

Given two fuzzy sets, say $A, A'$, on the same domain, a matching function $M$ compares them to get a degree of similarity, which is expressed as a real in the $[0, 1]$ interval. We refer to $M$ as the Matching Function in the sequel. Formally, $M : \mathcal{F}(X) \times \mathcal{F}(X) \rightarrow [0, 1]$.

Example 2.2. Let $X$ be a non-empty set and $A, A' \in \mathcal{F}(X)$. Below we list a few of the matching functions employed in the literature.

- Zadeh [12]: $M_Z(A, A') = \max_{x \in X} \min(A(x), A'(x))$.
- Magrez - Smets [8]: Given a fuzzy negation $N$, $M_M(A, A') = \max_{x \in X} \min(N(A(x)), A'(x))$.
- Measure of Subsethood [7]: For an $I \in \mathcal{I}$, $M_S(A, A') = \min I(A'(x), A(x))$.

2.3 Modification Function $J$

Let $A'$ be the fuzzified input and $s = M(A, A') \in [0, 1]$, a measure of the compatibility of $A'$ to $A$.

The modification function $J$ is again a function from $[0, 1]^2$ to $[0, 1]$ and, given the rule (1), modifies $B \in \mathcal{F}(Y)$ to $B' \in \mathcal{F}(Y)$ based on $s$, i.e., the consequence in SBR, using the modification function $J$, is given by

$$B'(y) = J(s, B(y)) = J(M(A, A'), B(y)), \quad y \in Y.$$  

In AARS [11] the following modification operators have been used:

(i) $J_{ML}(s, B) = B'(x) = \min\{1, B(x)/s\}, x \in X$;
(ii) $J_{MVR}(s, B) = B'(x) = s \cdot B(x), x \in X$.

In CMI [4] and CDR [7] $J$ is taken to be a fuzzy implication operator. In fact, $J_{ML}(s, B) = I_{GG}(s, B)$, where $I_{GG}$ is the Goguen implication [1].

2.4 Aggregation Function $G$

In the case of multiple rules

$$R_i: \text{IF } \tilde{x} \text{ is } A_i \text{ THEN } \tilde{y} \text{ is } B_i, \quad i = 1, 2, \ldots, m,$$

we infer the final output by aggregating over the rules, using an associative operator $G : [0, 1]^2 \rightarrow [0, 1]$:

$$B'(y) = G_{i=1}^{m}\left(J\left(M(A_i, A'), B_i(y)\right)\right), \quad y \in Y. \quad (2)$$

Usually, $G$ is a $t$-norm, $t$-conorm or a uninorm [5].

3 Fuzzy Systems $\mathcal{F}$ based on SBR

An SBR fuzzy inference system can be represented by the quintuple $F = \{R(A_i, B_j), M, J, G, g\}$ where

- $R$ is the fuzzy if-then rule base formed from the fuzzy partitions $\{A_i\}, \{B_j\}$ on $X, Y$, respectively;
- $M$ is any matching function,
- $J$ is any modification function,
- $G$ is any aggregation function, and
- $g : \mathcal{F}(Y) \rightarrow Y$ is any defuzzifier.

We consider $F$ with the following assumptions on the different components / elements.

3.1 The Fuzzy Partitions $A_i, B_i$

We assume that the fuzzy sets $\{A_k\}_{k=1}^{m}$ partitioning the input and space $X$ are continuous, convex, of finite support and satisfy the following properties:

Normal: For any $k \in \mathbb{N}_n$, there exists $x_k \in U$ such that $A_k(x_k) = 1$;

Complete: For any $x \in U$, there exists $k \in \mathbb{N}_n$, such that $A_k(x) > 0$; and

Ruspini Partition: $\sum_{k=1}^{m} A_k(x) = 1$ for every $x \in X$.

Hence, $\{A_k\}_{k=1}^{m}$ are also consistent, i.e., if $A_j(x) = 1$ then $A_j(x) = 0$ for any $j \neq k$.

Similar assumptions hold on the fuzzy partition $\{B_k\}_{k=1}^{m}$ on the output space $Y$.

3.2 The Fuzzified Input $A'$

Let $x' \in X$ be the given input. We fuzzify it to an $A' \in \mathcal{F}(X)$ such that $A'$ attains normality at $x'$, i.e., $A'(x') = 1$. It is with this fuzzified input $A'$ the antecedents $A_i$ of the different rules are matched against. Moreover, it is assumed that the $A'$ also belongs to the same class of fuzzy sets with identical parameters as the $\{A_k\}$, i.e., if $\{A_k\}$ are all triangular fuzzy sets with supports of constant length and particular symmetry then so is $A'$. Thus $A'$ intersects only any two adjacent fuzzy sets $\{A_k\}$, i.e., $\text{Supp } A' \cap \text{Supp } A_k \neq \emptyset$ if and only if $k = i, i + 1$ for some $i \in \mathbb{N}_{n-1}$. Moreover if $x' = x_k$ for some $k$ then $A' = A_k$. 

3.3 The Operations $M, J, G$

We choose a matching function $M$ such that

- $M$ is **Continuous on Finite Support fuzzy sets**: For any $A \in F(X)$ and for any given $\epsilon$ there exists a $\delta > 0$ such that whenever $\|A' - A''\| < \delta$ for any $A', A'' \in F(X)$ then $|M(A, A') - M(A, A'')| < \epsilon$, where

  \[ \|A' - A''\| = \max \{ |\text{Supp } A' \setminus \text{Supp } (A' \cap A'')|, \\
  |\text{Supp } A'' \setminus \text{Supp } (A' \cap A'')| \}. \]

- $M$ is **Consistent**: For any $A' \in F(X)$ and the given fuzzy partition $\{A_k\}$ we have that

  \[ \sum_{k=1}^{n} M(A', A_k) \leq 1. \quad (3) \]

Clearly, the choice of $M$ now depends on the given fuzzy partition $\{A_k\}$.

For instance, if $\{A_k\}$ are given by symmetric triangular functions of constant support, then for any $A, A', A'' \in \{A_k\}$, the Zadeh’s matching function $M_Z$ is continuous w.r.t to $\|\|$. Similarly, let the fuzzy partition $\{A_k\}$ be as given in Section 3.1. Now, if $x' \in X$ is the input let $A' \in F(X)$ be the fuzzified input such that $A'$ attains normality at $x'$, i.e., $A'(x') = 1$. Then the matching function defined as $M(A', A) = A(x')$ for any $A \in F(X)$ has the property (3).

We choose the modification function $J$ to be a fuzzy implication, i.e., $J = I \in \mathcal{FI}$. For notational convenience we will denote it by " $\rightarrow$ " in the sequel.

The aggregation function $G$ is any t-norm $T$.

3.4 The Fuzzy Output $B'$

With the above operations $M, J, G$ the fuzzy output for a given input $x' \in X$ is given by (2) as follows:

\[ B'(y) = T_{k=1}^{n}[M(A', A_k) \rightarrow B_k(y)]. \quad (4) \]

By our assumption on $A', A_k$, viz., that $A'$ intersects only two adjacent fuzzy sets among the $\{A_k\}$, say $A_i, A_j$ with $j = i - 1$ or $i + 1$, we have that $M(A', A_k) = 0$ for all $k \neq i, j$. Note also that $I(0, y) = 0 \rightarrow y = 1$ for any $y \in [0, 1]$. Now, letting $N_y = N_n \setminus \{i, j\}$, the fuzzy output $B'(y)$ for any $y \in Y$ which is given by (4) becomes

\[ B'(y) = T \left[ T_{k \in N_y}(M(A', A_k) \rightarrow B_k(y)), \right. \]

\[ \left. M(A', A_i) \rightarrow B_i(y), M(A', A_j) \rightarrow B_j(y) \right] \]

\[ = T [M(A', A_i) \rightarrow B_i(y), M(A', A_j) \rightarrow B_j(y)] \]

\[ = T [s_i \rightarrow B_i(y), s_j \rightarrow B_j(y)], \quad \text{(SBR)} \]

where $s_i = M(A', A_i)$ and $s_j = M(A', A_j)$. Note that by our assumption on $M$, we have that $s_i + s_j \leq 1$.

3.5 The Defuzzified Output $g(x')$

We have chosen the modification function $J$ to be a fuzzy implication, i.e., $J = I \in \mathcal{FI}$. An $I \in \mathcal{FI}$ is said to satisfy the **ordering property** if for all $x, y \in [0, 1]$

\[ x \leq y \iff I(x, y) = x \rightarrow y = 1. \quad (OP) \]

Thus the class $\mathcal{FI}$ can be partitioned into those that have $(OP)$ and those that do not. Depending on whether the considered modification function $J$ has $(OP)$ or not, we define the defuzzification function $g$ appropriately so that $g$ is continuous. In the following, we discuss the explicit formulae for $g$ under the different cases. Note that $g$ is also known as the system function of the fuzzy system $F$.

3.5.1 When $J$ has $(OP)$

**Lemma 3.1.** Let $\mathcal{F} = \{R_{\{A_i, B_j\}}, M, J, G, g\}$ such that the fuzzy partitions $\{A_k\}, \{B_k\}$ and the operations $M, G$ are as given in Sections 3.1 - 3.4. If the modification function $J = I \in \mathcal{FI}$ with $(OP)$, the system function $g$ of $\mathcal{F}$ is defined as in (6) and is continuous.

**Proof.** Let $x' \in X$ be the given input and $A_i, A_j$ $(i < j)$ be the adjacent input fuzzy sets that the fuzzified input $A'$ intersects. The output fuzzy set $B'$ is given by (SBR). We consider the kernel of $B'$, i.e., $\text{Ker } B' = \{y : B'(y) = 1\}$. We choose the defuzzified output $y'$ such that it belongs to $\text{Ker } B'$. Since $\{B_k\}$ form a Rusmini partition, $B_i(y) + B_j(y) = 1$ for all $y \in Y$.

Since $T$ is a t-norm, we know that $T(p, q) = 1$ if and only if $p = 1$ and $q = 1$. Noting that $J$ has $(OP)$, i.e.,

\[ p \rightarrow q = 1 \iff p \leq q \text{ and } s_i + s_j \leq 1, \]

we have

\[ \text{Ker } B' = \{y : B'(y) = 1\} = \{y : s_i \leq B_i(y)\} \cap \{y : s_j \leq B_j(y)\}. \]

Let $\alpha_i = \min\{s_i : \alpha = 1\}$ and $\beta_j = \min\{s_j : \beta = 1\}$. Since $J$ has $(OP)$, clearly $\alpha_i = s_i$ and $\beta_j = s_j$.

By the continuity and convexity of $B_i, B_j$ there exist $a_i, b_i, a_j, b_j$ such that $B_i(a_i) = B_i(b_i) = s_i$ and $B_j(a_j) = B_j(b_j) = s_j$. By the monotonicity of the implication in the second variable, for every $y \in [a_i, b_i]$ we have that $s_i \rightarrow B_i(y) = 1$ and for every $y \in [a_j, b_j]$ we have that $s_j \rightarrow B_j(y) = 1$. Thus we have

\[ \{y : s_i \leq B_i(y)\} = [a_i, b_i] \text{ and } \{y : s_j \leq B_j(y)\} = [a_j, b_j], \]

and hence

\[ g(x') = T_{\alpha_i, \beta_j}(\alpha_i, \beta_j), \]

where $\alpha_i = \min\{s_i : \alpha = 1\}$ and $\beta_j = \min\{s_j : \beta = 1\}$.
and hence

\[
\text{Ker } B' = \{ y : B'(y) = 1 \} = [a_i, b_i] \cap [a_j, b_j]. \tag{5}
\]

**Claim:** Ker \( B' = [a_j, b_i] \neq \emptyset \).

Firstly, note that for any \( s_i \in [0, 1] \) by the normality of \( B_i \) we have that \( B_i(y_i) = 1 \) and hence \( y_i \in \{ y : s_i \leq B_i(y_i) \} = y_i \in [a_i, b_i] \neq \emptyset \). Similarly, \( y_j \in [a_j, b_j] \neq \emptyset \).

It suffices to show that \( a_j \leq b_i \) from whence \( \text{Ker } B' = [a_j, b_i] \).

Note that since \( i < j \), \( y_i < y_j \) and from \( a_j \in \text{Supp } B_j \) we have that \( y_i \leq a_j \leq y_j \). Similarly, \( y_i \leq b_i \leq y_j \).

Hence, \( y_i \leq a_j, b_i \leq y_j \).

Since \( B_j \) is monotonic on \([y_i, y_j]\),

\[
a_j > b_i \implies B_j(a_j) \geq B_j(b_i) \\
\implies s_j \geq 1 - B_i(b_i) \\
\implies s_j \geq 1 - s_i \\
\implies s_i + s_j \geq 1. 
\]

On the one hand, if \( s_i + s_j > 1 \), then it is a contradiction to the fact that \( s_i + s_j \leq 1 \), since \( M \) satisfies (3).

On the other hand,

\[
s_i + s_j = 1 \implies B_j(a_j) + B_j(b_i) = 1 \\
\implies B_j(a_j) = 1 - B_i(b_i) \\
\implies B_j(a_j) = B_j(b_i) \\
\implies b_i \in [a_j, b_i], \text{ i.e., } a_j \leq b_i.
\]

Now, we define \( g(x') \) as follows:

\[
g'(x') = g(x') = \frac{s_i b_i + s_j a_j}{s_i + s_j}. \tag{6}
\]

Clearly, from the continuity of \( B_i, B_j, M \) we have that \( g \) is continuous.

### 3.5.2 When \( J \) does not have (OP)

When \( J \) does not have (OP), we again consider two subcases.

An \( I \in \mathcal{FI} \) is said to have the pseudo-contrapositivity property (PCP) w.r.t. to 1 if the following holds:

For any \( \alpha, \beta \in [0, 1] \), whenever \( I(\alpha, \beta) = 1 \) (PCP) then \( I(1 - \alpha, \gamma) = 1 \), for some \( 0 < \gamma < 1 - \beta \).

**Lemma 3.2.** Let \( \mathcal{F} = \{ \mathcal{R}(A_i, B_i), M, J, G \} \) such that the fuzzy partitions \( \{ A_k \}, \{ B_k \} \) and the operations \( M, G \) are as given in Sections 3.1 - 3.4. If the modification function \( J = I \in \mathcal{FI} \) does not satisfy (OP) but satisfies (PCP), the system function \( g \) of \( \mathcal{F} \) is defined as in (6) and is continuous.

**Proof.** Let \( x' \in X \) be the given input. Continuing along the lines of the proof given in Lemma 3.1 we find that Ker \( B' \) is given as in (5). Note that since \( J \) does not satisfy (OP) \( \alpha_i, \beta_j \) can be less than \( s_i, s_j \), respectively.

**Claim:** Ker \( B' = [a_j, b_i] \neq \emptyset \).

Note, once again, that for any \( s_i \in [0, 1] \) by the normality of \( B_i \) we have that \( B_i(y_i) = 1 \) and hence \( y_i \in \{ y : s_i \rightarrow B_i(y_i) = 1 \} = [a_i, b_i] \neq \emptyset \). Similarly, \( y_j \in [a_j, b_j] \neq \emptyset \).

Also \( s_j \leq 1 - s_i \) and \( B_j(y) = 1 - B_i(y) \) and hence

\[
y \in [y_i, b_i] \implies s_i \rightarrow B_i(y) = 1 \\
\implies (1 - s_i) \rightarrow \gamma = 1, \\
\text{for some } 0 < \gamma < 1 - B_i(y) \\
\implies J \text{satisfies (PCP)} \\
\implies 1 - B_i(y) = 1 \\
\implies J \text{ is increasing in the second variable implies } s_j \rightarrow 1 - B_i(y) = 1 \\
\implies J \text{ is non-increasing in the first variable implies } y \in [a_j, y_j], \text{ hence the claim.} \]

As in the proof of Lemma 3.1 we can show that

\[
\text{Ker } B' = \{ y : B'(y) = 1 \} = [a_i, b_i] \cap [a_j, b_j] = [a_j, b_i], 
\]

and \( g \) defined as in (6) is continuous.

**Lemma 3.3.** Let \( \mathcal{F} = \{ \mathcal{R}(A_i, B_i), M, J, G \} \) such that the fuzzy partitions \( \{ A_k \}, \{ B_k \} \) and the operations \( M, G \) are as given in Sections 3.1 - 3.4. If the modification function \( J = I \in \mathcal{FI} \) does not satisfy either (OP) or (PCP), the system function \( g \) of \( \mathcal{F} \) is defined as in (7) and is continuous.

**Proof.** As in the proof of Lemma 3.2, we find that the Ker \( B' \) is given by (5). However, note that this intersection may be empty, i.e., Ker \( B' = \emptyset \). Hence, we define the defuzzification function \( g \) as the weighted average of the matching values and the values of \( y \) at which both \( B_i, B_j \) attain normality, i.e.,

\[
y' = g(x') = \frac{s_i y_i + s_j y_j}{s_i + s_j}. \tag{7}
\]

Clearly, \( g \) is continuous.

**Remark 3.4.** Note that with \( g \) either as in (6) or (7), if \( x' = x_k \in X \) we have that \( A' = A_k \) and we obtain \( B' = B_k \), i.e., \( g(x') = y_k \) and the interpolativity of the inference is preserved.
4 SBR Fuzzy Systems and Universal Approximation

In this section, we show that $\mathbb{F} = \{R(A_i, B_i), M, J, G, g\}$ such that the fuzzy partitions $\{A_i\}, \{B_i\}$ and the operations $M, J, G, g$ as given in Sections 3.1 - 3.5 are universal approximators, i.e., they can approximate any continuous function over a compact set to arbitrary accuracy.

In fact, the results in this section are largely based on the proof given by Li et al., [6], since under the assumptions on $\mathbb{F}$ the arguments used in the proof of [6], Theorem 3.4 can be applied here with suitable modifications.

**Theorem 4.1.** For any continuous function $f: [a, b] \rightarrow \mathbb{R}$ over a closed interval and an arbitrary given $\varepsilon > 0$, there is an SBR fuzzy system $\mathbb{F} = \{R(A_i, B_i), M, J, G, g\}$ with $M$ having the property (3), $J$ having (OP), $G$ being a $t$-norm and $g$ as given in (6) such that $\max_{x \in [a, b]} |f(x) - g(x)| \leq \varepsilon$.

**Proof.** Since $f$ is continuous, the image of $f$ is also a closed interval, let it be $[c, d]$. For any given $\varepsilon$, there exists an $N \in \mathbb{N}$ such that $\frac{d - c}{N} < \varepsilon/2$. Let $\varepsilon = \frac{d - c}{N}$.

Suppose $y_1 = c$, $y_2 = c + \varepsilon$, $y_3 = c + 2\varepsilon$, ..., $y_{N+1} = d$. Let us construct the fuzzy partition $\{B_k\}_{k=1}^{N+1}$ on the output space $Y$ as follows:

- the support of $B_1$ is $[y_1, y_2]$,
- the support of $B_{N+1}$ is $(y_N, y_{N+1}]$,
- the support of $B_k$ is $(y_{k-1}, y_{k+1})$ for $1 < k < N + 1$,
- every $B_k$ attains normality at $k$, i.e., $B_k(y_k) = 1$ for $k = 1, 2, \ldots, N + 1$.

Let us define $U_k \subset [a, b]$ as follows:

$$U_1 = f^{-1}\left([y_1, y_1 + \frac{2\varepsilon}{3}]\right),$$

$$U_k = f^{-1}\left([y_k - \frac{2\varepsilon}{3}, y_k + \frac{2\varepsilon}{3}]\right), \quad k = 2, \ldots, N,$$

$$U_{N+1} = f^{-1}\left([y_{N+1} - \frac{2\varepsilon}{3}, y_{N+1}]\right).$$

Clearly, by the continuity of $f$ we see that $\{U_1, U_2, \ldots, U_{N+1}\}$ forms an open cover of $[a, b]$ and $U_i \cap U_j \neq \emptyset$ if and only if $i$, $j$ are adjacent numbers.

Since $f$ is uniformly continuous over a compact set $[a, b]$, there exists $\delta > 0$, such that $|f(x) - f(x')| < \varepsilon/2$ whenever $|x - x'| < \delta$. Therefore, there exist finite numbers $a = x_1 < x_2 < \ldots < x_n = b$ such that $|x_i - x_{i+1}| < \frac{\delta}{2}$ for $i = 1, 2, \ldots, n - 1$ such that the following claim is true.

**Claim:** Any open interval $(x_i, x_{i+2})$ meets $\{U_k\}$ at most two adjacent elements, for $i = 1, 2, \ldots, n - 2$.

If possible let there exist $j_1 < j_2 < j_3$ such that at least three elements $U_{j_1}, U_{j_2}, U_{j_3}$ intersect with $(x_i, x_{i+2})$. Then $U_{j_1} \cap U_{j_2} \neq \emptyset$. Let us choose $a_m \in U_{j_m} \cap (x_i, x_{i+2})$ for $m = 1, 2, 3$. Then we have

$$f(a_m) \in \left(y_{j_m} - \frac{2\varepsilon}{3}, y_{j_m} + \frac{2\varepsilon}{3}\right),$$

and

$$|f(a_1) - f(a_3)| > \left|y_{j_3} - \frac{2\varepsilon}{3} - \left(y_{j_1} + \frac{2\varepsilon}{3}\right)\right| = \left(y_{j_3} - y_{j_1}\right) - \frac{4\varepsilon}{3}.

By the choice of $y_{j_1}, y_{j_3}$ we have $y_{j_3} - y_{j_1} \geq 2\varepsilon$ and hence

$$|f(a_1) - f(a_3)| > 2e - \frac{4\varepsilon}{3} = \frac{2e}{3} > \frac{e}{2}.

(8)

Again since $a_1, a_3 \in (x_i, x_{i+2})$ and

$$|x_{i+2} - x_i| \leq |x_{i+2} - x_{i+1}| + |x_{i+1} - x_i| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,

we have that $|a_1 - a_3| < \delta$. From this it follows that $|f(a_1) - f(a_3)| < \frac{e}{2}$, which is a contradiction to (8). \[\square\]

Now we construct the fuzzy sets $\{A_i\}_{i=1}^n$ that form a fuzzy partition on $X$ such that the support of $A_i$, $\text{Supp } A_i = (x_{i-1}, x_{i+1})$, each $A_i$ is convex, continuous, attains normality at $x_i$, i.e., $A_i(x_i) = 1$ and $M$ is continuous on $\{A_i\}_{i=1}^n$. Note that for notational convenience we assume there exist $x_0, x_{n+1}$ such that $|x_0 - x_1| < \frac{\delta}{2}$ and $|x_n - x_{n+1}| < \frac{\delta}{2}$.

The fuzzy rule base $R(A_i, B_i)$ is designed such that each rule $R_i$ is given as follows:

IF $\bar{x}$ is $A_i$ THEN $\hat{y}$ is $C_i$, \quad $i = 1, 2, \ldots, n$, \quad (9)

where the $C_i$ are chosen as follows:

For the given $A_i$ which attains normality at $x_i \in X$, there exists $k \in \{1, 2, \ldots, N + 1\}$ such that $x_i \in U_k$ and we let $C_i = B_k$ for this particular $k$. Note that there are at most two $U_k$ such that $x_i \in U_k$ by the construction of $U_k$.

Let an input $x' \in [a, b]$ be given. Clearly, $x' \in (x_i, x_{i+1})$ for some $i \in \mathbb{N}_0$. We fuzzify $x'$ to an appropriate $A'$ such that $|\text{Supp } A'| < \delta$ and $A'(x') = 1$. By the assumptions on the different components of
the SBR fuzzy system $F$, we see that $A'$ intersects with some $A_i, A_{i+1}$ and hence the output $y' = g(x')$ is given by (6) as follows:

$$y' = g(x') = \frac{s_{i+1}a_{i+1} + s_i b_i}{s_{i+1} + s_i},$$

where $a_{i+1}, b_i \in \text{Supp } C_i \cap \text{Supp } C_{i+1} \subset \text{Supp } C_i$ and $C_i = B_{k-1}, B_{k} \text{ or } B_{k+1}$. Without loss of generality, let us assume that $C_i = B_{k-1}$. Thus $[a_{i+1}, b_i] \subset [y_k, y_{k+2}]$ and hence $g(x') = y' \in [a_{i+1}, b_i] \subset [y_k, y_{k+2}]$. In fact, $y' \in [y_k, y_{k+1}]$ or $y' \in [y_{k+1}, y_{k+2}]$ and so, $\max \{|y' - y_k|, |y' - y_{k+1}|\} \leq \epsilon$.

Noting that $x'$ belongs to at most two adjacent open sets $U_k$ or $U_{k+1}$, we have that either $|f(x') - y_k| < \frac{2\epsilon}{3}$ or $|f(x') - y_{k+1}| < \frac{2\epsilon}{3}$. Thus,

$$|g(x') - f(x')| = |y' - f(x')| \leq \min \left\{|y' - y_k| + |y_k - f(x')|, \right.$$  
$$|y' - y_{k+1}| + |y_{k+1} - f(x')|\left\} \leq \epsilon + \frac{2\epsilon}{3} = \frac{5\epsilon}{3} < \frac{5\epsilon}{6} < \epsilon.$$  

\qed

**Theorem 4.2.** For any continuous function $f: [a, b] \rightarrow R$ over a closed interval and an arbitrary given $\epsilon > 0$, there is an SBR fuzzy system $F = \{A_i, B_i, \mathcal{R}, M, J, G, g\}$ with $M$ having the property (3), $G$ is any t-norm and either

- $J$ does not have (OP) but satisfies (PCP) and $g$ is as given in (6), or
- $J$ does not satisfy either (OP) or (PCP) and $g$ is as given in (7),

such that $\max_{x \in [a, b]} |f(x) - g(x)| \leq \epsilon$.

**Remark 4.3.** (i) From Theorem 4.2, with $g$ as given in (7) the SBR fuzzy system $F$ under the above assumptions becomes the TS-fuzzy system and hence the proof given here can be thought of as an alternate proof to the result on the universal approximation of TS-fuzzy systems.

(ii) In the case $M = M_{Z}, J = I \in \mathcal{T}, G = \min$, the fuzzy output $B'$ given by (2) of the SBR fuzzy system $F$ is equivalent to the $B'$ obtained from the BK-Subproduct inference, see [9] and hence these results show that even SISO BK-Subproduct inference mechanism is a universal approximator under the given assumptions.

**References**


