Exact Analysis of the Piecewise Linear Combiner for Decode and Forward Cooperation with Three Relays

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***Abstract*—Exact expressions are obtained for the bit error rate (BER) for coherent and noncoherent decode and forward (DF) cooperative systems with upto three relays between the source and destination. The piecewise linear (PL) combiner is employed at the receiver. BER analysis is done using a contour integral approach for evaluating the Gil-Pelaez integral involving the characteristic function (CF) of the decision variable. This removes restrictions on relay location, imposed by the direct approach. Simulation results are provided to support the analysis and the relay diversity gain is demonstrated through BER plots.**

***Index Terms*—Cooperative diversity, Gil-Pelaez, Residue theorem, PL, DF.**

I. Introduction

The performance of maximum likelihood (ML) decode and forward (DF) cooperative systems [1]–[5] is of considerable interest as it provides a benchmark for all other cooperative schemes. The piecewise linear (PL) combiner [4] and cooperative-maximal ratio combining (C-MRC) receiver [6] have been suggested as useful alternatives to the ML detector, which is nonlinear. While C-MRC relies on instantaneous channel knowledge on the source-relay link, PL requires the average BER on that link, which may be easier to obtain. Thus, there exist scenarios where PL may be preferred to C-MRC, leading to interest in its performance despite C-MRC achieving full diversity [6] and the PL combiner losing half the diversity order [4].

Exact bit error rate (BER) analysis for the piecewise linear (PL) combiner was provided in [7] and [4] respectively, for coherent and noncoherent detection for a single relay system employing DF. Since multirelay systems are known to have a higher diversity order [4], [6], it is important to have a measure of their actual performance. This is the focus of our work, and we obtain exact expressions for the BER for multirelay PL-DF systems. For noncoherent PL-DF in a two relay system, exact expressions for the BER were obtained in [8]. However, due to multiple integration regions in the BER analysis, the results in [8] were limited to the case when both relays were equidistant from the source, an impractical scenario. A superior contour integral approach [9] is employed in this paper to remove this restriction on the relay location [10] to obtain exact expressions for the BER for binary phase shift keying (BPSK) and binary frequency shift keying (BFSK) (coherent and noncoherent). We also obtain an exact expression for the BER for three relay systems, which is not available in the existing literature.

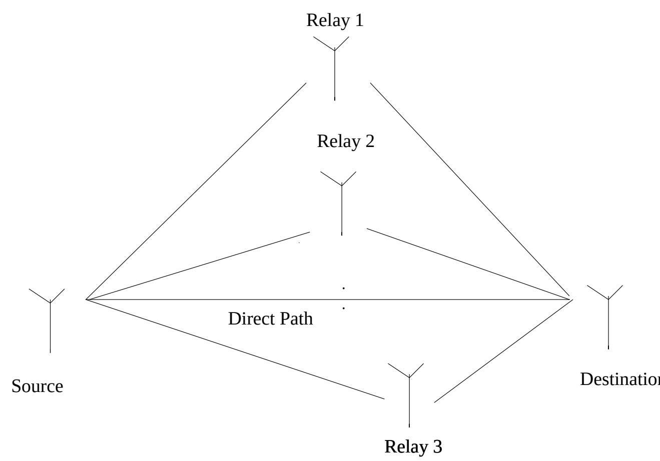

Fig. 1. Cooperative system with three relays.

Since the detection rule in [4] for higher order modulation schemes does not lend itself to meaningful analysis, our results are restricted to binary modulation and hence inapplicable to current high-rate systems, where combining schemes with more general modulation assumptions [11] can be employed. While a closed form BER expression is difficult for systems with more than three relays, our analysis indicates the feasibility of an algorithmic approach for its computation and also a a novel application of the residue theorem [12] through multiple contours.

II. System Model

We consider the model in [4], [7] with upto $N = 3$ relays, operating in full duplex mode with negligible processing delay, between the source and destination, as shown in Figure 1. The DF method is employed at the relay followed by ML decision at the destination. The modulation technique is BPSK and BFSK for coherent and noncoherent detection; all transmissions being on orthogonal channels experiencing Rayleigh fading. Henceforth, without loss of generality, the source (S), relay (R) and destination (D) parameters will be represented by the subscripts s, r and d respectively. For BPSK and BFSK, let $x_i \in \{1, -1\}$ and $x_i \in \{0, 1\}$, $i \in \{s, r\}$ be the respective bits transmitted at the source and rth relay, with powers E_s and E_r respectively. The received symbols on the S-D, S-R and R-D links for $r = 1, \ldots, N$ are respectively,

TABLE I: ML decision parameters for BPSK and coherent and noncoherent BFSK, $i \in \{s, r\}, m \in \{0, 1\}, \gamma_{r,s} = \frac{\Omega_{r,s} E_s}{N_0}, \gamma_{d,i} = \frac{\Omega_{d,i} E_i}{N_0}$

Modulation Scheme	ML Decision	X, Y_r	ϵ_r	α_i, β_i
BPSK	$X + \sum_{r=1}^{n} f(Y_r) \underset{-1}{\overset{1}{\gtrless}} 0$	$\frac{4\sqrt{E_i}\mathrm{Re}\{h_{d,i}^* y_{d,i}\}}{N_0}$	$\frac{1}{2}\left[1 - \left(1 + \gamma_{r,s}{}^{-1}\right)^{-\frac{1}{2}}\right]$	$\frac{1}{2}\left(\sqrt{1 + \frac{1}{\gamma_{d,i}}} \pm 1\right)$
Coherent BFSK	$X + \sum_{r=1}^{n} f(Y_r) \underset{0}{\overset{1}{\gtrless}} 0$	$\frac{2\sqrt{E_s}}{N_0}\mathrm{Re}\left[\left(y_{d,i}^0 - y_{d,i}^1\right) h_{d,i}^*\right]$	$\frac{1}{2}\left[1 - \left(1 + 2\gamma_{r,s}{}^{-1}\right)^{-\frac{1}{2}}\right]$	$\frac{1}{2}\left(\sqrt{1 + \frac{2}{\gamma_{d,i}}} \pm 1\right)$
Nonoherent BFSK	$X + \sum_{r=1}^{n} f(Y_r) \underset{0}{\overset{1}{\gtrless}} 0$	$\frac{\Omega_{d,i} E_i}{(\Omega_{d,i} E_i + N_0) N_0}\left(\lvert y_{d,i}^0\rvert^2 - \lvert y_{d,i}^1\rvert^2\right)$	$(2 + \gamma_{r,s})^{-1}$	$\frac{\left(1+\gamma_{d,i}^{-1}\right)}{\left[(m+(-1)^m x_s)\gamma_{d,i}+1\right]}$

BPSK: BFSK:

$$
\begin{aligned}
y_{d,s} &= \sqrt{E_s} h_{d,s} x_s + z_{d,s} & y_{d,s}^m &= (m + (-1)^m x_s)\sqrt{E_s} h_{d,s} + z_{d,s}^m \\
y_{r,s} &= \sqrt{E_s} h_{r,s} x_s + z_{r,s} & y_{r,s}^m &= (m + (-1)^m x_s)\sqrt{E_s} h_{r,s} + z_{r,s}^m \\
y_{d,r} &= \sqrt{E_r} h_{d,r} x_r + z_{d,r} & y_{d,r}^m &= (m + (-1)^m x_r)\sqrt{E_r} h_{d,r} + z_{d,r}^m,
\end{aligned}
\tag{1}
$$

where $z_{i,j}, z_{i,j}^m \sim \mathcal{CN}(0, N_0), m \in \{0, 1\}, h_{i,j} \sim \mathcal{CN}(0, \Omega_{i,j}), i \neq j, i \in \{r, d\}, j \in \{s, r\}$ represent the additive white Gaussian noise (AWGN) and the channel coefficients at the respective relays and the destination, m denoting the two BFSK channels. The link signal to noise ratio (SNR) is defined as $\gamma_{i,j} = \frac{\Omega_{i,j} E_j}{N_0}$.

A. ML Decision

The ML decision criterion at the destination for binary modulation is obtained from [4], [7], (Table I, cols. 2 & 3) where $\{^*\}$ denotes the complex conjugate operation and

$$
f(t) = \ln \frac{\delta + e^t}{1 + \delta e^t}, \quad 0 < \delta < 1. \tag{2}
$$

In $f(Y_r)$, $\delta_r = \frac{\epsilon_r}{1-\epsilon_r}$ for the rth relay, where ϵ_r is the average BER (Table I, col. 4) for the S-R link. Approximating $f(t)$ in (2), the PL combiner in ML-DF cooperation is [4]

$$
f(t) \approx \begin{cases} \ln \frac{1}{\delta} & t \geq \ln \frac{1}{\delta} \\ t & \ln \delta < t < \ln \frac{1}{\delta} \\ \ln \delta & t \leq \ln \delta \end{cases}. \tag{3}
$$

III. Problem Definition

Assuming equal probability of the symbols x_s, for $N = 1, 2, 3$, the average probability of error for the ML-DF cooperative diversity system in Figure 1 can be expressed as

$$
P_e = \begin{cases} \sum_{\mathbf{x}} \prod_{r=1}^{N} \epsilon_r^{\frac{1-x_r}{2}} (1 - \epsilon_r)^{\frac{1+x_r}{2}} P\left(X + \sum_{i=1}^{N} f(Y_i) < 0 | x_s = 1, \mathbf{x}\right) & \text{(BPSK)}, \\ \sum_{\mathbf{x}} \prod_{r=1}^{N} \epsilon_r^{1-x_r} (1 - \epsilon_r)^{x_r} P\left(X + \sum_{i=1}^{N} f(Y_i) < 0 | x_s = 1, \mathbf{x}\right) & \text{(BFSK)}, \end{cases} \tag{4}
$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is the set of all possible N-tuples formed by the symbols transmitted by the relays and X and Y_i are defined in Table I (col. 3). We wish to express P_e in closed form.

A. Statistics of X and Y_r

Dropping subscript s, the probability density function (PDF) and CF of X are [4], [7]

$$
p_X(x) = \begin{cases} g e^{-\beta x} & x \geq 0 \\ g e^{\alpha x} & x < 0 \end{cases} \quad \text{and}
$$

$$
\Phi_X(\jmath\omega) = \frac{g(\alpha + \beta)}{(\beta - \jmath\omega)(\alpha + \jmath\omega)}, \qquad g = \frac{\alpha\beta}{\alpha + \beta}, \ \alpha > 0, \beta > 0, \tag{5}
$$

with parameters α, β (Table I, col. 5) defined in terms of E and Ω. It is obvious that even Y_r (Table I, col. 3) has similar statistics. The CF of $V = f(X)$ for the PL approximation is [7]

$$
\Phi_V(\jmath\omega) = \left[1 - g\phi(\alpha, \jmath\omega) - g\phi(\beta, -\jmath\omega)\right], \text{where} \tag{6}
$$

$$
\phi(x, t) = \frac{t(1 - \delta^{x+t})}{x(x + t)}. \tag{7}
$$

B. Gil-Pelaez Inversion Formula

From the Gil-Pelaez theorem [13], the conditional probability in (4) can be expressed using (5) and (6) as [13]

$$
\begin{aligned}
P\left(X + \sum_{r=1}^{N} f(Y_r) < 0 | x_s = 1, \mathbf{x}\right) \\
&= \frac{1}{2} - \frac{1}{2\pi\jmath} \int_{-\infty}^{\infty} \frac{\Phi_X(\jmath\omega)}{\omega} \prod_{r=1}^{N} \Phi_{V_r}(\omega) d\omega, \\
&= \mathcal{A}\left(\Phi_{V_1}, \ldots, \Phi_{V_N}\right), \quad \text{say, where } V_r = f(Y_r).
\end{aligned}
\tag{8}
$$

C. Recursive BER

For the three relay system in Figure 1, from (8),

$$
\begin{aligned}
\mathcal{A}(\Phi_{V_1}, \Phi_{V_2}, \Phi_{V_3}) \\
&= \frac{1}{2} - \frac{1}{2\pi\jmath} \int_{-\infty}^{\infty} \frac{\Phi_X(\jmath\omega) \prod_{r=1}^{3} \Phi_{V_r}(\jmath\omega)}{\omega} d\omega
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&= \frac{1}{2} - \frac{1}{2\pi\jmath} \int_{-\infty}^{\infty} \frac{\Phi_X(\jmath\omega)}{\omega} d\omega + \sum_{\substack{l,r=1 \\ l<r}}^{3} \mathcal{A}(\Phi_{V_l}, \Phi_{V_r}) \\
&\quad - \sum_{r=1}^{3} \mathcal{A}(\Phi_{V_r}) + \frac{1}{2\pi\jmath} \int_{-\infty}^{\infty} \frac{\Phi_X(\jmath\omega) \mathcal{E}_3(\phi)}{\omega} d\omega.
\end{aligned}
\tag{10}
$$

upon substituting for $\Phi_{V_r}(\jmath\omega)$ from (6) and some algebra, with

$$\mathcal{A}(\Phi_{V_1},\Phi_{V_2}) = \sum_{r=1}^{2}\mathcal{A}(\Phi_{V_r}) - \frac{1}{2} + \frac{1}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)}{\omega}d\omega - \frac{1}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\mathcal{E}_2(\phi)}{\omega}d\omega, \tag{11}$$

$$\mathcal{A}(\Phi_{V_1}) = \frac{1}{2} - \frac{1}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)}{\omega}d\omega + \frac{1}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\mathcal{E}_1(\phi)}{\omega}d\omega \text{ and} \tag{12}$$

$$\mathcal{E}_r(\phi) = \prod_{l=1}^{r} g_l \sum_{k=1}^{2^r}\prod_{\substack{i\in\mathcal{I}_k\\ j\in\mathcal{J}_k}}\phi(\alpha_i,\jmath\omega)\phi(\beta_j,-\jmath\omega), \quad \mathcal{I}_k\cup\mathcal{J}_k = \{1,2,\ldots,r\}, \mathcal{I}_k\cap\mathcal{J}_k = \{\} \tag{13}$$

From (10) - (12), it is clear that a recursive relationship exists between the conditional BERs for systems with $N = 1, 2$ and 3. Further, it is sufficient to evaluate the integrals

$$J = \frac{1}{2} - \frac{1}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)}{\omega}d\omega \text{ and} \tag{14}$$

$$\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\mathcal{E}_r(\phi)}{\omega}d\omega = \sum_{k=1}^{2^r} I_k^r,$$
$$I_k^r = \frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\prod_{i\in\mathcal{I}_k,j\in\mathcal{J}_k}\phi(\alpha_i,\jmath\omega)\phi(\beta_j,-\jmath\omega)}{g\omega}d\omega, \tag{15}$$

where $g' = g\prod_{l=1}^{r} g_l$, to obtain expressions for the BER. This is done in the following section.

IV. BER Analysis

From [12, p. 296] and (5),

$$J = \frac{1}{2} - \frac{1}{2}\operatorname*{Res}_{z=0}\frac{\Phi_X(z)}{z} - \operatorname*{Res}_{z=-\alpha}\frac{\Phi_X(z)}{z} = \frac{g}{\alpha} \tag{16}$$

A. Single Relay, $N = 1$

For $r = 1$, (15) can be expressed as

$$\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\mathcal{E}_1(\phi)}{\omega}d\omega = \overbrace{\frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\phi(\beta_1,-\jmath\omega)}{g\omega}d\omega}^{I_1^1} + \overbrace{\frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\phi(\alpha_1,\jmath\omega)}{g\omega}d\omega}^{I_1^2}. \tag{17}$$

$\frac{\phi(\cdot,\cdot)}{\omega}$ has a removable singularity and does not contribute any poles. Further, for $w = Re^{\jmath\theta}, k = \pm 1$,

$$\left|\delta^{\jmath kw}\right| = \delta^{-kR\sin\theta + \jmath kR\cos\theta} < \infty,\ R\to\infty,\ -\pi < k\theta < 0,$$
$$(\because 0<\delta<1, \sin\theta > 0, 0<\theta<\pi)$$
$$\Rightarrow \quad |\phi(\beta_1,-\jmath w)| = \frac{|w||1-\delta^{\beta_1}\delta^{-\jmath w}|}{\beta_1|\beta_1-\jmath w|} < \infty, \quad 0<\theta<\pi \quad (ROC: \Im\{w\}>0),$$
$$|\phi(\alpha_1,\jmath w)| = \frac{|w||1-\delta^{\alpha_1}\delta^{\jmath w}|}{\alpha_1|\alpha_1+\jmath w|} < \infty, \quad -\pi<\theta<0, \quad (ROC: \Im\{w\}<0), \tag{18}$$

where ROC means region of convergence. Thus, I_1^1, I_1^2 converge and for $z = \jmath\omega$ [12, p. 277, 296],

$$\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\mathcal{E}_1(\phi)}{\omega}d\omega = g'\operatorname*{Res}_{z=-\alpha}\frac{\Phi_X(z)\phi(\beta_1,-z)}{gz} + g'\operatorname*{Res}_{z=\beta}\frac{\Phi_X(z)\phi(\alpha_1,z)}{gz} = g'\left[\frac{\phi(\alpha_1,\beta)}{\beta} - \frac{\phi(\beta_1,\alpha)}{\alpha}\right] \tag{19}$$

resulting in

$$\mathcal{A}(\Phi_{V_1}) = \frac{g}{\alpha} + g'\left[\frac{\phi(\alpha_1,\beta)}{\beta} - \frac{\phi(\beta_1,\alpha)}{\alpha}\right] \tag{20}$$

from (12), (14), (16) and (17), which is exactly equal to [7, (22)], validating the above approach.

B. Two Relays, $N = 2$

For $r = 2, \delta_1 < \delta_2$, computation of the integrals in (15) is demonstrated through I_3^2, where

$$I_3^2 = \frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\phi(\alpha_1,\jmath\omega)\phi(\beta_2,-\jmath\omega)}{g\omega}d\omega$$
$$= \underbrace{\frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{(\alpha+\beta)(\jmath\omega)^2\delta_2^{\beta_2-\jmath\omega}}{\alpha_1\beta_2\omega(\beta-\jmath\omega)(\alpha+\jmath\omega)(\alpha_1+\jmath\omega)(\beta_2-\jmath\omega)}d\omega}_{I_{31}^2} \quad (ROC: \Im\{w\}>0)$$
$$\underbrace{-\frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{(\alpha+\beta)(\jmath\omega)^2\left[1-\delta_1^{\alpha_1+\jmath\omega}+\delta_1^{\alpha_1}\delta_2^{\beta_2}\left(\frac{\delta_1}{\delta_2}\right)^{\jmath\omega}\right]}{\alpha_1\beta_2\omega(\beta-\jmath\omega)(\alpha+\jmath\omega)(\alpha_1+\jmath\omega)(\beta_2-\jmath\omega)}d\omega}_{I_{32}^2} \quad (ROC: \Im\{w\}<0) \tag{21}$$

after substituting for ϕ from (7) and simplifying.

$$\therefore \quad I_{31}^2 = g'\sum \operatorname{Res}_{z=-\alpha,-\alpha_1}\frac{(\alpha+\beta)z^2\delta_2^{\beta_2-z}}{\alpha_1\beta_2 z(\beta-z)(\alpha+z)(\alpha_1+z)(\beta_2-z)}$$
$$= \frac{g'}{\beta_2(\alpha-\alpha_1)}\left[\frac{\alpha\delta_2^{\beta_2+\alpha}}{\alpha_1(\beta_2+\alpha)} - \frac{(\alpha+\beta)\delta_2^{\beta_2+\alpha_1}}{(\beta+\alpha_1)(\beta_2+\alpha_1)}\right] \tag{22}$$

and

$$I_{32}^2 = -g'\sum\operatorname*{Res}_{z=\beta,\beta_2}\frac{g(\alpha+\beta)z^2\left[1-\delta_1^{\alpha_1+z}+\delta_1^{\alpha_1}\delta_2^{\beta_2}\left(\frac{\delta_1}{\delta_2}\right)^{z}\right]}{\alpha_1\beta_2 z(\beta-z)(\alpha+z)(\alpha_1+z)(\beta_2-z)} \quad \left(\because \frac{\delta_1}{\delta_2}<1\right)$$
$$= \frac{g'}{\alpha_1(\beta-\beta_2)}\left[\frac{(\alpha+\beta)}{(\alpha+\beta_2)(\alpha_1+\beta_2)} - \frac{\beta\left(1-\delta_1^{\alpha_1+\beta}+\delta_1^{\alpha_1+\beta}\delta_2^{\beta_2-\beta}\right)}{\beta_2(\alpha_1+\beta)}\right]. \tag{23}$$

The remaining integrals I_k^2 can be similarly evaluated (see Table II for expressions).

TABLE II: Integrals for $N = 2$

Integral	Integrand	Pole Location	Residue
I_1^2	$\frac{\Phi_X(z)\phi(\alpha_1,z)\phi(\alpha_2,z)}{z}$	$z=\beta$	$g\frac{\phi(\alpha_1,\beta)\phi(\alpha_2,\beta)}{\beta}$
I_2^2	$\frac{\Phi_X(z)\phi(\beta_1,-z)\phi(\beta_2,-z)}{z}$	$z=-\alpha$	$-g\frac{\phi(\beta_1,\alpha)\phi(\beta_2,\alpha)}{\alpha}$
I_{31}^2	$\frac{g'(\alpha+\beta)z^2\delta_2^{\beta_2-z}}{\alpha_1\beta_2 z(\beta-z)(\alpha+z)(\alpha_1+z)(\beta_2-z)}$	$z=-\alpha,-\alpha_1$	$\frac{g'}{\beta_2(\alpha-\alpha_1)}\left[\frac{\alpha\delta_2^{\beta_2+\alpha}}{\alpha_1(\beta_2+\alpha)}-\frac{(\alpha+\beta)\delta_2^{\beta_2+\alpha_1}}{(\beta+\alpha_1)(\beta_2+\alpha_1)}\right]$
I_{32}^2	$\frac{g'(\alpha+\beta)z^2\left[1-\delta_1^{\alpha_1+z}+\delta_1^{\alpha_1}\delta_2^{\beta_2}\left(\frac{\delta_1}{\delta_2}\right)^z\right]}{\alpha_1\beta_2 z(\beta-z)(\alpha+z)(\alpha_1+z)(\beta_2-z)}$	$z=\beta,\beta_2$	$\frac{g'}{\alpha_1(\beta-\beta_2)}\left[\frac{(\alpha+\beta)}{(\alpha+\beta_2)(\alpha_1+\beta_2)}-\frac{\beta\left(1-\delta_1^{\alpha_1+\beta}+\delta_1^{\alpha_1+\beta}\delta_2^{\beta_2-\beta}\right)}{\beta_2(\alpha_1+\beta)}\right]$
I_{41}^2	$\frac{g'(\alpha+\beta)z^2(1-\delta_2^{\alpha_2+z})}{\alpha_2\beta_1\omega(\beta-z)(\alpha+z)(\alpha_2+z)(\beta_1-z)}$	$z=\beta,\beta_1$	$\frac{g'}{\alpha_2(\beta-\beta_1)}\left[\frac{(\alpha+\beta)\left(1-\delta_2^{\alpha_2+\beta_1}\right)}{(\alpha+\beta_1)(\alpha_2+\beta_1)}-\frac{\beta\left(1-\delta_2^{\alpha_2+\beta}\right)}{\beta_1(\alpha_2+\beta)}\right]$
I_{42}^2	$\frac{g'(\alpha+\beta)z^2\left(-\delta_1^{\beta_1-z}+\delta_1^{\beta_1}\delta_2^{\alpha_2}\left(\frac{\delta_1}{\delta_2}\right)^{-z}\right)}{z(\beta-z)(\alpha+z)\alpha_2\beta_1(\alpha_2+z)(\beta_1-z)}$	$z=-\alpha,-\alpha_2$	$\frac{g'}{\beta_1(\alpha-\alpha_2)}\left[\frac{\alpha\left(-\delta_1^{\beta_1+\alpha}+\delta_1^{\beta_1+\alpha}\delta_2^{\alpha_2-\alpha}\right)}{\alpha_2(\beta_1+\alpha)}\right]$

C. Three Relays, $N = 3$

For $r = 3$, without loss of generality, we assume that $\delta_1 < \delta_2 < \delta_3$. From (15), we consider

$$I_5^3 = \frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{\Phi_X(\jmath\omega)\phi(\beta_1,-\jmath\omega)\phi(\alpha_2,\jmath\omega)\phi(\alpha_3,\jmath\omega)}{g\omega}d\omega. \quad (24)$$

Closed form expressions for the remaining integrals I_k^3 are listed in Table III.

Case I: For $\delta_1 < \delta_2\delta_3, I_5^3$ is split as (see ROCs below)

$$I_5^3 = \underbrace{\frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{(\alpha+\beta)(-\jmath\omega)^3\left[1-\delta_2^{\alpha_2+\jmath\omega}-\delta_3^{\alpha_3+\jmath\omega}+\delta_2^{\alpha_2}\delta_3^{\alpha_3}(\delta_2\delta_3)^{\jmath\omega}\right]}{(\beta-\jmath\omega)(\alpha+\jmath\omega)\omega\beta_1\alpha_2\alpha_3(\beta_1-\jmath\omega)(\alpha_2+\jmath\omega)(\alpha_3+\jmath\omega)}d\omega}_{I_{51}^3\,(ROC:\Im\{w\}<0)}$$
$$+\underbrace{\frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{(\alpha+\beta)(-\jmath\omega)^3\left[-\delta_1^{\beta_1-\jmath\omega}+\delta_1^{\beta_1}\delta_2^{\alpha_2}\left(\frac{\delta_2}{\delta_1}\right)^{\jmath\omega}+\delta_1^{\beta_2}\delta_3^{\alpha_3}\left(\frac{\delta_3}{\delta_1}\right)^{\jmath\omega}-\delta_1^{\beta_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\left(\frac{\delta_2\delta_3}{\delta_1}\right)^{\jmath\omega}\right]}{(\beta-\jmath\omega)(\alpha+\jmath\omega)\omega\beta_1\alpha_2\alpha_3(\beta_1-\jmath\omega)(\alpha_2+\jmath\omega)(\alpha_3+\jmath\omega)}d\omega}_{I_{52}^3\,(ROC:\Im\{w\}>0)} \quad (25)$$

$$\therefore I_{51}^3 = \operatorname*{Res}_{z=\beta,\beta_1}\frac{g'(\alpha+\beta)(-z)^3\left[1-\delta_2^{\alpha_2+z}-\delta_3^{\alpha_3+z}+\delta_2^{\alpha_2}\delta_3^{\alpha_3}(\delta_2\delta_3)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)} = -\frac{g'\phi(\alpha_2,\beta)\phi(\alpha_3,\beta)}{\beta_1(\beta_1-\beta)}-\frac{g'(\alpha+\beta)\phi(\alpha_2,\beta_1)\phi(\alpha_3,\beta_1)}{\beta_1(\beta-\beta_1)(\alpha+\beta_1)} \text{ and} \quad (26)$$

$$I_{52}^3 = \operatorname*{Res}_{z=-\alpha,-\alpha_2,-\alpha_3}\frac{g'(\alpha+\beta)(-z)^3\left[-\delta_1^{\beta_1-z}+\delta_1^{\beta_1}\delta_2^{\alpha_2}\left(\frac{\delta_2}{\delta_1}\right)^z+\delta_1^{\beta_2}\delta_3^{\alpha_3}\left(\frac{\delta_3}{\delta_1}\right)^z-\delta_1^{\beta_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\left(\frac{\delta_2\delta_3}{\delta_1}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)} = \frac{g'(\delta_1^{\beta_1+\alpha})\phi(\alpha_2,-\alpha)\phi(\alpha_3,-\alpha)}{\beta_1(\beta_1+\alpha)}. \quad (27)$$

Case II: For $\delta_1 > \delta_2\delta_3, I_5^3$ is split as

$$I_{51}^3 = \underbrace{\frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{(\alpha+\beta)(-\jmath\omega)^3\left[1-\delta_2^{\alpha_2+\jmath\omega}-\delta_3^{\alpha_3+\jmath\omega}+\delta_2^{\alpha_2}\delta_3^{\alpha_3}(\delta_2\delta_3)^{\jmath\omega}-\delta_1^{\beta_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\left(\frac{\delta_2\delta_3}{\delta_1}\right)^{\jmath\omega}\right]}{(\beta-\jmath\omega)(\alpha+\jmath\omega)\omega\beta_1\alpha_2\alpha_3(\beta_1-\jmath\omega)(\alpha_2+\jmath\omega)(\alpha_3+\jmath\omega)}d\omega}_{ROC:\Im\{w\}<0}$$
$$= \operatorname*{Res}_{z=\beta,\beta_1}\frac{g'(\alpha+\beta)(-z)^3\left[1-\delta_2^{\alpha_2+z}-\delta_3^{\alpha_3+z}+\delta_2^{\alpha_2}\delta_3^{\alpha_3}(\delta_2\delta_3)^z-\delta_1^{\beta_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\left(\frac{\delta_2\delta_3}{\delta_1}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)} \quad (28)$$
$$= -\frac{g'\phi(\alpha_2,\beta)\phi(\alpha_3,\beta)}{\beta_1(\beta_1-\beta)}+\frac{g'(\beta^2)\left(\delta_1^{\beta_1-\beta}\delta_2^{\alpha_2+\beta}\delta_3^{\alpha_3+\beta}\right)}{\beta_1\alpha_2\alpha_3(\beta_1-\beta)(\alpha_2+\beta)(\alpha_3+\beta)}-\frac{g'(\alpha+\beta)\phi(\alpha_2,\beta_1)\phi(\alpha_3,\beta_1)}{\beta_1(\beta-\beta_1)(\alpha+\beta_1)}+\frac{g'(\beta_1^2)(\alpha+\beta)\left(\delta_2^{\alpha_2+\beta_1}\delta_3^{\alpha_3+\beta_1}\right)}{\beta_1\alpha_2\alpha_3(\beta-\beta_1)(\alpha+\beta_1)(\alpha_2+\beta_1)(\alpha_3+\beta_1)}$$

and

$$I_{52}^3 = \underbrace{\frac{g'}{2\pi\jmath}\int_{-\infty}^{\infty}\frac{(\alpha+\beta)(-\jmath\omega)^3\left[-\delta_1^{\beta_1-\jmath\omega}+\delta_1^{\beta_1}\delta_2^{\alpha_2}\left(\frac{\delta_2}{\delta_1}\right)^{\jmath\omega}+\delta_1^{\beta_2}\delta_3^{\alpha_3}\left(\frac{\delta_3}{\delta_1}\right)^{\jmath\omega}\right]}{(\beta-\jmath\omega)(\alpha+\jmath\omega)\omega\beta_1\alpha_2\alpha_3(\beta_1-\jmath\omega)(\alpha_2+\jmath\omega)(\alpha_3+\jmath\omega)}d\omega}_{ROC:\Im\{w\}>0} = \operatorname*{Res}_{z=-\alpha,-\alpha_2,-\alpha_3}\frac{g'(\alpha+\beta)(-z)^3\left[-\delta_1^{\beta_1-z}+\delta_1^{\beta_1}\delta_2^{\alpha_2}\left(\frac{\delta_2}{\delta_1}\right)^z+\delta_1^{\beta_2}\delta_3^{\alpha_3}\left(\frac{\delta_3}{\delta_1}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)}dz,$$
$$= \frac{g'(\delta_1^{\beta_1+\alpha})\phi(\alpha_2,-\alpha)\phi(\alpha_3,-\alpha)}{\beta_1(\beta_1+\alpha)}-\frac{g'(\alpha^2)\left(\delta_1^{\beta_1+\alpha}\delta_2^{\alpha_2-\alpha}\delta_3^{\alpha_3-\alpha}\right)}{\beta_1\alpha_2\alpha_3(\beta_1+\alpha)(\alpha_2-\alpha)(\alpha_3-\alpha)}$$
$$-\frac{g'(\alpha_2^2)(\alpha+\beta)\left(\delta_1^{\beta_1+\alpha_2}\delta_3^{\alpha_3-\alpha_2}\right)}{\beta_1\alpha_2\alpha_3(\beta+\alpha_2)(\beta_1+\alpha_2)(\alpha-\alpha_2)(\alpha_3-\alpha_2)}-\frac{g'(\alpha_3^2)(\alpha+\beta)\left(\delta_1^{\beta_1+\alpha_3}\delta_2^{\alpha_2-\alpha_3}\right)}{\beta_1\alpha_2\alpha_3(\beta+\alpha_3)(\beta_1+\alpha_3)(\alpha-\alpha_3)(\alpha_2-\alpha_3)}. \quad (29)$$

TABLE III: Integrals for $N = 3$

Integral	Integrand	Pole Location	Residue
I^3_1	$\frac{g'(\alpha+\beta)\phi(\alpha_1,z)\phi(\alpha_2,z)\phi(\alpha_3,z)}{(\beta-z)(\alpha+z)z}$	$z=\beta$	$\frac{g'\phi(\alpha_1,\beta)\phi(\alpha_2,\beta)\phi(\alpha_3,\beta)}{\beta}$
I^3_2	$\frac{g'(\alpha+\beta)\phi(\beta_1,-z)\phi(\beta_2,-z)\phi(\beta_3,-z)}{(\beta-z)(\alpha+z)z}$	$z=-\alpha$	$-\frac{g'\phi(\beta_1,\alpha)\phi(\beta_2,\alpha)\phi(\beta_3,\alpha)}{\alpha}$
I^3_{31}	$\frac{g'(\alpha+\beta)(-z)^3\left[1-\delta_1^{\alpha_1+z}-\delta_2^{\alpha_2+z}+\delta_1^{\alpha_1}\delta_3^{\beta_3}\left(\frac{\delta_1}{\delta_3}\right)^z+\delta_2^{\alpha_2}\delta_3^{\beta_3}\left(\frac{\delta_2}{\delta_3}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\alpha_2\beta_3(\alpha_1+z)(\alpha_2+z)(\beta_3-z)}+\frac{g'(\alpha+\beta)(-z)^3\left[\delta_1^{\alpha_1}\delta_2^{\alpha_2}(\delta_1\delta_2)^z-\delta_1^{\alpha_1}\delta_2^{\alpha_2}\delta_3^{\beta_3}\left(\frac{\delta_1\delta_2}{\delta_3}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\alpha_2\beta_3(\alpha_1+z)(\alpha_2+z)(\beta_3-z)}$	$z=\beta,\beta_3$	$\frac{g'\phi(\alpha_1,\beta)\phi(\alpha_2,\beta)\phi(\beta_3,-\beta)}{\beta}-\frac{g'\beta^2\delta_3^{\beta_3-\beta}}{\alpha_1\alpha_2\beta_3(\alpha_1+\beta)(\alpha_2+\beta)(\beta_3-\beta)}-\frac{g'\beta_3^2(\alpha+\beta)}{\alpha_1\alpha_2\beta_3(\alpha_1+\beta_3)(\alpha_2+\beta_3)(\beta-\beta_3)(\alpha+\beta_3)}$
I^3_{32}	$\frac{g'(\alpha+\beta)(-z)^3\left(-\delta_3^{\beta_3-z}\right)}{(\beta-z)(\alpha+z)z\alpha_1\alpha_2\beta_3(\alpha_1+z)(\alpha_2+z)(\beta_3-z)}$	$z=-\alpha,-\alpha_1.-\alpha_2$	$\frac{g'\alpha^2\delta_3^{\beta_3+\alpha}}{\alpha_1\alpha_2\beta_3(\alpha_1-\alpha)(\beta_3+\alpha)(\alpha_2-\alpha)}+\frac{g'\alpha_1^2(\alpha+\beta)\delta_3^{\beta_3+\alpha_1}}{\alpha_1\alpha_2\beta_3(\beta+\alpha_1)(\alpha-\alpha_1)(\alpha_2-\alpha_1)(\beta_3+\alpha_1)}+\frac{g'\alpha_2^2(\alpha+\beta)\delta_3^{\beta_3+\alpha_2}}{\alpha_1\alpha_2\beta_3(\beta+\alpha_2)(\alpha-\alpha_2)(\alpha_1-\alpha_2)(\beta_3+\alpha_2)}$
I^3_{41}	$\frac{g'(\alpha+\beta)(-z)^3\left[1-\delta_1^{\alpha_1+z}-\delta_3^{\alpha_3+z}+\delta_1^{\alpha_1}\delta_2^{\beta_2}\left(\frac{\delta_1}{\delta_2}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\beta_2\alpha_3(\alpha_1+z)(\beta_2-z)(\alpha_3+z)}+\frac{g'(\alpha+\beta)(-z)^3\left[\delta_1^{\alpha_1}\delta_3^{\alpha_3}(\delta_1\delta_3)^z-\delta_1^{\alpha_1}\delta_2^{\beta_2}\delta_3^{\alpha_3}\left(\frac{\delta_1\delta_3}{\delta_2}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\beta_2\alpha_3(\alpha_1+z)(\beta_2-z)(\alpha_3+z)}$	$z=\beta,\beta_2$	$\frac{g'\phi(\alpha_1,\beta)\phi(\beta_2,-\beta)\phi(\alpha_3,\beta)}{\beta}-\frac{g'\beta\delta_2^{\beta_2-\beta}\phi(\alpha_3,\beta)}{\alpha_1\beta_2(\alpha_1+\beta)(\beta_2-\beta)}-\frac{g'\beta_2(\alpha+\beta)\phi(\alpha_3,\beta_2)}{\alpha_1\beta_2(\alpha_1+\beta_2)(\beta-\beta_2)(\alpha+\beta_2)}$
I^3_{42}	$\frac{g'(\alpha+\beta)(-z)^3\left[-\delta_2^{\beta_2-z}+\delta_2^{\beta_2}\delta_3^{\alpha_3}\left(\frac{\delta_3}{\delta_2}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\beta_2\alpha_3(\alpha_1+z)(\beta_2-z)(\alpha_3+z)}$	$z=-\alpha,-\alpha_1,-\alpha_3$	$-\frac{g'\alpha\delta_2^{\beta_2+\alpha}\phi(\alpha_3,-\alpha)}{\alpha_1\beta_2(\alpha_1-\alpha)(\beta_2+\alpha)}-\frac{g'\alpha_1(\alpha+\beta)\delta_2^{\beta_2+\alpha_1}\phi(\alpha_3,-\alpha_1)}{\alpha_1\beta_2(\beta+\alpha_1)(\alpha-\alpha_1)(\beta_2+\alpha_1)}$
$I^3_{51},\delta_1<\delta_2\delta_3$	$\frac{g'(\alpha+\beta)(-z)^3\left[1-\delta_2^{\alpha_2+z}-\delta_3^{\alpha_3+z}+\delta_2^{\alpha_2}\delta_3^{\alpha_3}(\delta_2\delta_3)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)}$	$z=\beta,\beta_1$	$-\frac{g'\phi(\alpha_2,\beta)\phi(\alpha_3,\beta)}{\beta_1(\beta_1-\beta)}-\frac{g'(\alpha+\beta)\phi(\alpha_2,\beta_1)\phi(\alpha_3,\beta_1)}{\beta_1(\beta-\beta_1)(\alpha+\beta_1)}$
$I^3_{51},\delta_1>\delta_2\delta_3$	$\frac{g'(\alpha+\beta)(-z)^3\left[1-\delta_2^{\alpha_2+z}-\delta_3^{\alpha_3+z}\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)}+\frac{g'(\alpha+\beta)(-z)^3\left[\delta_2^{\alpha_2}\delta_3^{\alpha_3}(\delta_2\delta_3)^z-\delta_1^{\beta_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\left(\frac{\delta_2\delta_3}{\delta_1}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)}$	$z=\beta,\beta_1$	$-\frac{g'\phi(\alpha_2,\beta)\phi(\alpha_3,\beta)}{\beta_1(\beta_1-\beta)}+\frac{g'\beta^2\left(\delta_1^{\beta_1-\beta}\delta_2^{\alpha_2+\beta}\delta_3^{\alpha_3+\beta}\right)}{\beta_1\alpha_2\alpha_3(\beta_1-\beta)(\alpha_2+\beta)(\alpha_3+\beta)}-\frac{g'(\alpha+\beta)\phi(\alpha_2,\beta_1)\phi(\alpha_3,\beta_1)}{\beta_1(\beta-\beta_1)(\alpha+\beta_1)}+\frac{g'\beta_1^2(\alpha+\beta)\left(\delta_2^{\alpha_2+\beta_1}\delta_3^{\alpha_3+\beta_1}\right)}{\beta_1\alpha_2\alpha_3(\beta-\beta_1)(\alpha+\beta_1)(\alpha_2+\beta_1)(\alpha_3+\beta_1)}$
$I^3_{52},\delta_1<\delta_2\delta_3$	$\frac{g'(\alpha+\beta)(-z)^3\left[-\delta_1^{\beta_1-z}+\delta_1^{\beta_1}\delta_2^{\alpha_2}\left(\frac{\delta_2}{\delta_1}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)}+\frac{g'(\alpha+\beta)(-z)^3\left[\delta_1^{\beta_2}\delta_3^{\alpha_3}\left(\frac{\delta_3}{\delta_1}\right)^z-\delta_1^{\beta_1}\delta_2^{\alpha_2}\delta_3^{\alpha_3}\left(\frac{\delta_2\delta_3}{\delta_1}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)}$	$z=-\alpha,-\alpha_2,-\alpha_3$	$\frac{g'\delta_1^{\beta_1+\alpha}\phi(\alpha_2,-\alpha)\phi(\alpha_3,-\alpha)}{\beta_1(\beta_1+\alpha)}$
$I^3_{52},\delta_1>\delta_2\delta_3$	$\frac{g'(\alpha+\beta)(-z)^3\left[-\delta_1^{\beta_1-z}+\delta_1^{\beta_1}\delta_2^{\alpha_2}\left(\frac{\delta_2}{\delta_1}\right)^z+\delta_1^{\beta_2}\delta_3^{\alpha_3}\left(\frac{\delta_3}{\delta_1}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\alpha_3(\beta_1-z)(\alpha_2+z)(\alpha_3+z)}$	$z=-\alpha,-\alpha_2,-\alpha_3$	$\frac{g'\delta_1^{\beta_1+\alpha}\phi(\alpha_2,-\alpha)\phi(\alpha_3,-\alpha)}{\beta_1(\beta_1+\alpha)}-\frac{g'\alpha^2\left(\delta_1^{\beta_1+\alpha}\delta_2^{\alpha_2-\alpha}\delta_3^{\alpha_3-\alpha}\right)}{\beta_1\alpha_2\alpha_3(\beta_1+\alpha)(\alpha_2-\alpha)(\alpha_3-\alpha)}-\frac{g'\alpha_2^2(\alpha+\beta)\left(\delta_1^{\beta_1+\alpha_2}\delta_3^{\alpha_3-\alpha_2}\right)}{\beta_1\alpha_2\alpha_3(\beta+\alpha_2)(\beta_1+\alpha_2)(\alpha-\alpha_2)(\alpha_3-\alpha_2)}-\frac{g'\alpha_3^2(\alpha+\beta)\left(\delta_1^{\beta_1+\alpha_3}\delta_2^{\alpha_2-\alpha_3}\right)}{\beta_1\alpha_2\alpha_3(\beta+\alpha_3)(\beta_1+\alpha_3)(\alpha-\alpha_3)(\alpha_2-\alpha_3)}$
$I^3_{61},\delta_1<\delta_2\delta_3$	$\frac{g'(\alpha+\beta)(z)^3\left[1-\delta_1^{\alpha_1+z}+\delta_1^{\alpha_1}\delta_2^{\beta_2}\left(\frac{\delta_1}{\delta_2}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\beta_2\beta_3(\alpha_1+z)(\beta_2-z)(\beta_3-z)}+\frac{g'(\alpha+\beta)(z)^3\left[\delta_1^{\alpha_1}\delta_3^{\beta_3}\left(\frac{\delta_1}{\delta_3}\right)^z-\delta_1^{\alpha_1}\delta_2^{\beta_2}\delta_3^{\beta_3}\left(\frac{\delta_1}{\delta_2\delta_3}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\beta_2\beta_3(\alpha_1+z)(\beta_2-z)(\beta_3-z)}$	$z=\beta,\beta_2,\beta_3$	$-\frac{g'\delta_1^{\alpha_1+\beta}\phi(\beta_2,-\beta)\phi(\beta_3,-\beta)}{\alpha_1(\alpha_1+\beta)}+\frac{g'\beta^2}{\alpha_1\beta_2\beta_3(\alpha_1+\beta)(\beta_2-\beta)(\beta_3-\beta)}+\frac{g'\beta_2^2(\alpha+\beta)}{\alpha_1\beta_2\beta_3(\alpha+\beta_2)(\alpha_1+\beta_2)(\beta-\beta_2)(\beta_3-\beta_2)}+\frac{g'\beta_3^2(\alpha+\beta)}{\alpha_1\beta_2\beta_3(\alpha+\beta_3)(\alpha_1+\beta_3)(\beta-\beta_3)(\beta_2-\beta_3)}$
$I^3_{61},\delta_1>\delta_2\delta_3$	$\frac{g'(\alpha+\beta)(z)^3\left[1-\delta_1^{\alpha_1+z}+\delta_1^{\alpha_1}\delta_2^{\beta_2}\left(\frac{\delta_1}{\delta_2}\right)^z+\delta_1^{\alpha_1}\delta_3^{\beta_3}\left(\frac{\delta_1}{\delta_3}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\beta_2\beta_3(\alpha_1+z)(\beta_2-z)(\beta_3-z)}$	$z=\beta,\beta_2,\beta_3$	$\frac{g'\beta^2\left(1-\delta_1^{\alpha_1+\beta}+\delta_1^{\alpha_1+\beta}\delta_2^{\beta_2-\beta}+\delta_1^{\alpha_1+\beta}\delta_3^{\beta_3-\beta}\right)}{\alpha_1\beta_2\beta_3(\alpha_1+\beta)(\beta_2-\beta)(\beta_3-\beta)}+\frac{g'(\alpha+\beta)\beta_2^2\left(1+\delta_1^{\alpha_1+\beta_2}\delta_3^{\beta_3-\beta_2}\right)}{\alpha_1\beta_2\beta_3(\alpha+\beta_2)(\alpha_1+\beta_2)(\beta-\beta_2)(\beta_3-\beta_2)}+\frac{g'(\alpha+\beta)\beta_3^2\left(1+\delta_1^{\alpha_1+\beta_3}\delta_2^{\beta_2-\beta_3}\right)}{\alpha_1\beta_2\beta_3(\alpha+\beta_3)(\alpha_1+\beta_3)(\beta_2-\beta_3)(\beta-\beta_3)}$
$I^3_{62},\delta_1<\delta_2\delta_3$	$\frac{g'(\alpha+\beta)(z)^3\left[-\delta_2^{\beta_2-z}-\delta_3^{\beta_3-z}+\delta_2^{\beta_2}\delta_3^{\beta_3}\left(\frac{1}{\delta_2\delta_3}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\beta_2\beta_3(\alpha_1+z)(\beta_2-z)(\beta_3-z)}$	$z=-\alpha,-\alpha_1$	$\frac{g'(\alpha+\beta)\phi(\beta_2,\alpha)\phi(\beta_3,\alpha)}{\alpha_1(\alpha_1-\alpha)}+\frac{g'(\alpha+\beta)\phi(\beta_2,\alpha_1)\phi(\beta_3,\alpha_1)}{\alpha_1(\beta+\alpha_1)(\alpha-\alpha_1)}-\frac{g'(\alpha+\beta)\alpha^2}{\alpha_1\beta_2\beta_3(\alpha_1-\alpha)(\beta_2+\alpha)(\beta_3+\alpha)}-\frac{g'(\alpha+\beta)\alpha_1^2}{\alpha_1\beta_2\beta_3(\beta+\alpha_1)(\alpha-\alpha_1)(\beta_2+\alpha_1)(\beta_3+\alpha_1)}$
$I^3_{62},\delta_1>\delta_2\delta_3$	$\frac{g'(\alpha+\beta)(z)^3\left[-\delta_2^{\beta_2-z}-\delta_3^{\beta_3-z}+\delta_2^{\beta_2}\delta_3^{\beta_3}\left(\frac{1}{\delta_2\delta_3}\right)^z-\delta_1^{\alpha_1}\delta_2^{\beta_2}\delta_3^{\beta_3}\left(\frac{\delta_1}{\delta_2\delta_3}\right)^z\right]}{(\beta-z)(\alpha+z)z\alpha_1\beta_2\beta_3(\alpha_1+z)(\beta_2-z)(\beta_3-z)}$	$z=-\alpha,-\alpha_1$	$\frac{g'\phi(\beta_2,\alpha)\phi(\beta_3,\alpha)}{\alpha_1(\alpha_1-\alpha)}-\frac{g'\alpha^2\left(1+\delta_1^{\alpha_1-\alpha}\delta_2^{\beta_2+\alpha}\delta_3^{\beta_3+\alpha}\right)}{\alpha_1\beta_2\beta_3(\alpha_1-\alpha)(\beta_2+\alpha)(\beta_3+\alpha)}-\frac{g'(\alpha+\beta)\alpha_1^2\left(\delta_2^{\beta_2+\alpha_1}+\delta_3^{\beta_3+\alpha_1}\right)}{\alpha_1\beta_2\beta_3(\beta+\alpha_1)(\alpha-\alpha_1)(\beta_2+\alpha_1)(\beta_3+\alpha_1)}$

TABLE III: Integrals for $N = 3$

Integral	Integrand	Pole Location	Residue
I_{71}^{3}	$\frac{g'(\alpha+\beta)(z)^3\left[1-\delta_2^{\alpha_2+z}+\delta_2^{\alpha_2}\delta_3^{\beta_3}\left(\frac{\delta_2}{\delta_3}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\beta_3(\beta_1-z)(\alpha_2+z)(\beta_3-z)}$	$z=\beta,\beta_1,\beta_3$	$-\frac{g'\phi(\alpha_2,\beta)\phi(\beta_3,-\beta)}{\beta_1(\beta_1-\beta)}+\frac{g'\beta^2\delta_3^{\beta_3-\beta}}{\beta_1\alpha_2\beta_3(\beta_1-\beta)(\alpha_2+\beta)(\beta_3-\beta)}-\frac{g'(\alpha+\beta)\phi(\alpha_2,\beta_1)\phi(\beta_3,-\beta_1)}{\beta_1(\alpha+\beta_1)(\beta-\beta_1)}+\frac{g'\beta_1^2(\alpha+\beta)\delta_3^{\beta_3-\beta_1}}{\beta_1\alpha_2\beta_3(\alpha+\beta_1)(\beta-\beta_1)(\alpha_2+\beta_1)(\beta_3-\beta_1)}+\frac{g'\beta_3^2(\alpha+\beta)}{\beta_1\alpha_2\beta_3(\alpha+\beta_3)(\beta_1-\beta_3)(\alpha_2+\beta_3)(\beta-\beta_3)}$
I_{72}^{3}	$\frac{g'(\alpha+\beta)(z)^3\left[-\delta_1^{\beta_1-z}-\delta_3^{\beta_3-z}+\delta_1^{\beta_1}\delta_2^{\alpha_2}\left(\frac{\delta_2}{\delta_1}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\beta_3(\beta_1-z)(\alpha_2+z)(\beta_3-z)}+\frac{g'(\alpha+\beta)(z)^3\left[\delta_1^{\beta_1}\delta_3^{\beta_3}\left(\frac{1}{\delta_1\delta_3}\right)^z-\delta_1^{\beta_1}\delta_2^{\alpha_2}\delta_3^{\beta_3}\left(\frac{\delta_2}{\delta_1\delta_3}\right)^z\right]}{(\beta-z)(\alpha+z)z\beta_1\alpha_2\beta_3(\beta_1-z)(\alpha_2+z)(\beta_3-z)}$	$z=-\alpha,-\alpha_2$	$-\frac{g'\alpha^2\delta_1^{\beta_1+\alpha}}{\beta_1\alpha_2\beta_3(\beta_1+\alpha)(\alpha_2-\alpha)(\beta_3+\alpha)}+\frac{g'\delta_1^{\beta_1+\alpha}\phi(\alpha_2,-\alpha)\phi(\beta_3,\alpha)}{\beta_1(\beta_1+\alpha)}-\frac{g'(\alpha+\beta)\alpha_2^2\delta_3^{\beta_3+\alpha_2}}{(\beta+\alpha_2)\beta_1\alpha_2\beta_3(\beta_1+\alpha_2)(\alpha-\alpha_2)(\beta_3+\alpha_2)}$
I_{81}^{3}	$\frac{g'(\alpha+\beta)(z)^3\left(1-\delta_3^{\alpha_3+z}\right)}{(\beta-z)(\alpha+z)z\beta_1\beta_2\alpha_3(\beta_1-z)(\beta_2-z)(\alpha_3+z)}$	$z=\beta,\beta_1,\beta_2$	$\frac{g'(\beta)\phi(\alpha_3,\beta)}{\beta_1\beta_2(\beta_1-\beta)(\beta_2-\beta)}+\frac{g'(\alpha+\beta)(\beta_1)\phi(\alpha_3,\beta_1)}{\beta_2(\alpha+\beta_1)(\beta-\beta_1)(\beta_2-\beta_1)}+\frac{g'(\alpha+\beta)(\beta_2)\phi(\alpha_3,\beta_2)}{\beta_1(\alpha+\beta_2)(\beta-\beta_2)(\beta_1-\beta_2)}$
I_{82}^{3}	$\frac{g'(\alpha+\beta)(z)^3\left(1-\delta_3^{\alpha_3+z}\right)\left(-\delta_1^{\beta_1-z}-\delta_2^{\beta_2-z}+\delta_1^{\beta_1-z}\delta_2^{\beta_2-\jmath\omega}\right)}{(\beta-z)(\alpha+z)z\beta_1\beta_2\alpha_3(\beta_1-z)(\beta_2-z)(\alpha_3+z)}$	$z=-\alpha,-\alpha_3$	$-\frac{g'\phi(\alpha_3,-\alpha)\phi(\beta_1,\alpha)\phi(\beta_2,\alpha)}{\alpha}+\frac{g'(\alpha)\phi(\alpha_3,-\alpha)}{\beta_1\beta_2(\beta_1+\alpha)(\beta_2+\alpha)}$

V. Results and Discussion

The channel fading follows the path loss model $\Omega_{i,j} \propto \frac{1}{L_{i,j}^4}$, $i, j \in \{s, r, d\}$ with $L_{i,j}$ being the distance between the nodes i and j [4]. Also, $\Omega_{r,s} = \frac{\Omega_{d,s}}{l_r^4}$ and $\Omega_{d,r} = \frac{\Omega_{d,s}}{(1-l_r)^4}$, where $l_r = \frac{L_{r,s}}{L_{d,s}}$. The average system SNR $= \frac{\Omega_{d,s}\left(E_s+\sum_{r=1}^{N} E_r\right)}{N_0}$, $N = 1, 2, 3$. All the following plots are with respect to this parameter. The closed form BER expression in (4) is used along with (8) - (15) and Tables II and III to plot the theoretical BER for $N = 2, 3$, and verify it through bit level simulations performed on all the links to estimate the average probability of error at the destination through the decision rules given in Table I (col. 2). The analytical expressions are similar for both BPSK and BFSK, and the numerical values of α and β that figure in these expressions, are readily obtained from Table I (col. 5), where $\gamma = \frac{\Omega E}{N_0}$ is the link SNR.

A comparison of the analytical and simulation results for the BER performance for $N = 3$ is shown in Figure 2. All nodes are assumed to transmit with equal power. Plots are available for BPSK and coherent and noncoherent BFSK for different relay positions (l_r). As expected, BPSK performs best, and coherent detection outperforms noncoherent detection for all cases considered. In Figure 3, the BER plots for BPSK based cooperation with $N = 1, 2$ and 3, for various relay locations are shown. The total power used for the case of multiple relays was kept equal to that for the case of single relay. The plots suggest that using multiple relays results in a better BER performance with no additional power requirement. The gain due to relay diversity is thus evident.

VI. Conclusions and Future Work

In this paper, exact expressions for the BER for a PL-DF cooperative diversity system employing upto three relays were obtained for the Rayleigh fading channel. This was done through a contour integral approach for a CF based inversion formula for the probability of error, where multiple contours were intelligently employed to facilitate application of the residue theorem. Besides being simpler, this approach allowed for the evaluation of the BER for arbitrary relay locations. From the analysis carried out in this paper, it is evident that for each additional relay, a recursive relation exists for the theoretical BER, and an algorithm for evaluating the BER for arbitrary number of relays seems feasible.

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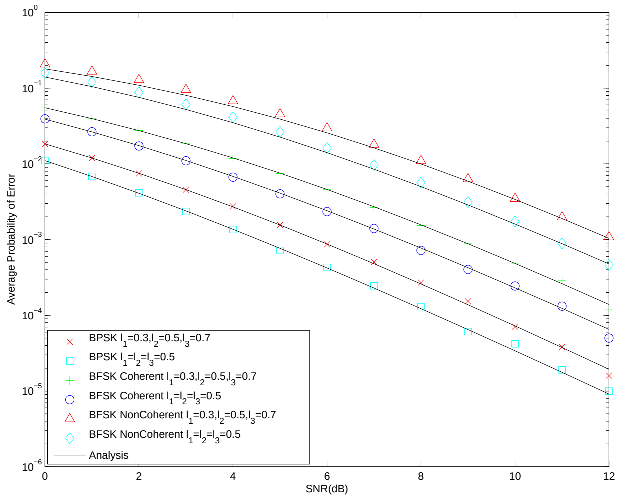

Fig. 2. Comparison of simulation and analytical results for the three relay system.

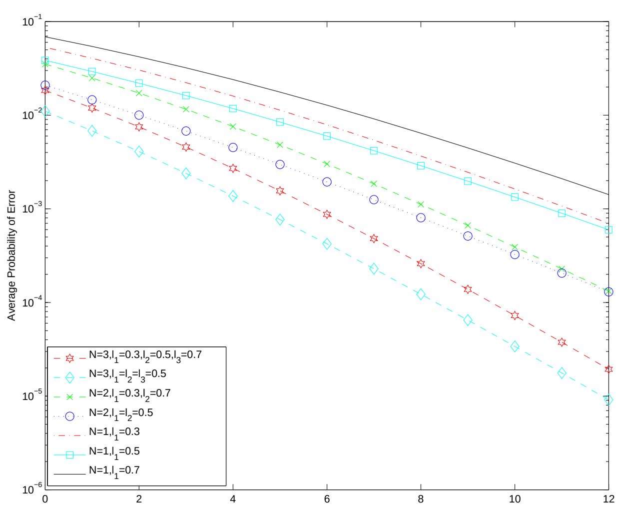

Fig. 3. Performance of BPSK for multiple relays.