Algebraic proofs for converse theorems for a cyclic quadrilateral

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ABSTRACT
Proofs of some theorems related to cyclic quadrilaterals are provided using coordinate geometry and trigonometry. Through this approach, constructions and proofs using contradiction are avoided.

KEYWORDS
cyclic quadrilateral, coordinate geometry, trigonometry

1. Introduction

Many proofs in Euclidean geometry are based on contradiction. For example, Figure 1 was used in [1] to prove the following theorem:
"If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle (i.e. they are concyclic)"
The proof given in [1] uses the theorem
"Angles in the same segment of a circle are equal"
So, a theorem is being used to prove its converse in an abstract manner, that may not be satisfactory to the reader. In these notes, we provide an alternative approach that relies on high school coordinate geometry, trigonometry and algebra. This involves a tradeoff between constructions, that are trial based, and algebra with trigonometry, which may be of interest to students. We show that this approach can be used in other converse theorems as well, e.g. [1].
"If sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic"

2. Preliminaries

Definition 2.1. Let

$$\mathbf{u}_i = \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix}, \quad \theta_i > \theta_j, i > j, i, j \in \{1, 2, 3, 4\}. \tag{1}$$

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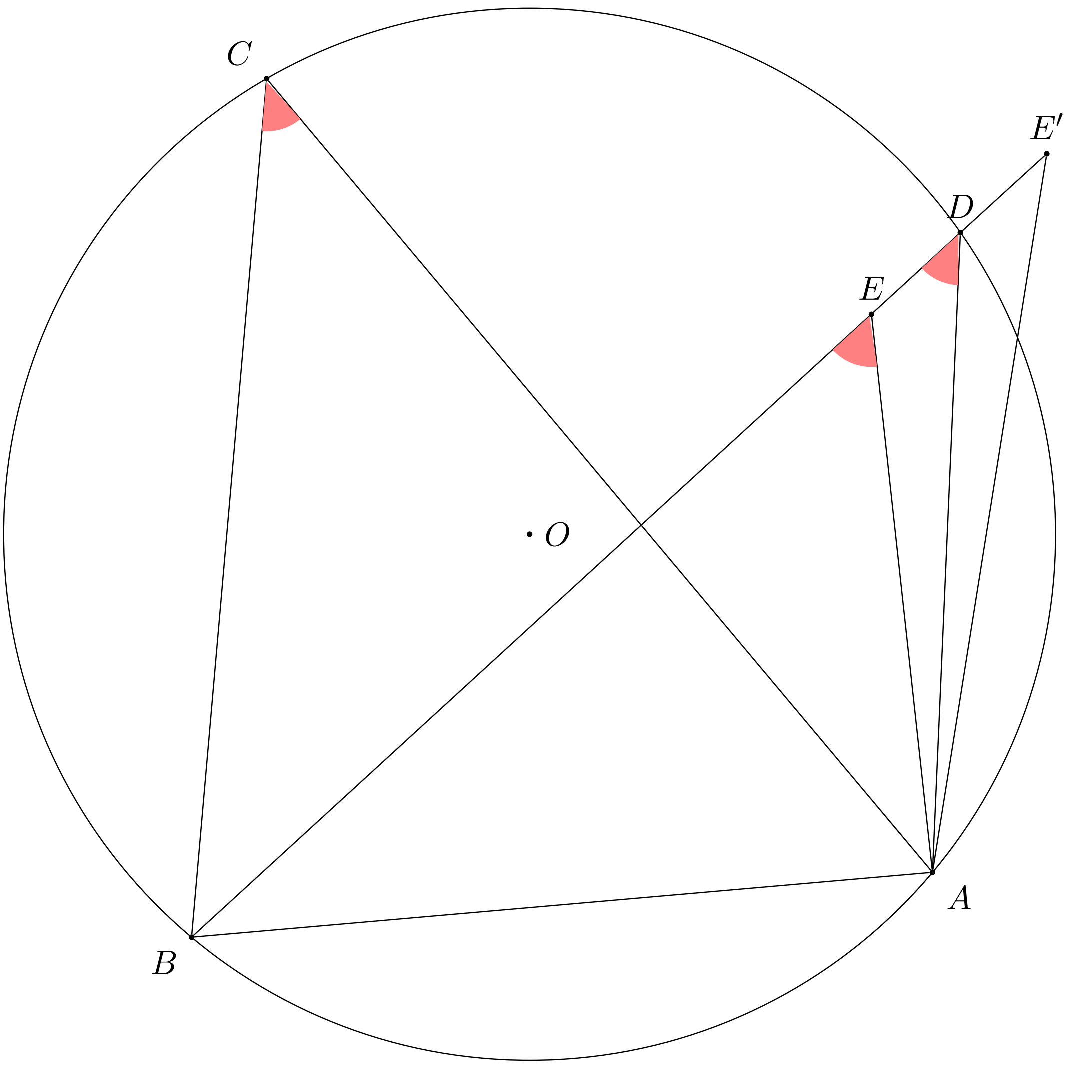

Figure 1. To show that $k = r$.

and

$$p_{ij} = p_{ji} \triangleq \mathbf{u}_i^\top \mathbf{u}_j \tag{2}$$

$$= \begin{cases} \cos(\theta_i - \theta_j) & i \neq j \\ 1 & i = j \end{cases} \tag{3}$$

Lemma 2.2.

$$\frac{1 - p_{ij} - p_{jk} + p_{ik}}{2\sqrt{(1 - p_{ij})(1 - p_{jk})}} = \operatorname{sign}(\theta_i - \theta_j)\operatorname{sign}(\theta_k - \theta_j)\cos\left(\frac{\theta_i - \theta_k}{2}\right) \tag{4}$$

$$\frac{1 + p_{ij} + p_{jk} + p_{ik}}{2\sqrt{(1 + p_{ij})(1 + p_{jk})}} = \operatorname{sign}(\pi - |\theta_i - \theta_j|)\operatorname{sign}(\pi - |\theta_k - \theta_j|)\cos\left(\frac{\theta_i - \theta_k}{2}\right) \tag{5}$$

where

$$\mathrm{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{6}$$

Proof. From (3), using the expression for the difference of cosines and difference of sines [2],

$$\begin{aligned} 1 - p_{ij} - p_{jk} + p_{ik} &= 1 - \cos(\theta_i - \theta_j) - \cos(\theta_j - \theta_k) + \cos(\theta_i - \theta_k) \\ &= 2\left[\sin^2\left(\frac{\theta_i - \theta_j}{2}\right) + \sin\left(\frac{\theta_j - \theta_i}{2}\right)\sin\left(\frac{\theta_i + \theta_j}{2} - \theta_k\right)\right] \\ &= 2\sin\left(\frac{\theta_i - \theta_j}{2}\right)\left[\sin\left(\frac{\theta_i - \theta_j}{2}\right) - \sin\left(\frac{\theta_i + \theta_j}{2} - \theta_k\right)\right] \\ &= 4\sin\left(\frac{\theta_i - \theta_j}{2}\right)\sin\left(\frac{\theta_k - \theta_j}{2}\right)\cos\left(\frac{\theta_i - \theta_k}{2}\right) \end{aligned} \tag{7}$$

Similarly, using the expression for the sum of cosines [2],

$$\begin{aligned} 1 + p_{ij} + p_{jk} + p_{ik} &= 1 + \cos(\theta_i - \theta_j) + \cos(\theta_j - \theta_k) + \cos(\theta_i - \theta_k) \\ &= 2\left[\cos^2\left(\frac{\theta_i - \theta_j}{2}\right) + \cos\left(\frac{\theta_i - \theta_j}{2}\right)\cos\left(\frac{\theta_i + \theta_j}{2} - \theta_k\right)\right] \\ &= 2\cos\left(\frac{\theta_i - \theta_j}{2}\right)\left[\cos\left(\frac{\theta_i - \theta_j}{2}\right) + \cos\left(\frac{\theta_i + \theta_j}{2} - \theta_k\right)\right] \\ &= 4\cos\left(\frac{\theta_i - \theta_j}{2}\right)\cos\left(\frac{\theta_k - \theta_j}{2}\right)\cos\left(\frac{\theta_i - \theta_k}{2}\right) \end{aligned} \tag{8}$$

Substituting from (7),

$$\frac{1 - p_{ij} - p_{jk} + p_{ik}}{2\sqrt{(1 - p_{ij})(1 - p_{jk})}} = \frac{4\sin\left(\frac{\theta_i - \theta_j}{2}\right)\sin\left(\frac{\theta_k - \theta_j}{2}\right)\cos\left(\frac{\theta_i - \theta_k}{2}\right)}{2\sqrt{2\sin^2\left(\frac{\theta_i - \theta_j}{2}\right)}\sqrt{2\sin^2\left(\frac{\theta_j - \theta_k}{2}\right)}} \tag{9}$$

$$= \mathrm{sign}(\theta_i - \theta_j)\,\mathrm{sign}(\theta_k - \theta_j)\cos\left(\frac{\theta_i - \theta_k}{2}\right) \tag{10}$$

Similarly, substituting from (8),

$$\frac{1 + p_{ij} + p_{jk} + p_{ik}}{2\sqrt{(1 + p_{ij})(1 + p_{jk})}} = \frac{4\cos\left(\frac{\theta_i - \theta_j}{2}\right)\cos\left(\frac{\theta_k - \theta_j}{2}\right)\cos\left(\frac{\theta_i - \theta_k}{2}\right)}{2\sqrt{2\cos^2\left(\frac{\theta_i - \theta_j}{2}\right)}\sqrt{2\cos^2\left(\frac{\theta_j - \theta_k}{2}\right)}} \tag{11}$$

$$= \mathrm{sign}(\pi - |\theta_i - \theta_j|)\,\mathrm{sign}(\pi - |\theta_k - \theta_j|)\cos\left(\frac{\theta_i - \theta_k}{2}\right) \quad \square \tag{12}$$

Lemma 2.3. *$x = \pm 1$ are roots of the quartic equation*

$$\frac{(1-p_{24}-p_{34}+p_{23})^2}{4(1-p_{24})(1-p_{34})} = \frac{\left(x^2-xp_{24}-xp_{34}+p_{23}\right)^2}{(x^2+1-2xp_{24})(x^2+1-2xp_{34})} \tag{13}$$

Proof. (1) For

$$|\theta_2-\theta_4| < \pi, |\theta_3-\theta_4| < \pi, \tag{14}$$

susbtituting $x = \pm 1$ in

$$\frac{1-p_{24}-p_{34}+p_{23}}{2\sqrt{(1-p_{24})(1-p_{34})}} = \frac{x^2-xp_{24}-xp_{34}+p_{23}}{\sqrt{(x^2+1-2xp_{24})(x^2+1-2xp_{34})}} \tag{15}$$

leads to the identities in Lemma 2.2,

(2) For

$$|\theta_2-\theta_4| > \pi \cap |\theta_3-\theta_4| < \pi \text{ or } |\theta_2-\theta_4| < \pi \cap |\theta_3-\theta_4| > \pi, \tag{16}$$

susbtituting $x = 1$ in

$$\frac{1-p_{24}-p_{34}+p_{23}}{2\sqrt{(1-p_{24})(1-p_{34})}} = \frac{x^2-xp_{24}-xp_{34}+p_{23}}{\sqrt{(x^2+1-2xp_{24})(x^2+1-2xp_{34})}} \tag{17}$$

leads to the identities in Lemma 2.2, Similarly, susbtituting $x = -1$ in

$$\frac{1-p_{24}-p_{34}+p_{23}}{2\sqrt{(1-p_{24})(1-p_{34})}} = -\frac{x^2-xp_{24}-xp_{34}+p_{23}}{\sqrt{(x^2+1-2xp_{24})(x^2+1-2xp_{34})}} \tag{18}$$

leads to the identities in Lemma 2.2.

□

Conjecture 2.3. *The remaining 2 roots of* (13) *are either complex or negative.*

Discussion: Let

$$\begin{aligned} f(x) &= \frac{x^2-xp_{24}-xp_{34}+p_{23}}{\sqrt{(x^2+1-2xp_{24})(x^2+1-2xp_{34})}}, \\ c &= \frac{1-p_{24}-p_{34}+p_{23}}{2\sqrt{(1-p_{24})(1-p_{34})}}. \end{aligned} \tag{19}$$

Representative plots[1] for

$$y = f(x) - c, y = f(x) + c \text{ and } y = [f(x)]^2 - c^2 \tag{20}$$

are available in Figures 2 and 3 for respectively. The roots of $y = 0$ for each of the functions in (20) are summarized in Table 2.

[1]Numerous plots were generated, yileding similar results

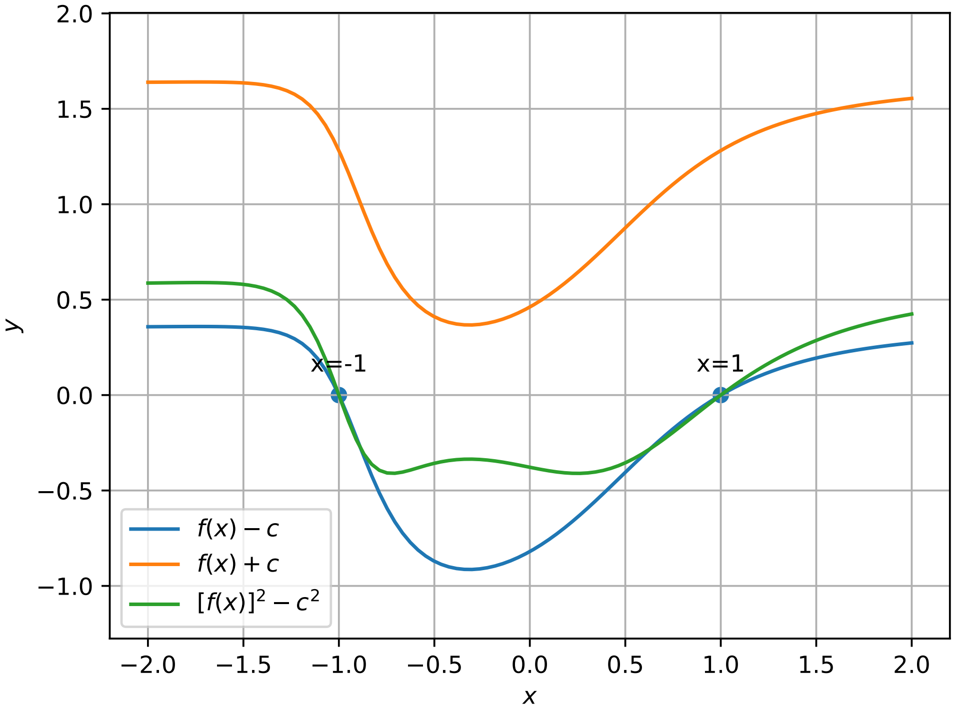

Figure 2. $\theta_1 = 354.29585899°, \theta_2 = 292.79715691°, \theta_3 = 192.4751066°, \theta_4 = 163.18664766°$. $f(x) = c$ has roots at $x = \pm 1$. $f(x) = -c$ has complex roots. The quartic has only two real roots at $x = \pm 1$. Behaviour of the roots is similar for other angles satisfying (14)

.

<table>
<tr><th>Figure No.</th><th>Angle Constraints</th><th>Roots of $f(x) = c$</th><th>Roots of $f(x) = -c$</th><th>Roots of the quartic $[f(x)]^2 = c^2$</th></tr>
<tr><td>2</td><td>$|\theta_2 - \theta_4| < \pi$
$\cap$
$|\theta_3 - \theta_4| < \pi$</td><td>$x = \pm 1$</td><td>Complex</td><td rowspan="2">Roots of $f(x) = \pm c$</td></tr>
<tr><td>3</td><td>$|\theta_2 - \theta_4| > \pi$
$\cap$
$|\theta_3 - \theta_4| < \pi$
or
$|\theta_2 - \theta_4| < \pi$
$\cap$
$|\theta_3 - \theta_4| > \pi$</td><td>$x_1 = 1, x_2 < 0$</td><td>$x_1 = -1$</td></tr>
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Table 2. Summary of the observations from Figs. 2 and 3

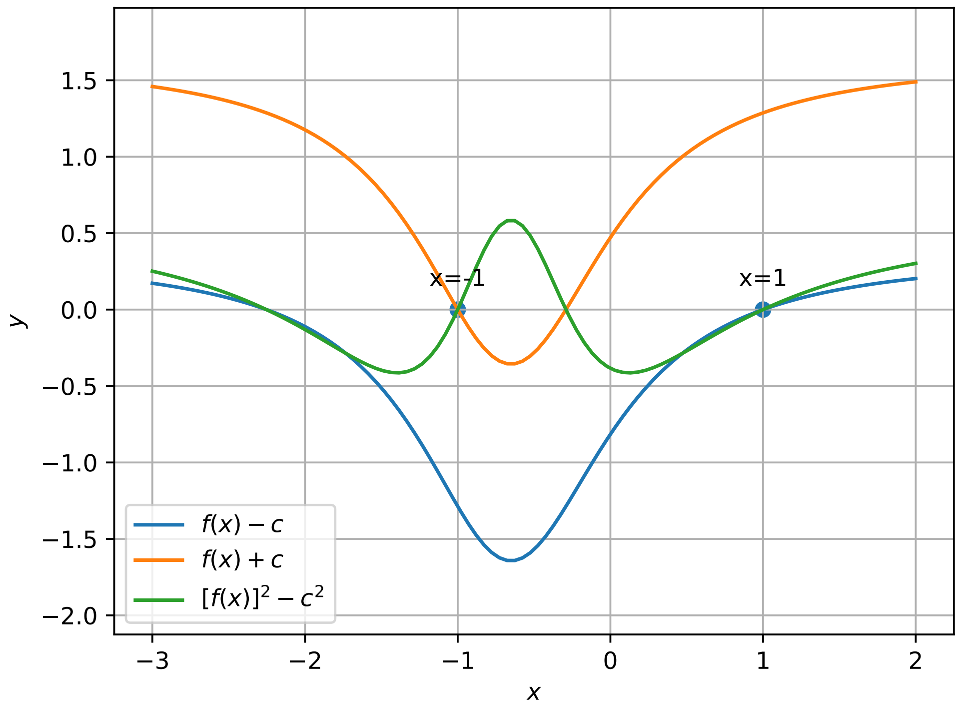

Figure 3. $\theta_1 = 338.29055322°, \theta_2 = 216.93519181°, \theta_3 = 117.0226174°, \theta_4 = 57.8273199°$. $f(x) = c$ has a root at $x = 1$ and one negative root. $f(x) = -c$ has two real roots with a root at $x = -1$. The quartic has roots at $x = \pm 1$ along with two more real roots. Behaviour of the roots is similar for other angles satisfying (16)

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Corollary 2.3. *$x = 1$ is the only positive root of*

$$\frac{1 - p_{24} - p_{34} + p_{23}}{2\sqrt{(1 - p_{24})(1 - p_{34})}} = \frac{x^2 - xp_{24} - xp_{34} + p_{23}}{\sqrt{(x^2 + 1 - 2xp_{24})(x^2 + 1 - 2xp_{34})}} \tag{21}$$

Proof. From Lemma 2.3 and Conjecture 2.3, we observe that (15) always has a root at $x = 1$ which is the only positive root of (13). The remaining root(s) of (13) and consequently (15) are either negative or complex. Thus, $x = 1$ is the only positive root of (15).

3. Main result

Theorem 3.1. *In Figure 4, if*

$$\angle BAC = \angle BDC = \theta, \tag{22}$$

then $ABCD$ is a cyclic quadrilateral

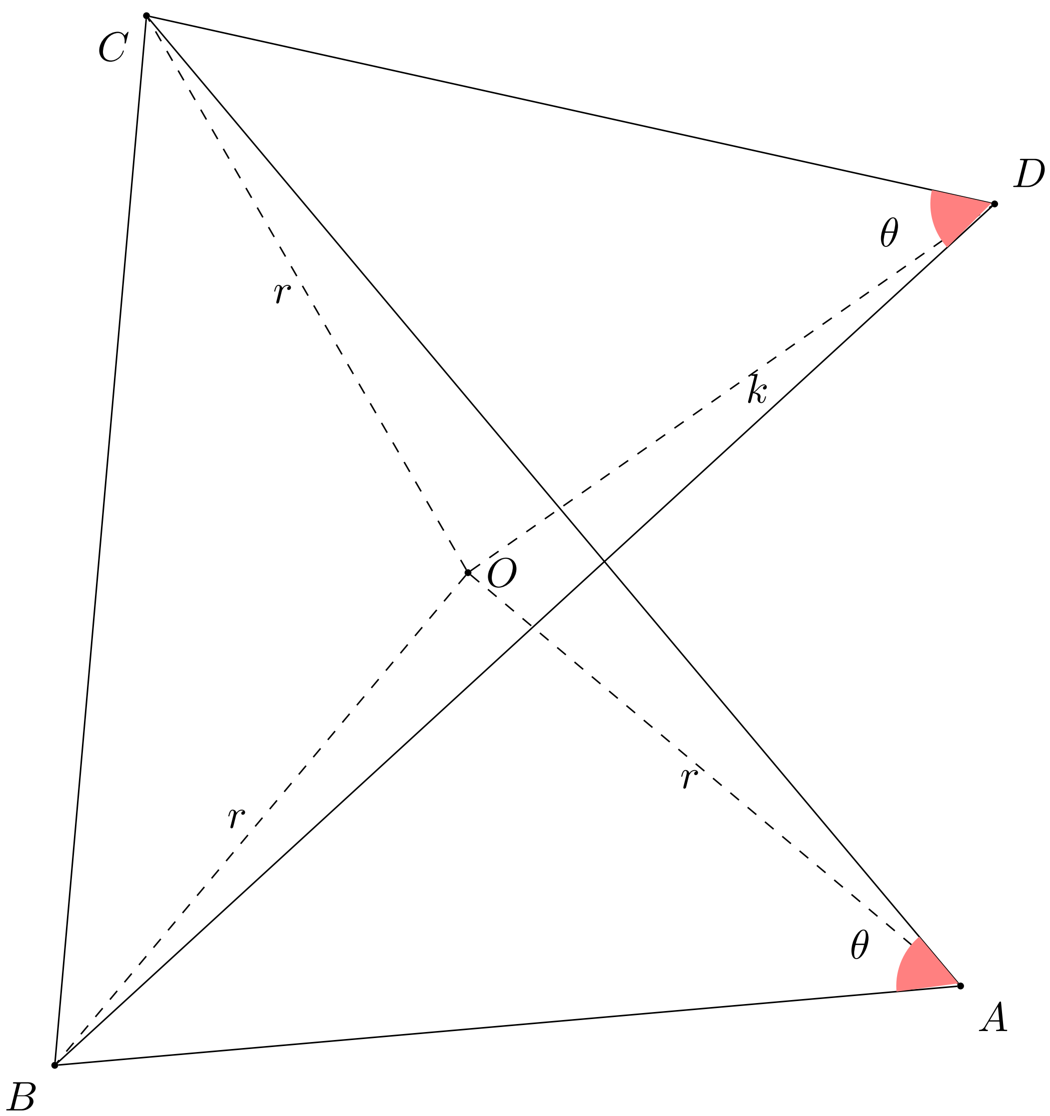

Figure 4. To show that $k = r$.

Proof. Let

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{23}$$

be the circumcentre of $\triangle ABC$ and let r be the radius. Without loss of generality, assuming that

$$\mathbf{A} = r \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = r\mathbf{u}_1, \mathbf{B} = r \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = r\mathbf{u}_2 \tag{24}$$

$$\mathbf{C} = r \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} = r\mathbf{u}_3, \mathbf{D} = \kappa \begin{pmatrix} \cos\theta_4 \\ \sin\theta_4 \end{pmatrix} = \kappa\mathbf{u}_4, \quad \theta_1 > \theta_2 > \theta_3 > \theta_4, \tag{25}$$

for $\mathbf{u}_i$ defined in (1), we need to show that

$$\kappa = r. \tag{26}$$

in Figure 4. The inner product of the vectors AB and AC is defined as [3]

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C}) = \|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\| \cos \angle BAC \tag{27}$$

which yields

$$\cos \angle BAC = \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \tag{28}$$

Similarly,

$$\cos \angle BDC = \frac{(\mathbf{D} - \mathbf{B})^\top (\mathbf{D} - \mathbf{C})}{\|\mathbf{D} - \mathbf{B}\| \|\mathbf{D} - \mathbf{C}\|} \tag{29}$$

From (22),

$$\angle BAC = \angle BDC \tag{30}$$
$$\implies \cos \angle BAC = \cos \angle BDC \tag{31}$$
$$\implies \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} = \frac{(\mathbf{D} - \mathbf{B})^\top (\mathbf{D} - \mathbf{C})}{\|\mathbf{D} - \mathbf{B}\| \|\mathbf{D} - \mathbf{C}\|} \tag{32}$$

upon substituting from (28) and (29). In (32),

$$\because (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C}) = r^2 (\mathbf{u}_1 - \mathbf{u}_2)^\top (\mathbf{u}_1 - \mathbf{u}_3) \tag{33}$$
$$= r^2 \left(\|\mathbf{u}_1\|^2 - \mathbf{u}_1^\top \mathbf{u}_3 - \mathbf{u}_1^\top \mathbf{u}_2 + \mathbf{u}_2^\top \mathbf{u}_3 \right) \tag{34}$$
$$= r^2 (1 - p_{12} - p_{13} + p_{23}) \tag{35}$$

upon substituting from (24)-(25) and using Lemma 2.1. Similarly,

$$(\mathbf{D} - \mathbf{B})^\top (\mathbf{D} - \mathbf{C}) = \kappa^2 - r\kappa p_{24} - r\kappa p_{34} + r^2 p_{23} \tag{36}$$

$$\|\mathbf{A} - \mathbf{B}\|^2 = (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = 2r^2 (1 - p_{12}) \tag{37}$$
$$\|\mathbf{A} - \mathbf{C}\|^2 = (\mathbf{A} - \mathbf{C})^\top (\mathbf{A} - \mathbf{C}) = 2r^2 (1 - p_{13}) \tag{38}$$

and

$$\|\mathbf{D} - \mathbf{B}\|^2 = (\mathbf{D} - \mathbf{B})^\top (\mathbf{D} - \mathbf{B}) = \kappa^2 + r^2 - 2\kappa r p_{24} \tag{39}$$
$$\|\mathbf{D} - \mathbf{C}\|^2 = (\mathbf{D} - \mathbf{C})^\top (\mathbf{D} - \mathbf{C}) = \kappa^2 + r^2 - 2\kappa r p_{34} \tag{40}$$

Substituting from (35) - (40) in (32),

$$\frac{(\mathbf{A}-\mathbf{B})^\top(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|}=\frac{(\mathbf{D}-\mathbf{B})^\top(\mathbf{D}-\mathbf{C})}{\|\mathbf{D}-\mathbf{B}\|\,\|\mathbf{D}-\mathbf{C}\|} \tag{41}$$

$$\implies \frac{r^2(1-p_{12}-p_{13}+p_{23})}{\sqrt{2r^2(1-p_{12})}\sqrt{2r^2(1-p_{13})}}=\frac{\kappa^2-r\kappa p_{24}-r\kappa p_{34}+r^2p_{23}}{\sqrt{\kappa^2+r^2-2\kappa r p_{24}}\sqrt{\kappa^2+r^2-2\kappa r p_{34}}} \tag{42}$$

which can be expressed as

$$\frac{1-p_{12}-p_{13}+p_{23}}{2\sqrt{(1-p_{12})(1-p_{13})}}=\frac{x^2-xp_{24}-xp_{34}+p_{23}}{\sqrt{(x^2+1-2xp_{24})(x^2+1-2xp_{34})}} \tag{43}$$

upon substituting

$$x=\frac{\kappa}{r}>0. \tag{44}$$

From (4) for $i=2, j=1, k=3$,

$$\frac{1-p_{12}-p_{13}+p_{23}}{2\sqrt{(1-p_{12})(1-p_{13})}}=\operatorname{sign}(\theta_2-\theta_1)\operatorname{sign}(\theta_3-\theta_1)\cos\left(\frac{\theta_2-\theta_3}{2}\right) \tag{45}$$

$$=\cos\left(\frac{\theta_2-\theta_3}{2}\right)\quad \because \theta_1>\theta_2>\theta_3 \tag{46}$$

Similarly, for $i=2, j=4, k=3$ in (4)

$$\frac{1-p_{24}-p_{34}+p_{23}}{2\sqrt{(1-p_{24})(1-p_{34})}}=\operatorname{sign}(\theta_2-\theta_4)\operatorname{sign}(\theta_3-\theta_4)\cos\left(\frac{\theta_2-\theta_3}{2}\right) \tag{47}$$

$$=\cos\left(\frac{\theta_2-\theta_3}{2}\right),\quad \because \theta_2>\theta_3>\theta_4 \tag{48}$$

From (46) and (48),

$$\frac{1-p_{12}-p_{13}+p_{23}}{2\sqrt{(1-p_{12})(1-p_{13})}}=\frac{1-p_{24}-p_{34}+p_{23}}{2\sqrt{(1-p_{24})(1-p_{34})}} \tag{49}$$

and (43) can be expressed as

$$\frac{1-p_{24}-p_{34}+p_{23}}{2\sqrt{(1-p_{24})(1-p_{34})}}=\frac{x^2-xp_{24}-xp_{34}+p_{23}}{\sqrt{(x^2+1-2xp_{24})(x^2+1-2xp_{34})}} \tag{50}$$

By definition, $x>0$ from (44). From Corollary 2.3, it follows that

$$x=1, \tag{51}$$

is the only possible a solution of (15) resulting in (26). Hence, D lies on the circumcircle of $\triangle ABC$ and $ABCD$ is a cyclic quadrilateral.

4. Application

Interestingly, the approach used in the proof of Theorem 3.1 can be used to prove the following theorem as well.

Theorem 4.1. *In Figure 5, if*

$$\angle ABC + \angle ADC = 180^\circ \tag{52}$$

then $ABCD$ is a cyclic quadrilateral.

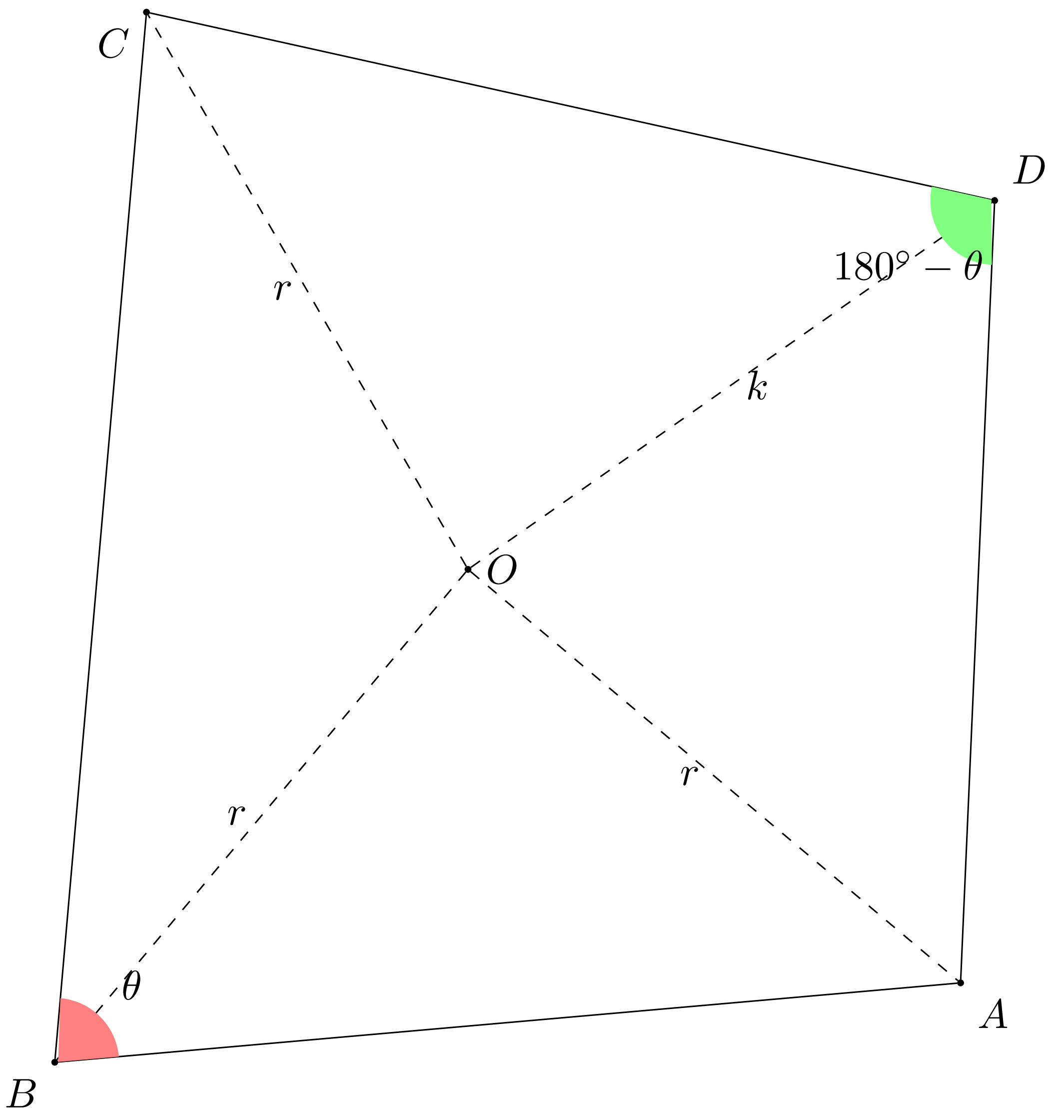

Figure 5. To show that $k = r$.

Proof. We follow the same notations as in Theorem 3.1 and assume that **O** is the circumcentre of $\triangle ABC$. Then we need to show that $k = r$ in Figure 5. Using the inner product and (52),

$$\angle ABC = \cos\theta = \frac{(\mathbf{B}-\mathbf{A})^\top (\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\| \, \|\mathbf{B}-\mathbf{C}\|} \tag{53}$$

$$\angle ADC = \cos(180-\theta) = -\cos\theta = \frac{(\mathbf{D}-\mathbf{A})^\top (\mathbf{D}-\mathbf{C})}{\|\mathbf{D}-\mathbf{A}\| \, \|\mathbf{D}-\mathbf{C}\|} \tag{54}$$

Thus,

$$\frac{(\mathbf{B}-\mathbf{A})^\top (\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\| \, \|\mathbf{B}-\mathbf{C}\|} = -\frac{(\mathbf{D}-\mathbf{A})^\top (\mathbf{D}-\mathbf{C})}{\|\mathbf{D}-\mathbf{A}\| \, \|\mathbf{D}-\mathbf{C}\|} \tag{55}$$

which can be expressed using (41)-(43) as

$$\frac{1-p_{12}-p_{23}+p_{13}}{2\sqrt{(1-p_{12})(1-p_{23})}} = -\frac{x^2 - xp_{14} - xp_{34} + p_{13}}{\sqrt{(x^2+1-2xp_{14})(x^2+1-2xp_{34})}} \tag{56}$$

Substituting $i=1, j=2, k=3$ in (4)

$$\begin{aligned} \frac{1-p_{12}-p_{23}+p_{13}}{2\sqrt{(1-p_{12})(1-p_{23})}} &= \operatorname{sign}(\theta_1-\theta_2)\operatorname{sign}(\theta_3-\theta_2)\cos\left(\frac{\theta_1-\theta_3}{2}\right) \\ &= -\cos\left(\frac{\theta_1-\theta_3}{2}\right) \quad \because \theta_1 > \theta_2 > \theta_3 \end{aligned} \tag{57}$$

Similarly, substituting $i=1, j=4, k=3$ in (4)

$$\frac{1-p_{14}-p_{43}+p_{13}}{2\sqrt{(1-p_{14})(1-p_{43})}} = \operatorname{sign}(\theta_1-\theta_4)\operatorname{sign}(\theta_3-\theta_4)\cos\left(\frac{\theta_1-\theta_3}{2}\right) \tag{58}$$

$$= \cos\left(\frac{\theta_1-\theta_3}{2}\right) \quad \because \theta_1 > \theta_3 > \theta_4 \tag{59}$$

Thus, from (57) and (59),

$$\frac{1-p_{12}-p_{23}+p_{13}}{2\sqrt{(1-p_{12})(1-p_{23})}} = -\frac{1-p_{14}-p_{43}+p_{13}}{2\sqrt{(1-p_{14})(1-p_{43})}} \tag{60}$$

$$\implies \frac{1-p_{14}-p_{43}+p_{13}}{2\sqrt{(1-p_{14})(1-p_{43})}} = \frac{x^2 - xp_{14} - xp_{34} + p_{13}}{\sqrt{(x^2+1-2xp_{14})(x^2+1-2xp_{34})}} \tag{61}$$

upon substituting (60) in (56). From Corollary 2.3, it follows that

$$x = 1. \tag{62}$$

Thus, $k = r$, D lies on the circumcircle of $\triangle ABC$ and $ABCD$ is a cyclic quadrilateral.

References

[1] NCERT, *Mathematics*, 1st ed., chap. Circles, Textbook for Class IX, National Council of Educational Research and Training, New Delhi (2019).

[2] NCERT, *Mathematics*, 1st ed., chap. Trigonometric Functions, Textbook for Class XI, National Council of Educational Research and Training, New Delhi (2019).

[3] NCERT, *Mathematics*, 1st ed., chap. Probability, Textbook for Class X, National Council of Educational Research and Training, New Delhi (2019).