Almost multiplicative functions
on a class of Banach algebras

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Almost multiplicative function

A linear map \( \phi : A \rightarrow \mathbb{C} \) is said to be **multiplicative** if

\[
\phi(ab) = \phi(a)\phi(b) \quad \text{for all} \quad a, b \in A.
\]

**Definition (almost multiplicative function)**

A linear map \( \phi : A \rightarrow \mathbb{C} \) is said to be **amf** if there exists a \( \delta > 0 \) such that

\[
|\phi(ab) - \phi(a)\phi(b)| \leq \delta \|a\| \|b\| \quad \text{for all} \quad a, b \in A.
\]

- Originated from perturbation theory.
- **amf** are continuous.
- AMNM Algebra
Condition spectrum

\[ \sigma(a) := \{ \lambda \in \mathbb{C} : \lambda - a \in Sing(A) \} , \]

Definition (\( \varepsilon \)-Condition spectrum (0 < \( \varepsilon \) < 1))

\[ \sigma_\varepsilon(a) := \left\{ \lambda \in \mathbb{C} : \|\lambda - a\| \| (\lambda - a)^{-1} \| \geq \frac{1}{\varepsilon} \right\} \]

1. \( \sigma(a) \subseteq \sigma_\varepsilon(a) \), for every \( a \in A \) and for every \( \varepsilon > 0 \). The two spectrum coincides if and only if \( a \) is a scalar multiple of identity.

2. If \( \lambda \in \sigma_\varepsilon(a) \) then \( |\lambda| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \|a\| \).

3. If \( \lambda \in \sigma_\varepsilon(a) \) then there exists a \( b \in Sing(A) \) such that

\[ \|b\| \leq \varepsilon \|\lambda - a\| , \quad \lambda \in \sigma(a + b) . \]
Theorem

Let $A$ be complex commutative Banach algebra with unit $1$ and let $\phi$ be a $\delta$-amf on $A$ and $\phi(1) = 1$. Then

$$\phi(a) \in \sigma_\delta(a) \quad \forall a \in A.$$
Assumption: The class of complex commutative Banach algebras with this property

\[ (*) \quad \forall a \in \text{Inv}(A), \exists b \in \text{Sing}(A) \text{ such that } \|a - b\| = \frac{1}{\|a^{-1}\|}. \]

Example: Function algebras

Lemma

Let $A$ be a complex commutative Banach algebra satisfying $(*)$ and let $\lambda \in \sigma_\varepsilon(a)$. Then,

\[ d(\lambda, \sigma(a)) \leq \frac{2\varepsilon}{1 - \varepsilon} \|a\|. \]
Theorem

Let $A$ be a complex commutative unital Banach algebra with the property given in $(\ast)$. Let $a \in A$ and $\lambda \in \sigma_\epsilon(a)$. Then, there exists an almost $\delta$-amf $\psi$ such that $\psi(1) = 1$ and $\lambda = \psi(a)$, where

$$\delta = \alpha(3 + \alpha), \quad \alpha = \frac{2\epsilon^2 \| a \|}{(1 - \epsilon)m}, \quad m = \inf \{ \| z - a \| : z \in \mathbb{C} \}. $$
Theorem

Let $A$ be a function algebra and $\phi : A \to \mathbb{C}$ be a linear function. If $\phi(a) \in \sigma_\epsilon(a)$ for every $a$ in $A$. Then $\phi$ is $\delta$-amf, where

$$\delta = \log \left( \kappa^{-1} \right)^{-1} 2(2\kappa + 1) \quad \text{with} \quad \kappa = \frac{2\epsilon}{1 - \epsilon}.$$

Theorem (GKŽ Theorem)

Let $A$ be complex Banach algebra and $\phi : A \to \mathbb{C}$ be a linear map with $\phi(1) = 1$. If, for every $a \in A$,

$$\phi(a) \in \sigma(a)$$

then $\phi$ is multiplicative.
Conclusion

1. If \( \phi \) is \( \delta \)-amf, then \( \phi(a) \in \sigma_\delta(a) \) for all \( a \) in \( A \).
2. If \( \lambda \in \sigma_\epsilon(a) \), then \( \lambda = \phi(a) \) for some \( \delta(\epsilon) \)-amf \( \phi \).
3. If \( \phi \) is linear and
\[ \phi(a) \in \sigma_\epsilon(a) \quad \forall a \in A, \]
then \( \phi \) is \( \delta \)-amf for some \( \delta(\epsilon) \).
References

