Post Correspondence Problem - Example

Here is an instance $\langle M, w \rangle$ of A_{TM}.

- input string $w = 0100$
- machine M has set of states $Q = \{q_0, q_1, q_2, \ldots q_7, q_a\}$
- machine M has start state $q_0 \in Q$ and accept state $q_a \in Q$
- machine M has tape alphabet $\Gamma = \{0, 1, 2, 3, \sqcup\}$
- machine M has transition function δ. A few of the transitions of M are:

$$\delta(q_0, 0) = (q_7, 2, R),\ \delta(q_7, 1) = (q_2, 3),\ \delta(q_2, 0) = (q_a, 0, R)$$

- machine M cleans up after itself: if it reaches state q_a it cleans the tape and sits at the first tape-square.

Observation: The configurations of M on w are:

$$\begin{aligned}
C_0 &= \mathsf{q_0 0100} \\
C_1 &= \mathsf{2 q_7 100} \\
C_2 &= \mathsf{23 q_2 00} \\
C_3 &= \mathsf{230 q_a 0} \\
C_4 &= \mathsf{230 q_a} \\
C_5 &= \mathsf{23 q_a} \\
C_6 &= \mathsf{2 q_a} \\
C_7 &= \mathsf{q_a}
\end{aligned}$$

Hence the (accepting) computation history of M on w is

$$\#\mathsf{q_0 0100}\#\mathsf{2q_7 100}\#\mathsf{23q_2 00}\#\mathsf{230q_a 0}\#\mathsf{230q_a}\#\mathsf{23q_a}\#\mathsf{2q_a}\#\mathsf{q_a}\#\#$$

The instance of **MPCP**, P' (which is a finite set of dominoes) is obtained by applying the 7 rules to $\langle M, w\rangle$.

1. Rule 1 adds $\left[\frac{\#}{\#q_0 0100\#}\right]$ as the "first" domino of P'
2. Rule 2 adds $\left[\frac{q_0 0}{2q_7}\right]$, $\left[\frac{q_7 1}{3q_2}\right]$ and $\left[\frac{q_2 0}{0q_a}\right]$
3. Rule 3 adds nothing (in our case)
4. Rule 4 adds $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{3}{3}\right]$ and $\left[\frac{\sqcup}{\sqcup}\right]$
5. Rule 5 adds $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$
6. Rule 6 adds $\left[\frac{0q_a}{q_a}\right]$, $\left[\frac{q_a 0}{q_a}\right]$, $\cdots$ $\left[\frac{3q_a}{q_a}\right]$ and $\left[\frac{q_a 3}{q_a}\right]$
7. Rule 7 adds $\left[\frac{q_a\#\#}{\#}\right]$

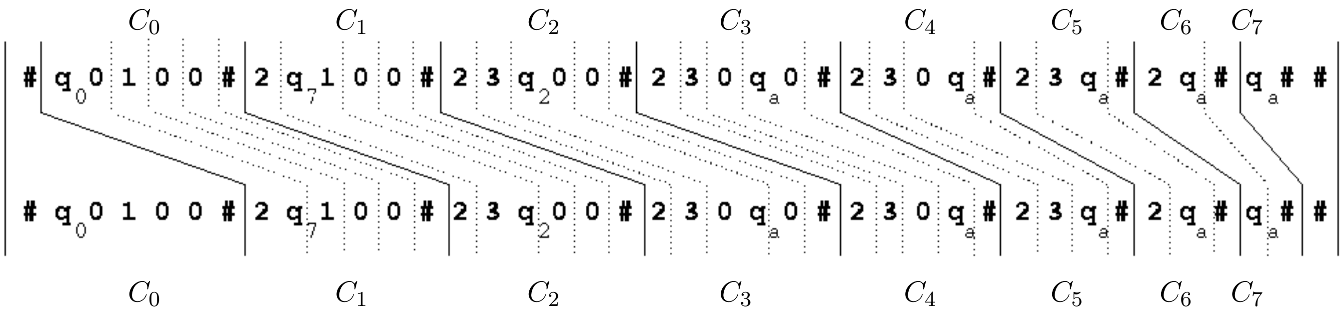
Thus in this example, we have

$$P' = \left\{ \left[\frac{\#}{\#q_0 0100\#}\right], \left[\frac{q_0 0}{2q_7}\right], \left[\frac{q_7 1}{3q_2}\right], \left[\frac{q_2 0}{0q_a}\right], \left[\frac{0}{0}\right], \ldots, \left[\frac{\sqcup}{\sqcup}\right], \left[\frac{\#}{\#}\right], \left[\frac{\#}{\sqcup\#}\right], \left[\frac{0q_a}{q_a}\right], \cdots \left[\frac{q_a 3}{q_a}\right], \left[\frac{q_a\#\#}{\#}\right] \right\}$$

Claim: Given M, w, construct P' using the rules. Then

P' has a match beginning with first domino $\Leftrightarrow$ M has an accepting computation history on w.

In our example, M does accept w. The match in P' is just the accepting computation history:



which is essentially

$$\left[\frac{\#}{\#q_0 0100\#}\right] \left[\frac{q_0 0}{2q_7}\right] \left[\frac{1}{1}\right] \left[\frac{0}{0}\right] \left[\frac{0}{0}\right] \left[\frac{\#}{\#}\right] \left[\frac{2}{2}\right] \left[\frac{q_7 1}{3q_2}\right] \left[\frac{0}{0}\right] \cdots \left[\frac{q_a\#\#}{\#}\right]$$

Conversely, if a match beginning with first domino does exist in P' then it must be the accepting computation history.