CS 4510 : Automata and Complexity : Cook-Levin Theorem

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Valid Windows

Let $A \in \text{NP}$ and let N be a single tape NTM which decides A. Let $N = (Q, \Sigma, \Gamma, \delta, q_s, q_a, q_r)$. For an input of length n, let the running time of N be upper bounded by $n^k - 3$. Define $\Delta = Q \cup \Gamma \cup \{\#\}$. Here are a list of valid 2×3 windows for the Cook-Levin Theorem reduction. We use the symbols $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}$ to denote tape symbols from $\Gamma \cup \{\#\}$ and q, r to denote states from Q.

- When the state symbol is in the middle of the top row.
  - When $(r, \mathtt{d}, R) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma \cup \{\#\}$.

b	q	a
b	d	r

  - When $(r, \mathtt{d}, L) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma$.

b	q	a
r	b	d

  - When $(r, \mathtt{d}, L) \in \delta(q, \mathtt{a})$.

#	q	a
#	r	d

- When the state symbol is in the right end of the top row.
  - If $\exists q \in Q, \mathtt{a} \in \Gamma$ such that $(r, \mathtt{d}, R) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma \cup \{\#\}$, $c \in \Gamma$.

b	c	q
b	c	d

  - If $\exists \mathtt{a}, \mathtt{d} \in \Gamma$ such that $(r, \mathtt{d}, L) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma \cup \{\#\}$, $c \in \Gamma$.

b	c	q
b	r	c

- When the state symbol is in the left end of the top row.
  - When $(r, \mathtt{d}, R) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma \cup \{\#\}$.

q	a	b
d	r	b

  - If $\exists r \in Q$ such that $(r, \mathtt{d}, L) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma \cup \{\#\}$, $\mathtt{c} \in \Gamma$.

q	a	b
c	d	b

  - When $(r, \mathtt{d}, L) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma \cup \{\#\}$. This is a valid window because this situation can occur, when this represents the left-most end of the tape.

q	a	b
r	d	b

- No state symbol in the top row.
  - For all $\mathtt{a} \in \Gamma \cup \{\#\}$, $\mathtt{b}, \mathtt{c} \in \Gamma$.

a	b	c
a	b	c

  - For all $\mathtt{a}, \mathtt{b} \in \Gamma$, $\mathtt{c} \in \Gamma \cup \{\#\}$.

a	b	c
a	b	c

  - If $\exists q \in Q, \mathtt{d} \in \Gamma$ such that $(r, \mathtt{d}, R) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma$, $\mathtt{c} \in \Gamma \cup \{\#\}$.

a	b	c
r	b	c

  - If $\exists q, r \in Q$ such that $(r, \mathtt{d}, L) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma$, $\mathtt{c} \in \Gamma \cup \{\#\}$.

a	b	c
d	b	c

  - If $\exists q \in Q, \mathtt{a}, \mathtt{d} \in \Gamma$ such that $(r, \mathtt{d}, L) \in \delta(q, \mathtt{a})$, for all $\mathtt{b} \in \Gamma \cup \{\#\}$, $\mathtt{c}, \mathtt{e} \in \Gamma$.

b	c	e
b	c	r

These are the list of **all** valid windows that can appear in the tableau of the machine N. Note that this is a constant that depends only on N and is independent of the input. The correctness of the table follows from two basic observations.

1. A tape symbol that is not adjacent to a state symbol in the top configuration should be unchanged in the following configuration. This is preserved because any such symbol which is not adjacent to a state symbol appears in the middle of the top row of a 2×3 valid window which has no state symbol on the top row. Moreover, in such a valid window, the middle symbol of the top row remains unchanged from top to bottom.

2. For the state symbols, and the tape symbols adjacent to the state symbol, they change based on the moves of the NTM N. This is reflected in the table because for every such possible combination, there is a window which has the state symbol in the middle of the top row. The valid windows list all possible moves of N, fixing all the three symbols in the bottom row of the window.

So we can conclude the following lemma.

Lemma 0.1. *For all $1 \leq i \leq n^k$, if the ith row of the table is a valid configuration C_i of N, then the following $(i+1)$th row is one of the valid successor configurations C_{i+1}.*

By using induction on i, and using the fact that the first row is the valid starting configuration of N on w, we can conclude that the tableau is one of the valid sequence of configurations that can occur when N computes on w. It follows that N accepts w iff there is a tableau which starts with the starting configuration and contains an accepting state.