Computation History of a Turing machine

Let $M = (Q, \Sigma, \Gamma, \delta_M, q_s, q_a, q_r)$ be a deterministic Turing machine and w be an input to M.

Accepting Computation History of M on w: A sequence of configurations$C_0, \cdots, C_t$ such that:

1. C_0 is the initial configuration of M on input w.
2. For all $1 \leq i \leq t$, the machine M moves from C_{i-1} to C_i.
3. C_t is an accepting configuration.

Represent an accepting computation history as a string over the alphabet $\Delta = Q \cup \Gamma \cup \{\#\}$ as:

$$C_0 \# C_1 \# \cdots \# C_t.$$

Not accepting computation histories

Fix a DTM M and an input w.

Let $\mathrm{NOT}-\mathrm{ACH}_{M,w} \;=\; \{z \;\in \Delta^* \mid z$ is not an accepting computationhistory of M on $w\}$.

Claim: $\mathrm{NOT}-\mathrm{ACH}_{M,w}$ is context-free and there is an algorithm to construct a context-free grammar $G_{M,w}$ for $\mathrm{NOT}-\mathrm{ACH}_{M,w}$.

Corollary: The following language is undecidable:

$$\mathrm{ALL}_{\mathrm{CFG}} \;=\; \{\langle G\rangle \mid L(G) \;=\; \Delta^*\}.$$

Proof of Corollary: Reduction from A_{TM} to $\overline{\mathrm{ALL}_{\mathrm{CFG}}}$ using the claim above.

Reduction from A_{TM} to $\overline{\mathrm{ALL}_{\mathrm{CFG}}}$

- Let $z \in \Delta^*$. The string is either a valid sequence of configurations or not.
- If z is not a valid sequence of configurations, $z \in \mathrm{NOT}-\mathrm{ACH}_{M,w}$.
- Suppose z is a valid sequence of configurations. If M does not accept w then $z \in \mathrm{NOT}-\mathrm{ACH}_{M,w}$.
- Therefore, if M does not accept w then $\mathrm{NOT}-\mathrm{ACH}_{M,w} = \Delta^*$.
- If M accepts w then $z \notin \mathrm{NOT}-\mathrm{ACH}_{M,w}$. In this case, $\mathrm{NOT}-\mathrm{ACH}_{M,w} \neq \Delta^*$.
- Let $G_{M,w}$ be the context-free grammar from the claim such that $L(G_{M,w}) = \mathrm{NOT}-\mathrm{ACH}_{M,w}$.
- Therefore, M accepts w if and only if $L(G_{M,w}) \neq \Delta^*$.

An algorithm to construct a context-free grammar for $\mathrm{NOT} - \mathrm{ACH}_{M,w}$

A string z in Δ^* is in $\mathrm{NOT} - \mathrm{ACH}_{M,w}$ if and only if it is the union of the following four languages:

1. z is not well-formed: does not start or end with a #, no state symbol between two # marks, more than one state symbol between two # marks.

   - Regular language.

2. z does not start correctly: the string between the first two # marks is not the initial configuration $q_s w_1 w_2 \ldots w_n$ of M on $w = w_1 w_2 \ldots w_n$.

   - Regular language.

3. z does not end correctly: the string between the last two # marks is not an accepting configuration - the state symbol in this string is not q_s.

   - Regular language.

4. There exists i such that C_{i+1} does not follow legally from C_i.

   - Nondeterministically guess i such that C_{i+1} does not follow legally from C_i.
   - Use the PDA (from an earlier construction) that on input $C_i \# C_{i+1} \in \Delta^*$ accepts iff C_{i+1} is not a valid immediate successor configuration of C_i on input w.

Construct a PDA $\mathcal{P}$ that non-deterministically checks for membership of z in one of the four context-free languages above.

Construct a CFG $G_{M,w}$ that is equivalent to the PDA $\mathcal{P}$.