# CS5660 Offline Exam1

20-Mar-2024 7pm-8:30pm

**NOTE:** Please write your ROLL NO. clearly on ALL answer sheets. Be extremely precise and formal in your derivations. Answers without justification will NOT be awarded marks. Don't use any advanced/sophisticated results not taught in this course.

1. For an $L$-smooth and convex function $f$, prove from first principles that:

$$\frac{1}{L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq \left(\nabla f(x) - \nabla f(y)\right)^\top (x - y) \leq L\|x - y\|_2^2.$$

   Clearly indicate which steps/results use convexity and which use smoothness etc. Hint: Consider functions $\phi_x(y) \equiv f(y) - \nabla f(x)^\top y$.

   [1+ 3 Marks]

   Using this inequality, present a convergence analysis of gradient descent for unconstrained minimization of $L-$smooth, convex objectives. Assume the step-size is $\frac{1}{L}$. Hint: Try to set-up recursion starting with $r_{k+1}^2$, then lower bounding $\|\nabla f(x^{(k)})\|$ and solving the recursion.

   [4 Marks]

2. Attempt this sequence of questions from section 6.4.4 in Beck's book and (almost :) covered in the lectures:

   (a) Let $f(x) \equiv -\log(x), x \in \mathbb{R}_{++}$. Derive a simplified expression for $prox_{\lambda f}(x)$ for a fixed (given, yet arbitrary) $x \in \mathbb{R}, \lambda > 0$.

   [2 Marks]

   (b) Consider the function $h(x) = -\sum_{i=1}^{n} \log(x_i), x \equiv \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Derive a simplified expression for $prox_{\lambda h}(x)$ for a fixed (given, yet arbitrary) $x \in \mathbb{R}^n, \lambda > 0$.

   [1 Mark]

   (c) Consider the set $C_\alpha \equiv \left\{ x \equiv \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}_{++}^n \;\middle|\; x_1 x_2 \ldots x_n \geq \alpha \right\}$, where $\alpha > 0$. Express this set in terms of $h$.

   [1 Mark]

   (d) Present a *simple* algorithm for computing $\Pi_{C_\alpha}(x)$ that uses your simplified expression for $prox_{\lambda h}(x)$.

   [4 Marks]

3. Let $C$ be a closed convex set and let $\Pi_C$ denote the projection onto $C$ operator. Express the prox operator of $f$, where $f$ is:

   (a) the support function of $C$,

   [2 Marks+2Marks]

   (b) defined by $f(x) = \frac{1}{2}\|x - \Pi_C(x)\|_2^2$ (projection error),

   [2 Marks]

in terms of $\Pi_C$. In case you use any theorem/result from this course, then repeat it's proof/derivation. In case you use any (advanced) theorem/result outside this course, then clearly write down the formal statement (no need to prove).

4. For any closed convex set $C \subset \mathbb{R}^n$ and any function $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that it's prox operator is well-defined, show that
$$\left(\Pi_C\left(prox_f(x)\right) - prox_f(x)\right)^\top \left(\Pi_C(x) - \Pi_C\left(prox_f(x)\right)\right) \geq 0 \; \forall \; x \in \mathbb{R}^n.$$

[2+2 Marks]