CS5590: Exam - 1

7:45pm-9:15pm, 03-Sep-2019

ROLL NO.________________

Note: Fill in the blanks/boxes appropriately such that the respective statements become true. While filling the blanks/boxes strictly follow the instructions in the respective question appearing immediately after/before the blank/box. You are free to use any standard mathematical symbols like $\pi, e, \Sigma, ||\cdot||, \log, \max$ etc. Answers that are not simplified enough, (correct) answers in wrong format, illegible writings, and those outside the blanks/boxes, will be ignored by the evaluator. Please attempt the problems in rough sheets first and prepare answers for all the blanks/boxes in rough. Then fair copy them in this sheet while respecting the boundaries of the blanks/boxes.

1. Consider a binary classification problem with input space $\mathcal{X} = \mathbb{R}^n$, and output space $\mathcal{Y} = \{-1, 1\}$. Consider a training set given by $= \{(x_0, 1), (-x_0, -1)\}$, where $x_0 \in \mathcal{X}$ $(\neq 0)$ is a given fixed point. It is proposed to employ the logistic loss and the linear inductive bias (without the norm-bound). Then, the simplified expression for the ERM problem is:

$$\min_{w \in \mathbb{R}^n} \left[\log\left(1 + e^{-w^T x_0}\right) \right].$$

Your expression in the previous blank must involve w, x_0 only.

[1 Mark]

In the box below argue that this optimization problem as no solution:

Since log, e are monotonic, the problem is equivalent to $\max_{w \in \mathbb{R}^n} w^T x_0 = \infty$ | This diverges! because obj. is a monotonic function of $\|w\|$. $\therefore$ No Soln.

[1 Mark]

Now, the inductive bias is changed to linear functions with norm-bound given by hyperparameter $W = 1$. Then, the optimal solution, $\hat{w}$, of the corresponding ERM problem in it's original form involving the hyperparameter[1] W is given by: $\hat{w} = x_0 / \|x_0\|$. This expression must involve x_0 alone.

[0.5 Mark]

The above exercise highlights yet another advantage of the norm-bounded linear functions over the set of all linear functions!

[1] Here, the ERM is not re-written in the Tikhonov form i.e., it is NOT rewritten in the regularized risk minimization form.

2. Assume you have a classification problem with 3 classes: '✠', '†', and '¶'. The loss function, l, you would employ if you were restricted to model only one real-valued function, f, is given by $l(x, \maltese, f) \equiv$ $\mathbb{1}\{w^T x > 0\}$, $l(x, \dagger, f) \equiv$ $\mathbb{1}\{w^T x < 0\} + \mathbb{1}\{w^T x > 1\}$, $l(x, \P, f) \equiv$ $\mathbb{1}\{w^T x < 1\}$.

[1.5 Marks; Practice set problem!]

Suppose you re allowed to model 3 real-valued functions, say, f, g, h. Then, the loss function you would employ is given by: $l\,(x, \maltese, (f, g, h)) \equiv$ $\max(0, 1 - [f(x) - \max(g(x), h(x))])$.

[2 Marks; Practice set problem!]

3. Consider a regression problem where the input space, $\mathcal{X} = \mathbb{R}^n$, and the output space, $\mathcal{Y} = \mathbb{R}$. It is proposed to use the inductive bias as the set of all affine functions:

$$\mathcal{G} \equiv \left\{ g \mid \exists\ w \in \mathbb{R}^n,\ b \in \mathbb{R}\ \ni\ g(x) = w^\top x - b\ \forall\ x \in \mathcal{X} \right\}.$$

The parameters are $w \in \mathbb{R}^n, b \in \mathbb{R}$. The loss to be used is the square loss. Let the input vectors in the training set be arranged as column vectors in the matrix $X_{n \times m}$, where m is the training set size. Let $y_{m \times 1}$ denote the vector with entries as the corresponding outputs in the training set. Let us denote the q-dimensional vector with all entries as unity by 1_q. Then, the simplified expression for the ERM optimization problem written in terms of $X, y, 1_m, w, b$ is given by:

$$\min_{w \in \mathbb{R}^n, b \in \mathbb{R}} \left[\frac{1}{m} \left(\|X^T w - b 1_m - y\|^2 \right) \right].$$

Your expression in the previous blank must not use explicit symbols for columns, entries of X, y. In other words, please employ vector operations rather than scalar ones.

[1 Mark]

Now, if the inductive bias is changed to norm-bounded affine functions:

$$\mathcal{G}_W \equiv \left\{ g \mid \exists\ w \in \mathbb{R}^n,\ b \in \mathbb{R},\ \|w\| \leq W\ \ni\ g(x) = w^\top x - b\ \forall\ x \in \mathcal{X} \right\},$$

the simplified expression for the ERM problem in Tikhonov form[2] turns out to be:

$$\min_{w \in \mathbb{R}^n, b \in \mathbb{R}} \left[\|X^T w - b 1_m - y\|^2 + \lambda \|w\|^2 \right].$$

Your expression in the previous blank must be in terms of $X, y, 1_m, w, b, \lambda$ only, where λ is the hyperparameter (that replaces W).

[0.5 Mark]

Now, by repeating the analysis done in the lecture for the case of linear/ridge regression, or otherwise, find an analytical expression for the optimal solution, $(\hat{w}, \hat{b})$, of this problem. The optimal $\hat{w}$ satisfies the following linear equalities:

$$\left(XX^T - \frac{X 1_m 1_m^T X^T}{m} + \lambda I_n \right) \hat{w} = \left(Xy - \frac{X 1_m 1_m^T y}{m} \right)$$

Your expression for the first of the previous two blanks must be only in terms of $X, \lambda, I_n, 1_m, m$, where I_n is the identity matrix of size n. And, the second must be in terms of $X, y, 1_m, m$.

[2] Regularized risk minimization form.

[1 Mark]

In the following box, please write a formal proof of why the matrix in the first of the previous two blanks, denoted by, say, P, is positive definite.

It is easy to see that P is symmetric. $\alpha^T P \alpha = \alpha^T X X^T \alpha - \frac{\alpha^T X 1_m 1_m^T X^T \alpha}{m} + \lambda \alpha^T \alpha$ Call $X^T \alpha = \gamma$	Now $(\gamma^T \gamma)(m) \geq (\gamma^T 1_m)^2$ By Cauchy Schwartz inequality. $\therefore \alpha^T P \alpha > 0$ (as $\lambda > 0, \alpha \neq 0$)

Hence, the optimal $\hat{w} = P^{-1}q$, where q denotes the vector in the second of the previous two blanks.

[1 Mark]

The optimal $\hat{b}$ is given by the expression: $\left(\hat{\omega}^T X 1_m - y 1_m\right)/m$. This expression must involve $\hat{w}, X, y, 1_m, m$ alone.

[0.5 Mark]