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Ans Let the dim. of symmetric matrices under question be $\underline{n}$.

Let I_n be the identity matrix of dim n.

It is easy to see that $f(-I_n) = -1 < 0$

(infact all eigenvalues are -1).

Since f takes negative values it can $\underline{\text{never}}$ be a norm. (for any $n \in \mathbb{N}$).

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Ans We begin with the claim:

$$f(A) = \max_{y \in \mathbb{R}^n} y^T A y$$
$$\text{s.t. } \|y\|_2^2 = 1$$

Proof of claim: From Spectral theorem it follows that

$$A = \sum_i \rho_i \gamma_i \gamma_i^T \text{ , where } \rho_i \in \mathbb{R}$$

& $\gamma_1 \ldots \gamma_n$ are orthogonal with unit columns.

$$\therefore f(A) = \max_{y \in \mathbb{R}^n} \sum_i \rho_i \, y^T \gamma_i \gamma_i^T y$$
$$\text{s.t. } \sum_i y_i^2 = 1$$

Let $\gamma_i^T y = z_i$. It is clear that $\sum_i z_i^2 = 1 \Leftrightarrow \sum_i y_i^2 = 1$
($\because \gamma_i$ are orthogonal basis)

$$\Rightarrow f(A) = \max_{z \in \mathbb{R}^n} \sum_i \rho_i z_i^2$$
$$\text{s.t. } \sum_i z_i^2 = 1$$

($\because$ "cows' knapsack problem")

$$= \max_i \rho_i$$

$=$

Also, $y^T A y = tr(y^T A y) = tr(A y y^T)$

Let $Y = y y^T \iff Y \in S_n$, rank$(Y) = 1$.

Then $\|y_2\|_2^2 = 1 \iff tr(Y) = 1$

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From above it follows that

$$f(A) = \max_{Y \in S_n} \langle A, Y \rangle$$

$$\underline{\underline{s.t.}} \quad \mathrm{tr}(Y) = 1, \ \mathrm{rank}(Y) = 1.$$

$\therefore$ f is support function of the set.

$$C \equiv \{ Y \in S_n \mid \mathrm{tr}(Y) = 1, \mathrm{rank}(Y) = 1 \}.$$

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$$f^*(-A) = \max_{M \in S_n} \langle A, M \rangle$$

$$\underline{\underline{\text{s.t.}}} \quad f(M) \le 1$$

Let $A = \sum_i \rho_i v_i v_i^T$ be the e.v.d.

$$f^*(-A) = \max_{M \in S_n} \sum_i \rho_i v_i^T M v_i \qquad \text{I}$$

$$\underline{\underline{\text{s.t.}}} \quad \underline{\max_{\|y\|_2^2 = 1} y^T M y \le 1}$$

$\hookrightarrow$ from Q2

Now if any $\rho_i < 0$, then choose $M \ni v_i^T M v_i \to -\infty$

$\therefore f^*(-A) = \infty$ whenever any $\lambda_j(A) < 0$

If $\rho_i \geq 0 \ \forall i$, then clearly

$$\text{II} \qquad f^*(-A) \le \sum_i \rho_i \quad \left(\because y^T M y \le 1 \ \forall \|y\|_2^2 = 1 \right)$$

But we can choose $M \ni v_i^T M v_i = 1 \ \forall i$

So all eigenvalues are 1.

But we can choose $\bar{M}$ $\Rightarrow$ $v_i^T \bar{M} v_i = 1 \;\forall i$

& all eigenvalues are 1.

i.e., $\bar{M} = \sum_i v_i v_i^T$.

with this choice RHS of I $= \sum_i \rho_i$

From I, II, $f^*(-A) = \sum_i \rho_i$.

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$$\therefore f^*(A) = \begin{cases} \text{trace}(-A) & \text{if } -A \succeq 0 \\ \infty & \text{else} \end{cases}$$

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Ans Since g is closed conic, we have:

$$g(x) = \max_{y \in C} \langle x, y \rangle \quad \text{for some set } C.$$

$$\Rightarrow g(\lambda x) = \max_{y \in C} \langle \lambda x, y \rangle \quad \forall \lambda \geq 0$$

$$\Rightarrow g(\lambda x) = \lambda \max_{y \in C} \langle x, y \rangle \quad \forall \lambda \geq 0$$

$$\Rightarrow g(\lambda x) = \lambda g(x) \quad \forall \lambda \geq 0$$

Hence Proved.

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Ans Let K be a polyhedral cone

$\iff K = \text{CONIC}(\{\gamma_1 \cdots \gamma_m\})$ for some $\gamma_1 \cdots \gamma_m$

Now, $K^* = \{ w / \langle w, \gamma \rangle \geq 0 \ \forall \gamma \in K \}$

$K^* = \{ w / \underbrace{\langle w, \gamma_i \rangle \geq 0 \ \forall \ i = 1 \text{ to } m}_{\text{denote by } S} \}$

Indeed, $S \subseteq K^*$ is trivial

& $K^* \subseteq S$ follows from linearity of $\langle \ \rangle$

$\therefore K^*$ has finite dual description.

Now let K have finite dual description

$\iff K = \{ w / \langle w, \gamma_i \rangle \geq 0 \}$ for some $\gamma_1 \cdots \gamma_m$.

But $K^* = \{ u / \langle w, u \rangle \geq 0 \ \forall \ w \in K \}$

It is trivial that $\gamma_i \in K^* \ \forall i$

$\iff \text{CONIC}(\{\gamma_1 \cdots \gamma_m\}) \subseteq K^*$

Claim: $K^* \subseteq \text{CONIC}(\{\gamma_1 \cdots \gamma_m\}) \iff K^*$ is polyhedral

$* \ \ \ \ \ \not\in \text{CONIC}(\{\gamma_1 \cdots \gamma_m\})$

Proof: Let $x \in K^*$ but $x \notin \text{CONIC}(\{v_1, \dots v_m\})$

$\Downarrow$ Sep. theorem

$\exists\, w \neq 0 \ni$

$\langle w, x \rangle < 0$

& $\langle w, x \rangle \geq 0 \ \forall x \in$

$w \in K \Longleftrightarrow$

$x \notin K^* \Longleftarrow$

Contradiction