CS5580 End Semester Exam
4-May-2019 (2:15pm-5:15pm)

Important Instructions:

- Please write your **ROLL No.** on your answer sheets, clearly.
- Please provide concise and relevant answers, while giving **rigorous** justifications.
- Always **simplify** your expressions as far as possible.
- In case you use a theorem/lemma from the course, then it is **required** to clearly write the theorem's statement. Please do not repeat the proof of such theorems/lemmas.

1. Consider the set $C \equiv \left\{x \in \mathbb{R}^n \mid \frac{1}{2}x^\top Px + q^\top x + r \leq 0\right\}$, where P is a square matrix of size n, $q \in \mathbb{R}^n$, and $r \in \mathbb{R}$.
   (a) (4 points) Show that C is convex whenever $P \succeq 0$ and C is non-empty.
   (b) (4 points) Is the converse true? i.e, C is convex implies $P \succeq 0$? If yes, provide a proof. If no, provide a counter example.
   (c) (2 points) Visualize (rough plot) this set in special case $n = 2$.

2. Consider the function $f \equiv \max(g, h)$, where, $g(x) \equiv \|x\|_p \ \forall\ x \in \mathbb{R}^n$, $h(x) \equiv \frac{1}{2}\|x - a\|_2^2 \ \forall\ x \in \mathbb{R}^n$. Assume $p \geq 1, a \in \mathbb{R}^n$.
   (a) (2 points) From first principles, derive a simplified expression for the sub-differential set of g at 0.
   (b) (1 point) Write a simplified expression for the sub-differential set of h at 0.
   (c) (4 points) Write a simplified expression for the sub-differential set of f at 0.
   (d) (3 points) Visualize (rough plot) this sub-differential set in the special case $n = 2$.

3. A mobile service provider company wishes to extend its coverage to a few scattered residential settlements. The technical team came up with a description of a relevant optimization problem in English: "Given 2-d coordinates of each settlement[1], find location(s) such that the worst-case (Euclidean) distance between that (to be found) location and any of the settlement is minimized".
   (a) Assume that the team approached you for faithfully[2] converting their informal English description into a valid formal Mathematical Program (MP). Now, faithfully write this problem as:
      i. (2 points) an unconstrained Convex Program. Let's call this `CP1`.
      ii. (2 points) a Convex Program where each constraint set[3] is an ice-cream cone. Let's call this `CP2`.
      iii. (4 points) a Semi-Definite Program (SDP) in standard primal form. Let's call this `CP3`. *Hint*: Use Schur complement lemma i.e., try to write a matrix that is a linear function of the primal variables such that its Schur complement[4] condition turns out to be equivalent to the ice-cream cone constraint set.

[1] Because the settlements are tiny (compared to the distances between them), the team approximated settlements by points. Since the settlements are close-by (compared to the size of the earth), the team represented these points using 2-d co-ordinates.
[2] Informally, faithfully means that the solution set of the MP is the solution set of the intended problem.
[3] If $g_i(x) \leq 0$ is the constraint, then $\{x \in \mathcal{X} \mid g_i(x) \leq 0\}$ is the constraint set. Here, $\mathcal{X}$ is the domain of the MP.
[4] Let $M \equiv \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$ be a block matrix where $A \succ 0$. Then, $M \succeq 0 \iff A - BC^{-1}B^\top \succeq 0$.

iv. (3 points) a Convex Program with differentiable objective and differentiable constraint functions. Let's call this `CP4`. Note that we only insist that the (optimal) solution sets of `CP1,CP2,CP3,CP4` are the same (as intended). We don't insist that their (optimal) values are the same[5].

(b) To show-off your expertise, let's say you began exploring dual problems:

i. (4 points) Write down a simplified expression for a Langrange dual problem of `CP4`. Simplify until you obtain a form for the Lagrange dual problem, where it is easy to inspect that the dual is indeed a convex program. Let's call this `CP5`.

ii. (4 points) Using the expression derived in lecture for a conic dual problem of a (generic) primal conic problem, write down a simplified expression for conic dual of a (generic) primal SDP. Hint: What is the adjoint for the linear transformation that exists in SDP primal?

iii. (4 points) Using the above, and by representing dual variables using block matrices, write a simplified expression for the conic dual of `CP3`. At this stage, your dual may still have conic-constraints on block matrices; but ensure that the size of each block matrix variable is not more than $n \times n$.

iv. (5 points) Further simplify your expression for the conic dual until you are left with variables that are only (Euclidean) vectors or scalars (i.e., all matrix variables must be eliminated. Hint: Use Schur complement lemma for elimination). Needless to say, at this stage your expression must not involve any conic-constraints (in space of matrices). Let's call this `CP6`.

v. (0 points) You tried to give an intuitive intrepretation for `CP5,CP6` and coded `CP1,CP5,CP6` in cvx to verify that your derivations are all correct.

(c) Encouraged by your progress, you embarked on writing down optimality conditions:

i. (2 points) Let x^* be a feasible point of `CP4` and λ^* be a feasible point of `CP5`. Using KKT theorems, write down the (additional) necessary and sufficient conditions for (x^*, λ^*) being an optimal primal-dual pair i.e., for being optimal for `CP4,CP5` respectively.

ii. (3 points) Let x^* be a feasible point of a (generic) primal conic problem and λ^* be a feasible point of it's conic dual. Write down simple necessary and sufficient conditions (additionally required) for (x^*, λ^*) being an optimal primal-dual pair i.e., for being optimal for the primal conic problem and it's conic dual respectively. What is the name of these conditions in the special case of Linear Program.

iii. (2 points) Using the above, provide simplified expression for necessary and sufficient conditions for (x^*, λ^*) being optimal for `CP3,CP6` respectively.

(d) (0 points) Impressed with your analysis, the company made you an attractive offer :)

[5] Neither do we insist that the optimal values are different!