CS5580: Practice Problems

Note: Work out all problems on your own. If you think your answer is NOT satisfactory/correct, then ask for hints from friend/instructor. You MUST use ONLY the definitions, results/theorems used during the lectures. Do not google for proofs/solutions as they might work with different definitions/axioms. Provide as rigorous proofs as you can.

Quiz 1

Topics: Definition of MP, Comparing MPs etc.

1. Show that

$$\begin{array}{cc} \max_{x\in\mathcal{X}} & f(x) \\ \text{s.t.} & g_i(x) \leq 0\ \forall\ i \end{array} = \begin{array}{cc} -\min_{x\in\mathcal{X}} & -f(x) \\ \text{s.t.} & g_i(x) \leq 0\ \forall\ i \end{array}$$

2. Show that

$$\begin{array}{cc} \min_{x\in\mathcal{X}} & f(x)+h(x) \\ \text{s.t.} & g_i(x) \leq 0\ \forall\ i \end{array} \geq \begin{array}{cc} \min_{x\in\mathcal{X}} & f(x) \\ \text{s.t.} & g_i(x) \leq 0\ \forall\ i \end{array} + \begin{array}{cc} \min_{x\in\mathcal{X}} & h(x) \\ \text{s.t.} & g_i(x) \leq 0\ \forall\ i \end{array}$$

3. Prove that

$$h\left(\begin{array}{cc} \min_{x\in\mathcal{X}} & f(x) \\ \text{s.t.} & g_i(x) \leq 0\ \forall\ i \end{array}\right) = \begin{array}{cc} \min_{x\in\mathcal{X}} & h(f(x)) \\ \text{s.t.} & g_i(x) \leq 0\ \forall\ i, \end{array}$$

whenever h is a monotonically non-decreasing continuous function.

4. Show that:

$$\begin{array}{ccccc} \min_{y\in\mathcal{Y}}\min_{x\in\mathcal{X}} & f(x,y) & = & \min_{x\in\mathcal{X}}\min_{y\in\mathcal{Y}} & f(x,y) \\ \text{s.t.} & g_i(x,y) \leq 0\ \forall\ i & & \text{s.t.} & g_i(x,y) \leq 0\ \forall\ i. \\ & & = & \min_{z\in\mathcal{Z}} & f(z) \\ & & & \text{s.t.} & g_i(z) \leq 0\ \forall\ i, \end{array}$$

where $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}, z = (x, y)$.

Quiz 2

Topics: Review of Euclidean Spaces

1. Let K, Q be two given pd matrices of sizes $m \times m, n \times n$ respectively. Consider the function $g : \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ defined by $g(X, Y) \equiv trace(X^\top KYQ)$. Prove or disprove that g is a valid inner-product over the vector space of $m \times n$ matrices.

2. What is the length of an orthogonal matrix of size n?

3. Consider the set of all permutation matrices. What is the diameter of this set ? Diameter of S is defined as $\max_{x\in S, y\in S} \|x - y\|$.

4. Consider the inner-product defined by the kernel $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Are $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$ orthogonal to each other according to this inner-product?

5. What is the angle between any two permutation matrices?

6. What is the projection of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ onto the set of all symmetric matrices?

Quiz 3

Topics: Linear and Affine Sets

Consider the standard space of Matrices of size n. Identify which of the following subsets are linear or affine sets. If linear/affine, then write down their dimension, and most efficient primal as well as most efficient dual representations too.

1. Set of all upper triangular matrices of size n.

2. Set of all orthogonal matrices of size n.

3. Set of all Hankel matrices of size n.

4. Set of all psd matrices of size n.

5. Set of all unit-trace matrices of size n.
6. Set of all unit-determinant matrices of size n.
7. Set of all null-symmetric matrices of size n.
8. Set of all matrices of size n with all row sums as unity.

Quiz 4

Topics: Basics of Cones

1. Consider the set $K_p \equiv \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \|x\|_p \leq y, x \in \mathbb{R}^n, y \in \mathbb{R} \right\}$, where $\|x\|_p$ denotes the p-norm of x ($\|x\|_p \equiv \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$ if $p \geq 1$ and $\|x\|_\infty \equiv \max_{i=1,\dots,n} |x_i|$). Show that K_p is a cone for all $p \geq 1, p = \infty$. What is the dual cone of K_p (simplify your expression as much as possible)? **Hint:** Holder's inequality. Are K_1, K_∞ polyhedral cones? Interested students may attempt below too:
   (a) Show that K_p is never polyhedral except at $p = 1, p = \infty$.
   (b) K_p is always closed.
   (c) K_p is always pointed. A cone K is pointed iff $a \in K, -a \in K \Rightarrow a = 0$.
2. Consider the set $K_M \equiv \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \|x\|_M \leq y, x \in \mathbb{R}^n, y \in \mathbb{R} \right\}$, where $\|x\|_M$ denotes the M-norm of x: $\|x\|_M \equiv \sqrt{x^\top M x}$ if $M \succ 0$. Show that K_M is a cone for all $M \succ 0$. What is the dual cone of K_M (simplify your expression as much as possible)? **Hint:** Use EVD of M in your simplifications. Interested students may attempt below too:
   (a) Show that $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \|x\|_M^2 \leq y, x \in \mathbb{R}^n, y \in \mathbb{R} \right\}$ is a cone.
3. Consider the gram-matrix-cone $\mathbb{G}_m$, which is defined as follows: A matrix $G \in \mathbb{G}_m$ iff $\exists$ vectors $v_1, \dots, v_m$ such that $G_{ij} = \langle v_i, v_j \rangle$. Is $\mathbb{G}_m$ a cone? **Hint**: Please refer definition of direct sum of spaces. Interested students may attempt the following too:

(a) Is $\mathbb{G}_m$ the same as the psd cone?

(b) Show that $\mathbb{G}_m$ is closed and pointed.

4. Consider the Euclidean-Distance-Matrix (EDM) cone $\mathcal{D}_m$, which is defined as follows: A matrix $D \in \mathcal{D}_m$ iff $\exists$ vectors $v_1, \ldots, v_m$ such that $D_{ij} = ||v_i - v_j||^2$. Is $\mathcal{D}_m$ a cone?

   (a) Interested students may ask yourself: Can you express $\mathcal{D}_m$ in terms of $\mathbb{G}_m$? In general, are matrices in EDM cone psd? If not, what are all the psd matrices in the EDM cone?

Quiz 5

Topics: Proofs in Cones, Basics of Convex sets

1. Prove that intersection of two polyhedral cones is a polyhedral cone.

2. Prove that once a cone has a line, then it must have all parallel lines to it through every point in the cone.

3. Prove that (Arbitrary) intersection of cones is a cone.

4. Union of cones need not be a cone. However, show that $CONIC(K_1 \cup K_2) = K_1 + K_2$.

5. Cartesian product of cones is a cone, and $(K_1 \times K_2)^* = K_1^* \times K_2^*$.

6. $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$.

7. Milutin-Dubovitski lemma: $(K_1 \cap K_2)^* = K_1^* + K_2^*$, for closed cones K_1, K_2 whose sum is also closed[1].

8. Let $C \equiv \{x \in \mathbb{R}^n \mid A_0 - (\sum_{i=1}^n x_i A_i) \succeq 0\}$, where A_i are all symmetric matrices of size m. Show that C is a convex set.

9. Let P be the set of probability density functions wrt. a fixed probability space. Show that P is a convex set.

[1] Proposition B.2.7 in [1].

10. Show that the hyperbolic cone, $\left\{x \mid x^\top M x \leq \left(c^\top x\right)^2, x \in \mathbb{R}^n, c^\top x \geq 0\right\}$, is a convex set.

11. Exercise 2.12 in Boyd's book.

Quiz 6a

Topics: Examples and proofs with Convex sets, 1-d characterization, Polar, Seperation theorem, Supporting Hyperplane, Tangent and Normal Cones

1. Exercises B.3, B.7 in [1].

2. Starting from the definition of convex sets as: C is convex iff $x \in C, y \in C \Rightarrow \lambda x + (1-\lambda)y \in C \ \forall \ \lambda \in [0,1]$; show by mathematical induction that $C = CONV(C)$.

3. For any set S show that $CONV(S)$ is the intersection of all convex sets that contain S.

4. Read and understand proof and statement of theorem B.2.5 in [1].

5. Using separation theorem prove the homogeneous Farkas Lemma B.2.1 in [1].

6. Derive simplified expressions for polars of Simplex, Spectrahedron, and the Birkhoff Polytope.

7. Formally derive simplified expressions for the tangent and normal cones of:

    (a) the psd cone at the origin? What are they at a given diagonal matrix with strictly positive entries?

    (b) Norm ball at some point on the boundary.

    (c) Simplex at some "vertex".

    (d) Spectrahedron at $\frac{1}{n}I_n$, where I_n is identity matrix of size n.

8. Formally derive simplified expression for (all) the supporting hyperplane(s) of:

(a) norm ball at some boundary point.

(b) Simplex at some vertex.

(c) Spectrahedron at $\frac{1}{n}I_n$, where I_n is identity matrix of size n.

9. Let f be a linear fractional function defined in equation (2.13) in [2]. Show the following:

   (a) If C is a convex set, then so is $f(C)$.

   (b) if C is a convex set, then its pre-image under f is also a convex set.

Quiz 6b

Topics: Multivariate Calculus, Linear and Affine functions

1. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be L_1-Lipschitz and $g : \mathbb{R}^n \mapsto \mathbb{R}$ be L_2-Lipschitz. Is $f \circ g$ Lipschitz continuous[2]? If so can you express the Lipschitz constant of the composition in terms of L_1, L_2?

2. Using expression for truncated Taylor series (first-order multivariate), prove that an everywhere differentiable f is L-Lipschitz if norm of gradient is upper-bounded by L.

   (a) Compute Lipschitz constant of $f(x) = \frac{1}{2}x^\top Px$ over the domain $||x|| \leq 1$. Express it in terms of maximum eigen-value, vector of P.

   (b) Compute Lipschitz constant of $g(x) = \sqrt{x^2 + 3}$.

3. Compute gradient of $f(X) = tr(X^\top BX)$. Is this a linear function?

4. Prove that functions with constant gradient are affine.

Quiz 7

Topics: Conic Functions: Norms, Semi-norms, Support Functions, Dual Norms, Dual functions.

[2]Important question for deep learning enthusiasts :)

1. Show that the function $f(X) \equiv \max_{y \in \mathbb{R}^n \setminus \{0\}} \frac{||Xy||_a}{||y||_b}$, where $a, b \geq 1$ and $||y||_a$ is the (entry-wise) l_a-norm, is a support function. Show that f defines a norm over matrices.

2. Show that a conic function, f, is a semi-norm if and only if f is such that a) it's domain is entire vector space b) it's closed b) it's non-negative c) it's even i.e., $f(-x) = f(x) \; \forall \; x \in dom(f)$.

3. Show that f defined by $f(M)$ is the sum of singular values of M is a conic function. Is f a norm? If not, find the dual of f, else find it's dual norm.

4. Show that support functions characterize closed convex sets i.e., $P = Q \Leftrightarrow \mathcal{S}_P = \mathcal{S}_Q$ whenever P,Q are closed convex sets.

Quiz 8

Topics: Primal and epigraph characterization of convex functions, Jensen's inequality, Sub-differentiability

1. Show that f is affine if and only if both $f, -f$ are convex.

2. Let $S \subseteq \mathbb{R}^n$ be a non-empty set. Show that the function f given by: $f(x) =$ the distance between x and the farthest point in C from x, is closed convex.

3. Let g, h be convex functions defined on R^n. Suppose h is bounded below and all its non-empty level-sets are bounded. Consider f defined by:
$$f(x) \equiv \min_{y \in \mathbb{R}^n} g(y) \;\; \text{s.t.} \;\; h(y) \leq x.$$
Express the domain of f in terms of things related to g and/or h. Is the domain of f (i) non-empty? (ii) bounded? (iii) convex? Is f a convex function?

4. Show that the function f given by $f(M) = trace(M^{-1})$, $M \succ 0$ is sub-differentiable everywhere. Hint: Guess a subgradient that satisfies the sub-gradient inequality and then verify. To help your guess, think what if the matrix is a positive number?

5. Express the sub-differential set of $h \equiv \max(f, g)$ in terms of those of f, g.

6. Write down a simplified expression for the sub-differential set of the dual function (assuming it exists). Hence derive the same for dual norm functions.

7. Let $f(x) \equiv ||x||$, where the norm is arbitrary. Find a simplified expression for $\partial f(0)$.

Quiz 9

Topics: First, Second order, dual characterization of convex functions, Conjugate

1. Exercises 3.1, 3.5,3.7,3.13,3.16,3.18,3.25,3.26,3.30,3.31,3.36a,3.36e,3.37 from [2].

Quiz 10

Topics: Sufficient conditions for boundedness, solvability, unique solvability of convex programs, Strict convexity, convex program solution characterization

References

[1] A. Ben-Tal and A. Nemirovski. Lectures On Modern Convex Optimization. Available freely at `https://www2.isye.gatech.edu/~nemirovs/LMCOLN2021WithSol.pdf`, 2021.

[2] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.