CS5560 : EXAM2

18/11/18

1) Suppose $p_\theta(x)$ can be written in canonical form, ~~then~~ Firstly since there is only a single parameter, a single canonical parameter should suffice.

So, let us assume,

$$p_\theta(x) = \frac{\theta}{\pi(x^2+\theta^2)} = h(x)e^{\omega\varphi(x) - A(\omega)}$$

for some $\omega = g(\theta) \in \mathbb{R}$
& some $\varphi(x) : \mathcal{X} \to \mathbb{R}$.

Taking log on both sides, we obtain:

$$\log\theta - \log(\pi(x^2+\theta^2)) = \log h(x) + \omega\varphi(x) - A(\omega). \quad (\forall x>0,\ \theta>0)$$

From this, it is clear that $h(x) = 1$ and $\omega\varphi(x) = -\log(\pi(x^2+\theta^2)) \quad (\forall x>0, \theta>0)$

$$\Rightarrow \omega\varphi(0) = -\log\pi\theta^2$$

So we have, $\varphi(x) = \frac{\varphi(0)\log(\pi(x^2+\theta^2))}{\log\pi\theta^2} \quad (\forall x>0,\ \theta>0)$

which is an impossibility (contradiction) as the LHS is not a function of θ & the RHS is a (non-linear) function of θ.

Hence Cauchy does not belong to exponential family!

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(2) The likelihood for IG is: $p_{a,b}(x) = \frac{b^a}{\Gamma(a)} x^{-(a+1)} e^{-b/x}, \quad x>0$
$a, b > 0.$

Let us define: $\varphi(x) = \begin{bmatrix} -\log(x) \\ -1/x \end{bmatrix}$, ~~and~~ $\omega = \begin{bmatrix} a \\ b \end{bmatrix}$, and $h(x) = \frac{1}{x}$.

It is easy to see that: $p_{a,b}(x) = h(x)e^{\omega^T\varphi(x)}/Z(\omega), \quad x>0$

& $Z(\omega)$ ~~must~~ be $1/\frac{b^a}{\Gamma(a)} = \cancel{\frac{\omega_2^{\omega_1}}{\Gamma(\omega_1)}} \frac{\Gamma(\omega_1)}{\omega_2^{\omega_1}}, \quad \omega_1, \omega_2 > 0.$

③ ⓐ Canonical likelihood function is $e^{\omega^T \varphi(x,y) - A(\omega)}$

where, $A(\omega) \equiv \log\left(\iint e^{\omega^T\varphi(x,y)}\, dx\,dy\right)$, for

is the cumulant function.

$$\log(p_\omega(D)) = \log\left(\prod_{i=1}^{m} p_\omega(x_i, y_i)\right) \quad (\because \text{iid samples})$$

$$= \log\left(e^{\omega^T \sum_{i=1}^{m} \varphi(x_i,y_i) - mA(\omega)}\right)$$

$$\Rightarrow \hat{\omega}_{MLE} \equiv \underset{\omega \in W}{\operatorname{argmax}} \log p_\omega(D)$$

$$= \underset{\omega \in W}{\operatorname{argmax}}\ \omega^T\left(\sum_{i=1}^{m} \varphi(x_i, y_i)\right) - mA(\omega). \quad \text{I}$$

Let us assume W (parameter space) is an open set, then, gradient is zero gives optimality condition if we additionally show that $A(\omega)$ is a convex function.

Now
$$\frac{\partial A(\omega)}{\partial \omega_j} = \frac{\partial}{\partial \omega_j} \log\left(\iint e^{\omega^T\varphi(x,y)}\, dx\,dy\right)$$

$$= \frac{\frac{\partial}{\partial \omega_j}\iint e^{\omega^T\varphi(x,y)}\,dx\,dy}{\iint e^{\omega^T\varphi(x,y)}\,dx\,dy} \quad (\because \text{chain rule})$$

$$= \frac{\iint \frac{\partial\left(e^{\omega^T\varphi(x,y)}\right)}{\partial \omega_j}\,dx\,dy}{\iint e^{\omega^T\varphi(x,y)}\,dx\,dy} \quad (\because \text{integral is absolutely convergent}) \text{ we can exchange.}$$

$$= \iint \frac{\varphi_j(x,y)\, e^{\omega^T\varphi(x,y)}}{\iint e^{\omega^T\varphi(x',y')}\,dx'\,dy'}\,dx\,dy$$

$$= E_\omega\left[\varphi_j(x,y)\right].$$

Also, $\dfrac{\partial^2 A(\omega)}{\partial \omega_k \partial \omega_j} = \dfrac{\partial}{\partial \omega_k} \displaystyle\iint \frac{\varphi_j(x,y)\, e^{\omega^T \varphi(x,y)}}{\iint e^{\omega^T \varphi(x',y')} dx' dy'} dx dy$

$$= \iint \frac{\varphi_j(x,y)\varphi_k(x,y)\, e^{\omega^T \varphi(x,y)}}{\iint e^{\omega^T \varphi(x',y')} dx' dy'} dx dy \quad (\because \text{ same arguments as above})$$

$$= \mathbb{E}_\omega\left[\varphi_j(X,Y)\, \varphi_k(X,Y)\right].$$

$\Rightarrow$ the Hessian of A is covariance matrix of $\varphi(X,Y)$. & hence is p.d.
$\Rightarrow$ A is convex.

Now gradient zero (I) gives:

$$\boxed{\mathbb{E}_{\omega_{MLE}}\left[\varphi(X,Y)\right] = \frac{\sum_{i=1}^{m} \varphi(x_i, y_i)}{m}}$$

(b) $p_\omega(y/x) \propto p_\omega(x,y)$ (i.e. only look at terms involving 'y') & rest are normalization const!

$$\propto e^{\omega^T(\psi(x) \otimes T(y))} \quad (\because \text{ ignoring normalization})$$

$$= e^{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \omega_{ij} \psi_i(x) T_j(y)} \quad (\because \text{ defn. of } \otimes \text{ \& } \omega_{ij} \text{ are entries of } \omega)$$

$$= e^{\sum_{j=1}^{n_2} \omega_j(x) T_j(y)}, \text{ where } \omega_j(x) = \sum_{i=1}^{n_1} \omega_{ij} \psi_i(x)$$

$\therefore$ Sufficient statistics of $p(y/x)$ are $T(y)$
& canonical parameters are $\begin{bmatrix} \omega_1(x) \\ \vdots \\ \omega_{n_2}(x) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n_1} \omega_{i1} \psi_i(x) \\ \vdots \\ \sum_{i=1}^{n_1} \omega_{in_2} \psi_i(x) \end{bmatrix}$

(c) Discriminative model:

$$p_w(y/x) \equiv e^{\sum_{j=1}^{n_2} (w_j^T \varphi(x)) T_j(y) - \bar{A}(w_1^T\varphi(x) \ldots w_{n_2}^T\varphi(x))}$$

$w_1, \ldots w_{n_2} \in \mathbb{R}^{n_1}$ are canonical parameters, $\bar{A}$ is cumulant fction with suff. statistics as $T(y)$.

(d)

$$\frac{\sum_{i=1}^{m} \varphi(x_i) \otimes E_{\hat{w}_{MLE}}[T(Y)/x_i]}{m} = \frac{\sum_{i=1}^{m} \varphi(x_i) \otimes T(y_i)}{m}$$

(partial moment match) (derived in lecture)

(e) In (a) we derived: $p_w(D) = e^{w^T(m\bar{\varphi}(D)) - mA(w)}$, where $\bar{\varphi}(D) \equiv \frac{\sum_{i=1}^{m}\varphi(x_i, y_i)}{m}$.

Looking at this, we guess the conjugate prior as:

$$p(w) = \frac{e^{w^T(\nu_0 u_0) - \nu_0 A(w)}}{Z(\nu_0, u_0)},$$

where sufficient statistics are $\begin{bmatrix} w \\ -A(w) \end{bmatrix}$

Canonical parameters are $\begin{bmatrix} \nu_0 u_0 \\ \nu_0 \end{bmatrix}$

(f) $p(w/D) \propto p(D/w)\, p(w)$

$$\propto e^{w^T(m\bar{\varphi}(D) + \nu_0 u_0) - A(w)(m+\nu_0)}$$

$$\Rightarrow p(w/D) = e^{w^T(m\bar{\varphi}(D) + \nu_0 u_0) - A(w)(m+\nu_0)} \Big/ Z\left(\nu_0 + m, \frac{m\bar{\varphi}(D) + \nu_0 u_0}{m+\nu_0}\right).$$

$$\Rightarrow \hat{\omega}_{MAP} = \underset{\omega \in W}{\operatorname{argmax}} \; \omega^T\left(m\bar{\varphi}(\mathcal{D}) + \nu_0 u_0\right) - A(\omega)\left(m+\nu_0\right)$$

$$\Rightarrow \boxed{E_{\hat{\omega}_{MAP}}\left[\varphi(x,y)\right] = \frac{\sum_{i=1}^{m} \varphi(x_i,y_i) + \nu_0 u_0}{m+\nu_0}}$$

(g) In lecture, this was derived:

$$\hat{p}_{BAM}(x,y) = \left(\int p(x,y/\omega)\, p(\omega/\mathcal{D})\, d\omega\right.$$

$$= \int \frac{e^{\omega^T\varphi(x,y) - A(\omega)}\; e^{\omega^T(m\bar{\varphi}(\mathcal{D}) + \nu_0 u_0) - A(\omega)(m+\nu_0)}}{Z\left(\nu_0+m, \frac{m\bar{\varphi}(\mathcal{D}) + \nu_0 u_0}{m+\nu_0}\right)}\, d\omega$$

$$= \frac{Z\left(\nu_0+m+1, \frac{m\bar{\varphi}(\mathcal{D}) + \nu_0 u_0 + \varphi(x,y)}{\nu_0+m+1}\right)}{Z\left(\nu_0+m, \frac{m\bar{\varphi}(\mathcal{D}) + \nu_0 u_0}{m+\nu_0}\right)}.$$

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(4) $$p(y/x) = \int p(y,z/x)\, dz$$

$$= \int p(y/z,x)\, p(z/x)\, dz$$

(∵ modeling assumption)

$$\propto \int p(y/z)\, p(x/z)\, p(z)\, dz$$

$$\propto \int e^{-\frac{1}{2}(y - \omega^T z - \omega_0)^2}\; e^{-\frac{1}{2}\|x - Vz - v_0\|^2}\; e^{-\frac{1}{2}\|z-\mu\|^2}\, dz$$

Note that we must not ignore terms involving y, z, rest of the terms can be safely ignored (for example terms that involve x & constant). Also it is clear that $y|x$ is gaussian distributed.

$$p(y|x) \propto e^{-\frac{(y-w_0)^2}{2}} \int e^{-\frac{1}{2} z^T (ww^T + V^TV + I) z + z^T \left(w(y-w_0) + V^T(x-v_0) + \mu \right)} dz$$

Let $\Sigma^{-1} \equiv (ww^T + V^TV + I)$ and let $u \equiv \Sigma\left((y-w_0)w + V^T(x-v_0) + \mu\right)$

then, $$p(y|x) \propto \left(e^{-\frac{(y-w_0)^2}{2}}\right)\left(\int e^{-\frac{1}{2}(z-\mu)^T \Sigma^{-1}(z-\mu)} dz\right)\left(e^{\frac{1}{2}\mu^T \Sigma^{-1}\mu}\right)$$

$$\propto e^{-\frac{(y-w_0)^2}{2} + \frac{1}{2}\left((y-w_0)w + V^T(x-v_0)+\mu\right)^T (ww^T+V^TV+I)^{-1}\left((y-w_0)w + V^T(x-v_0)+\mu\right)}$$

$$\propto e^{-\frac{(y-w_0)^2}{2}\left(1 - w^T(ww^T+V^TV+I)^{-1}w\right) + (y-w_0)w^T(ww^T+V^TV+I)^{-1}(V^T(x-v_0)+\mu)}$$

$$\propto e$$

$$\cancel{\Rightarrow p(y|x) \propto N\left(w_0, \frac{1}{1-w^T(ww^T+V^TV+I)^{-1}w}\right)}$$

$$\propto e^{-\frac{\left(y - w_0 - \frac{w^T(ww^T+V^TV+I)^{-1}(V^T(x-v_0)+\mu)}{(1-w^T(ww^T+V^TV+I)^{-1}w)}\right)^2}{2\left(1/(1-w^T(ww^T+V^TV+I)^{-1}w)\right)}}$$

$$\Rightarrow p(y|x) \sim N\left(w_0 + \frac{w^T(ww^T+V^TV+I)^{-1}(V^T(x-v_0)+\mu)}{1-w^T(ww^T+V^TV+I)^{-1}w}, \frac{1}{1-w^T(ww^T+V^TV+I)^{-1}w}\right)$$

(b) Conditional log-likelihood for D is:

$$= \sum_{i=1}^{m} \log\left(\int p(y_i|z)\, p(z|x_i)\, dz\right) \quad (\because \text{ modeling assumptions})$$

$$= \sum_{i=1}^{m} \log\left(\int \frac{p(y_i|z)\, p(z|x_i)}{q_i(z)}\, q_i(z)\, dz\right) \quad (\because \text{ auxiliary distributions})$$

$$\geq \sum_{i=1}^{m} \int \log\left(\frac{p(y_i/z)\,p(z/x_i)}{q_i(z)}\right) q_i(z)\, dz \quad \text{— (I)} \quad (\because \text{Jensen's inequality})$$

$$= \sum_{i=1}^{m} \int \log\left(\frac{p(z/x_i,y_i)\,p(y_i/x_i)}{q_i(z)}\right) q_i(z)\, dz .$$

$$= \sum_{i=1}^{m} \int q_i(z) \log\left(\frac{p(z/x_i,y_i)}{q_i(z)}\right) dz + \sum_{i=1}^{m} \log p(y_i/x_i)$$

∴ If equality is needed, then choose $\boxed{q_i^{(k)}(z) \equiv p^{(k-1)}(z/x_i,y_i)}$.

From (I),

$$\text{LHS} \geq \underbrace{\sum_{i=1}^{m} \int \left(\log p(y_i/z)\right) q_i(z)\, dz}_{(A)\ \text{function of } w,\, w_0} + \underbrace{\sum_{i=1}^{m} \int \left(\log (p(z/x_i))\right) q_i(z)\, dz}_{(B)\ \text{function of } V,\, V_0,\, \mu} \underbrace{- \sum_{i} \text{Entropy}(q_i)}_{\text{Constant.}}$$

$$(A) \longrightarrow \underset{w,\,w_0}{\text{argmax}} \sum_{i=1}^{m} \int -\frac{1}{2}\left(y_i - w^T z - w_0\right)^2 q_i(z)\, dz$$

$$= \underset{w,\,w_0}{\text{argmax}} \; -\frac{1}{2}\sum_{i=1}^{m}(y_i - w_0)^2 + \sum_{i=1}^{m}(y_i - w_0)\sum_{j=1}^{n} w_j \int z_j\, q_i(z)\, dz - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k \int z_j z_k\, q_i(z)\, dz$$

$$= \underset{w,\,w_0}{\text{argmax}} \; -\frac{1}{2}\sum_{i=1}^{m}(y_i - w_0)^2 + \sum_{i=1}^{m}(y_i - w_0)\, w^T u_i - \frac{1}{2}\sum_{i=1}^{m} w^T C_i w ,$$

where $u_i \equiv E_{q_i}[Z]$; $C_i \equiv E_{q_i}[ZZ^T]$; Solving grad = 0.

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ADDITIONAL SHEET

ROLL No.

Now, $p(z/x_i) \propto p(x_i/z)\, p(z)$

$$\propto e^{-\frac{1}{2}\left(z^T(V^TV+I)z - 2z^T\left(V^T(x_i-v_0)+\mu\right)\right)}$$

$\cancel{\Rightarrow p(z/x_i)}$

$\Rightarrow z/x_i \sim N(u_i, \Sigma_1)$, where $\Sigma_1 \equiv (V^TV+I)^{-1} \equiv S^{-1}$

$u_i \equiv \Sigma_1\left(V^T(x_i-v_0)+\mu\right)$

(B) $\longrightarrow \underset{V,v_0,\mu}{\arg\max} \sum_{i=1}^{m} \int -\frac{1}{2}(z-u_i)^T\Sigma_1^{-1}(z-u_i)\, q_i(z)\, dz - m\log|\Sigma_1|^{1/2}$

$$= \underset{V,v_0,\mu}{\arg\max} \sum_{i=1}^{m} \int -\frac{1}{2}\left(z^T S z\right) q_i(z)\, dz + \sum_{i=1}^{m} \int z^T\left(V^T(x_i-v_0)+\mu\right) q_i(z)\, dz$$

$$-\frac{1}{2}\sum_{i=1}^{m} u_i^T\Sigma_1^{-1}u_i - m\log|\Sigma_1|^{1/2}$$

constant!

$$= \underset{V,v_0,\mu}{\arg\max} -\frac{1}{2}\sum_{i=1}^{m}\left[\int z^T V^T V z\, q_i(z)\, dz - \cancel{\int z^T z\, q_i(z)\, dz}\right]$$

$$+ \sum_{i=1}^{m} a_i^T\left(V^T(x_i-v_0)+\mu\right) - \frac{1}{2}\sum_{i=1}^{m} u_i^T\Sigma_1^{-1}u_i - m\log|\Sigma_1|^{1/2}$$

" " "

$$= \underset{V,v_0,\mu}{\arg\max} -\frac{1}{2}\sum_{i=1}^{m}\sum_{j,k} v_j^T E_{q_i}[z_j z_k]\, v_k -$$

where v_k is k^{th} column of V.

Solve using gradient $=0$ & numerical procedure.