# CS5560 : Exam 1

(1) Let $p^*(y/x)$ be the likelihood being modeled (unknown). Let $p_\theta(y/x)$ denote the distribution with model parameter $\theta$. We choose the "distance" as expected KL divergence.

i.e. Our goal is:

$$\min_{\theta\in\Theta} \underbrace{E_{X\sim p^*(x)}\left[KL\left(p^*(y/x)\,\|\,p_\theta(y/x)\right)\right]}_{\text{marginal wrt. input in } p^*(x,y).}$$

$$= \min_{\theta\in\Theta} E_{x\sim p^*(x)}\left[E_{y/x\sim p^*(y/x)}\left[\log \frac{p^*(y/x)}{p_\theta(y/x)}\right]\right] \quad (\because \text{dfn. of KL})$$

$$= \min_{\theta\in\Theta} E_{(x,y)\sim p^*(x,y)}\left[\log \frac{p^*(y/x)}{p_\theta(y/x)}\right] \quad (\because \text{Total expectation rule})$$

$$\approx \min_{\theta\in\Theta} \frac{\sum_{i=1}^{m} \log \frac{p^*(y_i/x_i)}{p_\theta(y_i/x_i)}}{m} \quad (\because \text{finite sample approximation})$$

$$= -\max_{\theta\in\Theta} \sum_{i=1}^{m} \log p_\theta(y_i/x_i) \quad (\because \text{ignoring terms that are not function of } \theta \text{ \& writing as maximization})$$

$$= -\max_{\theta\in\Theta} \log\left(\underbrace{\prod_{i=1}^{m} p_\theta(y_i/x_i)}_{\text{Conditional likelihood of training data}}\right)$$

MCLE

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(2) $$p_w(y/x) \equiv \frac{e^{w_y^T\varphi(x)}}{\sum_{y\in Y} e^{w_{y'}^T\varphi(x)}}.$$ Parameters are $w_1,\ldots,w_c$ if $Y=\{1,\ldots,c\}$.

$$\text{MCLE} \longrightarrow \max_{w_1,\ldots w_c} \sum_{i=1}^{m} \log p_w(y_i/x_i) = \max_{w_1,\ldots w_c} \sum_{i=1}^{m} w_{y_i}^T\varphi(x_i) - \log\left(\sum_{y'\in Y} e^{w_{y'}^T\varphi(x)}\right) m$$

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(3) (a) Since 'y' takes discrete values, it is not a regression problem.
Since 'y' takes integer values, they satisfy a distance/total-order relation etc.
hence it is not a classification problem.

(b) Poisson distribution is best suited for counts where the expected count is constant.
(Binomial is also OK).

(c)
$$p_w(y/x) = e^{-(w^T\varphi(x))} \frac{(w^T\varphi(x))^y}{y!}, \text{ for } y = 0, 1, 2, \ldots$$
(replace mean parameter of Poisson by a function of $x$)

'$w$' is the parameter.

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(4) (a) MLE $\rightarrow$ $\max_{\theta > 0} \underbrace{\sum_{i=1}^{m} \log \frac{\theta}{\pi(x_i^2+\theta^2)}}_{g(\theta)}$, where $x_1, \ldots, x_m$ is training data.

$$g'(\theta) = 0 \iff \sum_{i=1}^{m} \frac{\theta^2}{x_i^2+\theta^2} = \frac{m}{2} \quad \text{--- (I)}$$

Note that each term in LHS (I) is continuous function of '$\theta$' taking values in $(0,1)$.
Hence LHS is a continuous function with range as $(0, m)$.
Hence it must at some value of $\theta_0$ that achieves $m/2 \in (0,m)$, by continuity of LHS.
$\therefore$ MLE problem is solvable, for any data.

Additionally, the terms in LHS are monotonic, hence the solution is unique. (say, $\theta_0$)

Only thing remaining to show is that ~~such~~ $g$ is maximized at $\theta_0$.

$$g''(\theta_0) = \frac{-m}{\theta_0^2} - 2\sum_{i=1}^{m} \frac{(x_i^2-\theta_0^2)}{(x_i^2+\theta_0^2)^2} = -2\sum_{i=1}^{m}\left(\frac{1}{x_i^2+\theta_0^2} + \frac{x_i^2-\theta_0^2}{(x_i^2+\theta_0^2)^2}\right) \quad (\because \text{by eqn. (I)})$$

$$= -2\sum_{i=1}^{m} \frac{x_i^2}{(x_i^2+\theta_0^2)^2} < 0.$$ Hence Proved.

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4 (b) Let's compute $E[x] = \int_{-\infty}^{\infty} \frac{x\theta}{\pi(x^2+\theta^2)} dx$,

Since integrand is an odd function, its value must be zero or undefined.

(in this case you can show that it is undefined)

In either case, $E[x]$ is not usable for method of moments.

Similarly, $E[X^{2k+1}]$ is also not usable for any $k \in \mathbb{N}$.

Now, $$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2\theta}{\pi(x^2+\theta^2)} dx = \int_{-\infty}^{\infty} \frac{(x^2+\theta^2-\theta^2)\theta}{\pi(x^2+\theta^2)} dx$$

$$= \int_{-\infty}^{\infty} \frac{\theta\, dx}{\pi} - \theta^2 \int_{-\infty}^{\infty} \frac{\theta}{\pi(x^2+\theta^2)} dx$$

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$$= \infty - \theta^2 \quad (\because \text{ given a valid likelihood})$$

$$= \infty$$

Since $x^{2k} > x^2$ for any $k \in \mathbb{N}$ and $|x| > 1$, we have that $E[X^{2k}] = \infty$ for all $k \in \mathbb{N}$.

Hence none of the "natural" moments can be employed. This is the difficulty.

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