CS5560: Exam 3 Solutions

23/11/18

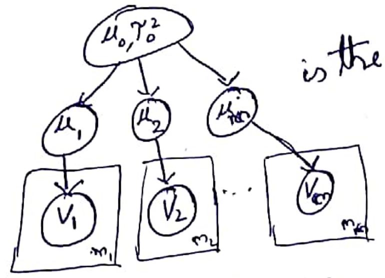
(1) ML-II hyperparameter estimation:

$$(\hat{\mu}_0, \hat{\tau}_0^2) \equiv \underset{\mu_0, \tau_0^2}{\operatorname{argmax}}\; p(\mathcal{D}/\mu_0, \tau_0^2)$$

$$= \underset{\mu_0, \tau_0^2}{\operatorname{argmax}} \int \cdots \int p(\mathcal{D}/\mu_1, \ldots \mu_m, \mu_0, \tau_0^2)\, p(\mu_1, \ldots \mu_m/\mu_0, \tau_0^2)\, d\mu_1 \ldots d\mu_m$$

($\because$ Total prob. rule, assume 'm' is no. faculty)

$$= \underset{\mu_0, \tau_0^2}{\operatorname{argmax}} \int \cdots \int p(\mathcal{D}_1/\mu_1) \ldots p(\mathcal{D}_m/\mu_m)\, p(\mu_1/\mu_0, \tau_0^2) \cdots p(\mu_m/\mu_0, \tau_0^2)\, d\mu_1 \ldots d\mu_m$$

($\because$ 

is the Bayesian model!)

$$= \underset{\mu_0, \tau_0^2}{\operatorname{argmax}} \prod_{i=1}^{m} \left[\int p(\mathcal{D}_i/\mu_i)\, p(\mu_i/\mu_0, \tau_0^2)\, d\mu_i \right]$$

($\because$)

$$= \underset{\mu_0, \tau_0^2}{\operatorname{argmax}} \prod_{i=1}^{m} \left[\int \prod_{j=1}^{m_i} p(v_{ij}/\mu_i)\, p(\mu_i/\mu_0, \tau_0^2)\, d\mu_i \right] \longrightarrow \text{5 marks}$$

$$= \underset{\mu_0, \tau_0^2}{\operatorname{argmax}} \prod_{i=1}^{m} \left[\frac{\int e^{-\frac{1}{2}\frac{\sum_{j=1}^{m_i}(v_{ij}-\mu_i)^2}{\sigma^2} - \frac{(\mu_i-\mu_0)^2}{2\tau_0^2}}\, d\mu_i}{\tau_0} \right]$$

$$= \underset{\mu_0, \tau_0^2}{\operatorname{argmax}} \prod_{i=1}^{m} \frac{e^{-\frac{\left(\frac{\sum_{j=1}^{m_i} v_{ij}}{m_i} - \mu_0\right)^2}{2(\tau_0^2+\sigma^2)}}}{\sqrt{2\pi(\tau_0^2+\sigma^2)}}$$

(by completing squares. Like exercises done before. See (5.90) in book)

$$= \underset{\mu_0, \tau_0^2}{\operatorname{argmax}} \frac{e^{-\sum_{i=1}^{m}\left(\frac{\sum_{j=1}^{m_i} v_{ij}}{m_i} - \mu_0\right)^2 / 2(\tau_0^2+\sigma^2)}}{\left(\sqrt{2\pi(\tau_0^2+\sigma^2)}\right)^m}$$

Similar to MLE with Gaussian model & data $\frac{\sum_{j=1}^{m_1} v_{1j}}{m_1}, \ldots \frac{\sum_{j=1}^{m_m} v_{mj}}{m_i}$.

$$\Rightarrow \hat{\mu}_0 = \frac{\sum_{i=1}^{m}\left(\sum_{j=1}^{m_i} v_{ij}/m_i\right)}{m}, \quad \hat{\tau}_0^2 + \sigma^2 = \frac{\sum_{i=1}^{m}\left(\left(\sum_{j=1}^{m_i} v_{ij}/m_i\right) - \hat{\mu}_0\right)^2}{m}$$

(sample mean) (sample variance)

$\rightarrow$ 2.5 marks.

~~$p(\mu_1 \ldots \mu_m / \mathcal{D}, \hat{\mu}_0, \hat{\tau}_0) \propto p(\mathcal{D}/\mu_1 \ldots \mu_m)$~~

$$p(\mu_1 \ldots \mu_m / \mathcal{D}, \hat{\mu}_0, \hat{\tau}_0^2) \propto \prod_{i=1}^{m} \prod_{j=1}^{m_i} p(v_{ij}/\mu_i)\, p(\mu_i/\hat{\mu}_0, \hat{\tau}_0^2)$$ (∵ Same Baye's net as earlier)

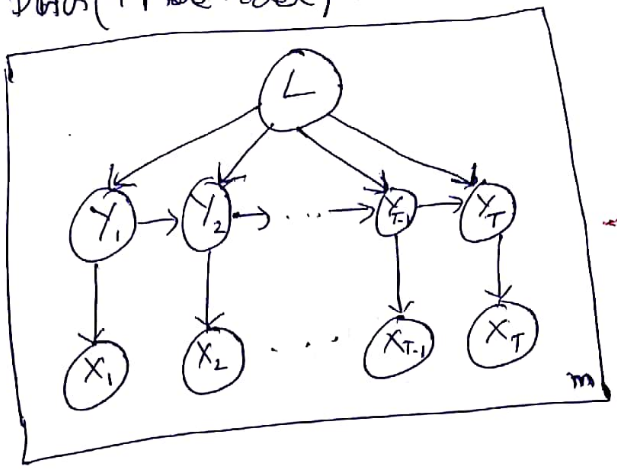
$$\propto \prod_{i=1}^{m} e^{-\frac{1}{2}\left[\frac{\sum_j (v_{ij}-\mu_i)^2}{\sigma^2} + \frac{(\mu_i - \hat{\mu}_0)^2}{\hat{\tau}_0^2}\right]}$$

$$\propto \prod_{i=1}^{m} e^{-\frac{1}{2}\frac{\left(\mu_i - \frac{\left(\frac{m\bar{v}_i}{\sigma^2} + \frac{\hat{\mu}_0}{\hat{\tau}_0^2}\right)}{\left(\frac{1}{\sigma^2} + \frac{1}{\hat{\tau}_0^2}\right)}\right)^2}{\frac{1}{1/\sigma^2 + \frac{1}{\hat{\tau}_0^2}}}}$$ (∵ Completing squares in μ_i)

$$\Rightarrow \hat{\mu}_{i\mathcal{B}}^{MAP} = \frac{\frac{m\bar{v}_i}{\sigma^2} + \frac{\hat{\mu}_0}{\hat{\tau}_0^2}}{1/\sigma^2 + \frac{1}{\hat{\tau}_0^2}}$$

$\longrightarrow$ 5 marks.

② Here is the DAG (+ plate model):



5 marks

L modeled by Multinoulli → parameters: $\theta = (\theta_1, \theta_2, \theta_3) \geq 0$ $\sum_{i=1}^{3} \theta_i = 1$.

$Y_1 / L=l$ is modeled by Multinoulli → parameters $\pi_l = (\pi_{l1}, \dots, \pi_{lc}) \geq 0$ $\sum_{i=1}^{c} \pi_{li} = 1$

$Y_t / Y_{t-1}, L=l$ is modeled by Transition prob. matrix i.e. parameters A_1, A_2, A_3. by Multinoullis

each is a stochastic matrix.

$X_t / Y_t = s, L=l$ is modeled using exponential family model with parameters

e.g. $w_{11}, \dots w_{1c}$, → language 1

$w_{21}, \dots w_{2c}$, → " 2

$w_{31}, \dots w_{3c}$. → " 3

↓ state 1 . . . ↓ state c

2 marks

$$p(x) = \sum_{\forall y \forall l} p(x, y, l)$$

$$= \sum_{\forall y \forall l} p(x_1/y_1) \cdots p(x_T/y_T)\, p(y_1/l)\, p(y_2/y_1, l) \cdots p(y_T/y_{T-1}, l)\, p(l)$$

(∵ chain rule for above BN)

$$\log(p(\mathcal{D})) = \sum_{i=1}^{m} \log p(x_i)$$

$$= \sum_{i=1}^{m} \log\left(\sum_{l=1}^{3} p(x_i/l)\, p(l)\right)$$

This is same as that for Mixture model.

$$\therefore\ q_i(l)^{(k)} \equiv p^{(k-1)}(l/x_i)$$

$$\hat{\theta}_l^{(k)} = \frac{\sum_{i=1}^{m} q_i^{(k)}(l)}{\sum_{l=1}^{3}\sum_{i=1}^{m} q_i^{(k)}(l)} \rightarrow \text{2 marks}$$

Same as in mixture model. But, computationally $p(l/x_i)$ may be a challenge!

The other terms are:

$$\underset{\pi_l, A_l, w_{l1} \dots w_{lC}}{\operatorname{argmax}} \sum_{i=1}^{m} q_i(l) \log p(x_i/l) \qquad \forall l = 1,2,3.$$

$$= \underset{\pi_l, A_l, w_{l1} \dots w_{lC}}{\operatorname{argmax}} \sum_{i=1}^{m} q_i(l) \log\left(\sum_{\forall y} p(x_{i1}/y_1) \dots p(x_{iT}/y_T)\, p(y_1/l)\, p(y_2/y_1, l) \dots p(y_T/y_{T-1}, l)\right)$$

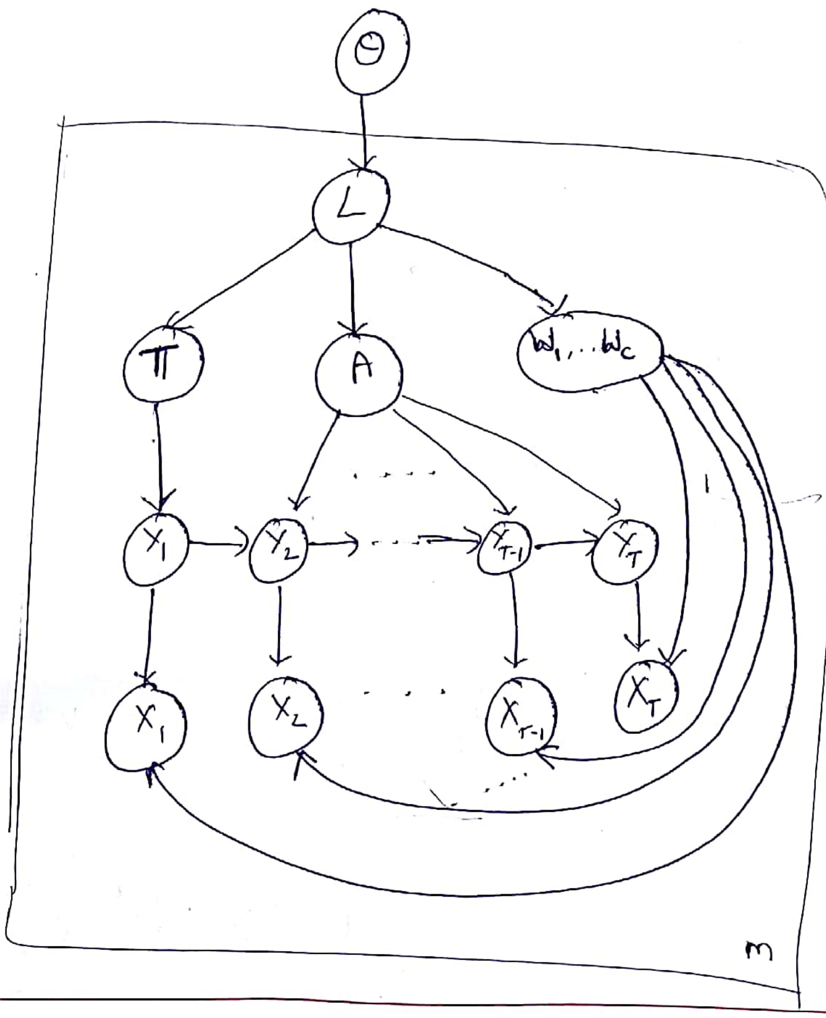
(This is like weighted samples ~~for~~ MLE with HMM!)

$$\therefore\ \hat{\pi}_{ls}^{(k)} = \frac{\sum_{i=1}^{m} q_i^{(k)}(l)\, p^{(k-1)}(y_{i1} = s / x_i, l)}{\sum_{s'=1}^{C}\sum_{i=1}^{m} q_i^{(k)}(l)\, p^{(k-1)}(y_{i1} = s' / x_i, l)} \rightarrow \text{2 marks}$$

$$\hat{A}_{ljk}^{(k)} = \frac{\sum_{i=1}^{m}\sum_{t=2}^{T_i} q_i^{(k)}(l)\, p^{(k-1)}(y_{i,t-1} = j, y_{it} = k / x_i, l)}{\sum_{j', k'} \text{''}} \rightarrow \text{2 marks}$$

$$E_{\hat{w}_{lj}}[\varphi(X_j)] = \frac{\sum_{i=1}^{m}\sum_{t=1}^{T_i} q_i^{(k)}(l)\, p^{(k-1)}(y_{it} = j / x_i)\, \varphi(x_{it})}{\sum_{i=1}^{m}\sum_{t=1}^{T_i} q_i^{(k)}(l)\, p^{(k-1)}(y_{it} = j / x_i)} \rightarrow \text{2 marks}$$

Here is a DAG for the Bayesian model :



5 marks

(3) $I_g(\mathcal{G})$: we need CI ~~for~~ of the form: $A, \boxed{?} \perp\!\!\!\perp \boxed{?} \,/\, J$

Since any path from A to others is d-separated by J, because of 'v' structure, there is no CI that is true.

$\therefore$ From $I_g(\mathcal{G})$ we have empty set! —— 2 marks

$I_o(\mathcal{G}) \rightarrow$ ~~$D \perp\!\!\!\perp A / G$~~ $A \perp\!\!\!\perp B$

$C \perp\!\!\!\perp (A,B)$

$D \perp\!\!\!\perp (A,C) \,/\, B$ → 1.5 marks

~~$I_e(\mathcal{G}) \rightarrow A \perp\!\!\!\perp (B,C,D,E,F,H,I)$~~

~~$B \perp\!\!\!\perp (A,C,E)$~~

~~$\perp\!\!\!\perp (A,B,D,E,F,H,J)$~~ ~~1.5 marks~~

$I_\ell(g) \longrightarrow E \perp\!\!\!\perp (A,C,D,F,G,J) / B$

$F \perp\!\!\!\perp (A,B,D,E,G,H,J) / C$ → 1.5 marks

$G \perp\!\!\!\perp (B,C,E,F,H,I) / (A,D)$

---

(4) Idea is to write p in canonical form and look at the factors. ~~Scope will not exist and~~

For Gaussian a pairwise-scope will not exist iff corresponding covariance inverse term is zero!

It is easy to verify that only cofactor of 1, 3 position is zero ∴ the only scope absent must be X, Z

$\Rightarrow H = $ $X - Y - Z$ → is the perfect I MAP → 6 marks

~~Good~~ $I_g(H) = \{X \perp\!\!\!\perp Z / Y\}$ ~~is the~~ → 1.5 marks

---

(5) Least no. factors are possible iff only maximal cliques are selected.

∴ $\{C, D\}$ $\{S, L, J\}$

$\{D, I, G\}$ $\{S, I\}$

$\{G, J, H\}$

$\{G, L, J\}$

only 6 factors!

5 marks