A Lattice Boltzmann Simulation of Three-Dimensional Displacement Flow of Two Immiscible Liquids in a Square Duct

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1 Introduction

The displacement of one fluid by another under the action of a pressure gradient is known as displacement flow. This paper investigates the three-dimensional displacement flow of a liquid originally occupying a square duct by another immiscible liquid of different density and viscosity, which is injected from the inlet of the channel. The work is an extension of our previous study, in which the displacement flow in a 2D channel was simulated [1]. The displacement flow of one fluid by another can be found in many industrial and geological applications, such as transportation of crude oil in pipelines [2], oil recovery, food processing, coating, flow of lava inside the earth [3], etc. In the latter case, the temperature gradient also plays an important role in the flow dynamics. In porous media or in a Hele–Shaw cell, the displacement of a highly viscous fluid by a less viscous one becomes unstable, forming various instability patterns, known as viscous fingering [4].

Due to their relevance to practical applications, displacement flow has been studied by considering miscible [5–15] as well as immiscible fluids [1,16–20]. In a miscible system, Goyal and Meiburg [7] numerically studied the displacement of a larger viscosity fluid by a less viscous one in a Hele–Shaw cell and found qualitatively similar flow fields observed in the experiment of Petitjeans and Maxworthy [8]. These results are also in good agreement with the theoretical predictions of LaJeunesse et al. [21]. They observed that the two-dimensional instability patterns become 3D at higher flow rates. Taghavi et al. [11,12] studied the displacement flow of two miscible liquids in a system having significant buoyancy effects. They observed Kelvin–Helmholtz–like instabilities at low imposed velocities in the exchange flow dominated regime. Sahu et al. [9,10] studied the pressure-driven flow of two miscible liquids of different densities and viscosities in an inclined channel. They investigated the effects of Reynolds number, Schmidt number, Froude number, and angle of inclination. They also studied the effects of an imposed infinitesimal disturbance on the flow by conducting a linear stability analysis. The effects of temperature gradient and wall heating on a similar system were included in Sahu et al. [22].

The flow characteristics in the above-mentioned systems are very complex, even without the presence of a sharp interface, where one could have a sudden change in the fluid properties. Often the traditional computational methods, which solve the Navier–Stokes equation, are used to study these problems. Immiscible systems are expensive computationally, due to the interfacial dynamics, which happen in a very narrow region. A big task while handling such systems is to track the dynamically changing interface, which requires additional computational effort. Due to this, a very few computational studies have been carried out on immiscible systems. The instabilities in immiscible systems have been studied mainly in core-annular/three-layer/two-layer configuration by several researchers (see the extensive review by Joseph et al. [2] and a recent review on viscosity stratified flows by Govindarajan and Sahu [23]). Considering a two-layer Couette/Poiseuille flow, Yih [24] was the first to show that long waves on a viscosity interface can be unstable at any Reynolds number. Since then, several researchers (for example, Refs. [25–27]) have studied the effects of viscosity ratio, density ratio, and height of the interface in such systems and have investigated both the long- and short-wave instabilities by conducting linear stability analyses.

During the past few decades, lattice Boltzmann method (LBM) has emerged as a promising alternative technique for fluid flow simulations [28]. There are mainly four distinctly different LBM approaches for multiphase flows: the color segregation method of Gunstensen et al. [29], the method of Shan and Chen [30], the free energy approach of Swift et al. [31], and the method of He and coworkers [32–34]. These methods have been previously used by a few researchers [1,17,20] to study the displacement flows of two immiscible liquids. The effects of viscosity ratio, surface tension, wettability, and capillary number was discussed by Chin et al. [18], Grosfils et al. [19], Kang et al. [17], and Dong et al. [20], respectively. Due to the low characteristic Reynolds number considered, these studies did not observe any interfacial instabilities. Using a LBM approach, Langaa and Yeomans [35] studied the fingering phenomenon of two immiscible fluids with different viscosities and investigated the effects of surface tension on the flow.
dynamics. Recently, Redapangu et al. [1] revisited the displacement flow of one liquid by another immiscible liquid in a two-dimensional channel using the two-dimensional version of a two-phase LBM based on the Zhang et al. [32] approach.

In spite of the large number of studies carried out on displacement flows, to the best of our knowledge, none of them have examined the pressure-driven displacement flow of immiscible fluids in three-dimensional duct at moderate-to-large Reynolds numbers. A three-dimensional study is necessary in order to understand the complete picture of the instabilities. Riaz and Meiburg [36] and Oliveira and Meiburg [37] numerically studied three-dimensional displacement flow of two miscible fluids in porous media and Hele–Shaw cell, respectively. They found that some of the flow features in 3D are qualitatively different from those obtained in 2D simulation. Similarly, Haliez and Magnaudet [38] investigated the effects of buoyancy in inclined channel/pipe and found that the vortical structures are more coherent in two-dimensional geometry than those in three-dimensional geometry.

In the present work, we study the three-dimensional pressure-driven displacement flow of two immiscible liquids of different densities and viscosities in a square duct using a multiphase lattice Boltzmann method. To achieve high computational efficiency, we implemented our LBM algorithm on a graphics processing unit (GPU) [39]. Our present GPU-based LBM solver is 25 times faster than the corresponding central processing unit–based code. In this paper, we have investigated the effects of channel inclination, viscosity, and density contrasts. The flow behavior observed in 3D is compared with that obtained in a 2D channel.

The rest of the paper is organized as follows. Details of the problem formulation are given in Sec. 2, the LBM approach used to carry out the computations is discussed in Sec. 3, the results of this study are provided in Sec. 4, and concluding remarks are given in Sec. 5.

2 Formulation

We consider the pressure-driven displacement of one liquid (“liquid 2”: viscosity \( \mu_2 \) and density \( \rho_2 \)) by another liquid (“liquid 1”: viscosity \( \mu_1 \) and density \( \rho_1 \)). The liquids are immiscible and are assumed to be incompressible Newtonian. The schematic of the geometry is shown in Fig. 1. A three-dimensional rectangular coordinate system, \((x, y, z)\), is used to model the flow dynamics where \( x, y, \) and \( z \) denote the axial, vertical, and the azimuthal coordinates, respectively. The inlet and the outlet of the channel are located at \( x = 0 \) and \( L \), respectively. The rigid and impermeable walls of the channel are located at \( y = 0, H \) and \( z = 0, W \). Here, we consider \( W = H \), and the aspect ratio of channel \( L/H \) is 32. \( \theta \) is the angle of inclination measured with the horizontal. \( g \) is the acceleration due to gravity. The components of gravity \( g_x (\equiv g \sin \theta) \) and \( g_z (\equiv g \cos \theta) \) act in the negative axial and negative vertical directions, respectively. The gravity component in the azimuthal direction, \( g_\phi \), equals to zero. In the initial configuration, the channel portions from \( 0 \leq x \leq H/4 \) and \( H/4 \leq x \leq L \) are filled with liquid 1 and liquid 2, respectively.

At \( t = 0 \), the liquids inside the channel are stationary when a fully developed flow due to a pressure gradient is imposed at the inlet. This is given by

\[
\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = Re \frac{dp}{dx}
\]

where \( u \) is the velocity component in the axial direction and \( p \) is the pressure. The velocity components in the vertical \( (v) \) and azimuthal \( (w) \) directions are zero at the inlet. The volumetric flow rate, \( Q \), is maintained constant for all the simulations performed in the present study, and the imposed pressure-gradient is calculated from the prescribed volumetric flow rate. The no-slip and no-penetration boundary conditions are imposed at the walls, and the Neumann condition is prescribed at the outlet \((x = L)\). We define the Reynolds number based on the viscosity of liquid 1, \( Re = \frac{Q \rho_1}{\mu_1} \). Viscosity ratio, \( m \), is defined as the ratio of viscosity of liquid 2 to the viscosity of liquid 1, \( (i.e., \ m \equiv \mu_2/\mu_1) \). For a value of \( m \) above 1, the displacing liquid is less viscous than the liquid inside the channel and vice versa for \( m \) below 1. The density contrasts between the liquids are measured by a dimensionless quantity, \( At \equiv (\rho_2 - \rho_1)/(\rho_2 + \rho_1) \); thus, \( At > 0 \) indicates a lighter liquid entering the channel. The gravity effects can be characterized by Froude number \( (Fr) \), defined as \( Fr \equiv Q/HW/\sqrt{At} \). The magnitude of surface tension is measured by \( \kappa \), which is related to surface tension, \( \sigma \) (please see Sec. 3). The surface tension is characterized by capillary number, \( Ca \equiv (\mu_1 / \sigma \beta H) \). The dimensionless time scale is represented as \( t \equiv H^2/W/Q \).

3 Numerical Method

The simulations are performed with the two-phase lattice Boltzmann method (LBM) previously proposed by He and coworkers [32] and used in recent study of Sahu and Vanka [41]. A three-dimensional, 15-velocity lattice model (D3Q15) is used. The LBM simulates fluid flow problems using discrete density distribution functions. In this method, two distribution functions (index distribution function \((f)\) and the pressure distribution function \((g)\)) are used to track the interface and to calculate the macroscopic properties of the fluids. The evolution equations of distribution functions are given by

\[
\begin{align*}
\frac{f_x(x + e_x \delta t, t + \delta t) - f_x(x, t)}{\tau} &= -f_x(x, t) - f_y(x, t) \partial_x \phi(x, y) - f_z(x, t) \partial_z \phi(x, y) - f_y(x, t) \partial_y \phi(x, y) - f_z(x, t) \partial_z \phi(x, y) \\
\frac{2\tau - 1}{2\tau} (e_x - \mathbf{u}) \cdot \nabla \phi(x, y) &+ \Gamma_x(\mathbf{u}) \delta t \\
\frac{2\tau - 1}{2\tau} (e_z - \mathbf{u}) \cdot \nabla \phi(x, y) &+ \Gamma_z(\mathbf{u}) \delta t \\
\frac{2\tau - 1}{2\tau} (e_y - \mathbf{u}) \cdot \nabla \phi(x, y) &+ \Gamma_y(\mathbf{u}) \delta t \\
\end{align*}
\]

Here, \( \mathbf{u} = (u, v, w) \) represents the three-dimensional velocity field, \( \delta t \) is the time step, \( z \) is the lattice direction (shown in Fig. 2), and...
\[ \tau \] is the relaxation time. The kinematic viscosity, \( \nu \), is related to the relaxation time by the expression \( \nu = (\tau - 1/2) \delta t c_s^2 \).

In D3Q15 lattice, lattice velocities and the weighing coefficients are given by

\[
[e_1, e_2, \ldots, e_{15}] = 
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{bmatrix}
\] (4)

\[
[t_1, t_2, \ldots, t_{15}] = 
\begin{bmatrix}
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\
72 & 72 & 72 & 72 & 72 & 72 & 72 & 72 & 72 & 72 & 72 & 72 & 72 & 72 & 72
\end{bmatrix}
\] (5)

respectively. \( \Gamma (u) \) is a function of macroscopic velocity \( u \), which is given by

\[
\Gamma (u) = t_s \left[ 1 + \frac{e_s \cdot u}{c_s^2} + \frac{(e_s \cdot u)^2}{2c_s^4} - \frac{u^2}{2c_s^2} \right] \] (6)

The equilibrium distribution functions, \( f_s^{eq} \) and \( g_s^{eq} \), are given by

\[
f_s^{eq} = t_s \phi \left[ 1 + \frac{e_s \cdot u}{c_s^2} + \frac{(e_s \cdot u)^2}{2c_s^4} - \frac{u^2}{2c_s^2} \right] \] (7)

\[
g_s^{eq} = t_s \left[ p + \rho c_s^2 \left( \frac{e_s \cdot u}{c_s^2} + \frac{(e_s \cdot u)^2}{2c_s^4} - \frac{u^2}{2c_s^2} \right) \right] \] (8)

where \( c_s^2 = 1/3 \).

The term \( \nabla \psi (\phi) \) mimics the intermolecular interactions for nonideal gases or dense fluids, which plays an important role in phase segregation. Thus, it needs to be computed accurately; a fourth order compact scheme is used to discretize \( \nabla \psi (\phi) \) and \( \nabla \psi (\rho) \) [42]. We use the Carnahan-Starling equation of state for \( \psi (\phi) \), which describes nonideal gases and fluids [43,44],

\[
\psi (\phi) = c_2 \phi \left[ 1 + \phi + \frac{\phi^2}{(1-\phi)^2} - a\phi^2 \right] (9)
\]

where \( a \) determines the strength of molecular interactions. The critical value of Carnahan-Starling equation of state is \( a_c = 3.53374 \). Choosing a value of \( a > a_c \), one could capture phase separation of immiscible liquids; however, selecting a very large \( a \) value can create numerical error [32]. Therefore, we chose \( a = 4 \) in the present study. The term \( \psi (\rho) \) is given by \( p - \rho c_s^2 \). The terms \( F_s \) and \( G \) in Eq. (3) represent the surface tension force and the gravity forces, respectively,

\[
F_s = \kappa \phi \nabla^2 \phi \quad \text{and} \quad G = (\rho - \rho_w)g
\] (10)

where \( \kappa \) is the magnitude of surface tension and \( \rho_w = (\rho_1 + \rho_2)/2 \).

The surface tension, \( \sigma \), can be related to \( k \) as follows [45]:

\[
\sigma = \kappa \left( \frac{\partial \phi}{\partial \zeta} \right)^2 d \zeta = \kappa I(a)
\] (11)

where \( \zeta \) is the direction normal to the interface. Zhang et al. [32] obtained an analytical expression for \( I(a) \) to calculate surface tension, which is given by

\[
I(a) = \frac{0.1518(a - a_c)^{1.5}}{1 + 3.385(a - a_c)^{0.5}}
\] (12)

The index function \( (\phi) \), pressure \( (\rho) \), and the velocity field \( (u) \) are calculated from the distribution functions as

\[
\phi = \sum \phi_s (13)
\]

\[
p = \sum (x - \frac{1}{2} u \cdot \nabla \phi) \delta t (14)
\]

\[
\rho_ uc_s^2 = \sum \phi_s (12) (F_s + G) \delta t (15)
\]

The liquid density and kinematic viscosity can be estimated from the index function using the following equations:

\[
\rho (\phi) = \rho_2 + \frac{\phi - \phi_2}{\phi_1 - \phi_2} (\rho_1 - \rho_2) (16)
\]

\[
\nu (\phi) = \nu_2 + \frac{\phi - \phi_2}{\phi_1 - \phi_2} (\nu_1 - \nu_2) (17)
\]

Here, \( \nu_1 \) and \( \nu_2 \) are the kinematic viscosities of liquids 1 and 2, respectively. \( \phi_1 \) and \( \phi_2 \) are minimum and maximum values of the index function; in the present study, \( \phi_1 \) and \( \phi_2 \) are given values of 0.02381 and 0.2508, respectively [32]. In Sec. 4, we discuss the results obtained from three-dimensional simulations followed by comparison of the results with those obtained from two-dimensional simulations.
4 Results and Discussion

We start presenting our results by first showing the results of grid independence tests. The temporal variation of the dimensionless mass of liquid 2, $M_2/M_0$, is plotted for different grids in Fig. 3. Here, $M_i$ represents the instantaneous mass of liquid $i$ and is defined as $M_i = \rho_i \int_0^H \int_0^{2\pi} \int_0^L (\phi - \phi_1)/(\phi_2 - \phi_1) \, dx \, dy \, dz$. $M_0(=\rho_2(\phi - \phi_1)/(\phi_2 - \phi_1)HWL)$ is the mass of liquid 2 initially occupying the channel at $t = 0$. The parameters used to generate this plot are $Re = 100, m = 10, At = 0.2, Fr = 5, Ca = 29.9$, and $\theta = 45$ deg. In Fig. 3, it can be seen that $M_2/M_0$ decrease monotonically with time as liquid 2 is displaced by liquid 1. The dotted line represents the analytical expression for plug-flow displacement (given by $M_i/M_0 = 1 - \theta H/L$).

The spatiotemporal evolution of the isosurface of $\phi$ in the range 0.023–0.25 (which represents the interface between the fluids) is shown in Fig. 5 for the parameter values the same as those used to generate Fig. 3. This set of parameter values represents a situation when a less viscous and less dense liquid displaces a highly viscous and highly dense liquid. In this system, due to the imposed pressure gradient at the inlet, a “finger” of liquid 1 penetrates into the region of liquid 2. As the channel is inclined at an angle $\theta = 45$ deg, the motion induced by the imposed pressure gradient is opposed by the flow, due to the gravitational force (proportional to $g_x$) in the negative axial direction. Therefore, the flow dynamics are due to the competition between the imposed pressure gradient and the gravity force. At the same time, the component of the gravitational force in the vertical direction (proportional to $g_y$) acts to segregate the two liquids. The finger becomes asymmetrical because of this force. We observed that, for $t \geq 10$, the interface separating the two liquids becomes unstable, forming nonlinear interfacial waves that grow in time. At later times ($t > 18$ for this set of parameter values), the instabilities of cork-screw pattern are observed. This pattern is the consequence of the Yih-type instabilities [24] in the linear regime. By conducting a linear stability analysis in a miscible core-annular flow configuration, Selvam et al. [46] found that, at low diffusivity, the cork-screw mode becomes more unstable than the axisymmetric mode. The cork-screw-type pattern was also observed by Scoffoni et al. [47] in miscible displacement flow in a vertical pipe.

Next, we discuss the effects of angles of inclination, $\theta$, by plotting the contours of $\phi$ at $t = 20$ in the $x$-$y$ plane at $z = W/2$ and in $x$-$z$ plane at $y = H/2$ in Figs. 6(a) and 6(b), respectively. The rest of the parameter values are $Re = 100, At = 0.2, m = 10, Fr = 1$, and $Ca = 29.9$. The plots in panel (a) of this figure are the views in the $z$ direction. It can be seen that the intensity of the instabilities increases with increasing angles of inclination. Also, due to the decrease in the gravitational force in the $y$ direction, for higher angles of inclination, core-annular structures form as the finger...
inclination, the pressure gradient dominates the flow dynamics, creating a core-annular configuration, as shown in Fig. 7 for \( \theta > 60 \) deg.

In Figs. 8(a) and 8(b), the axial variation of normalized average viscosity, \( \bar{\mu}_{xz} \equiv \mu_{xz} / \mu_{x0} \), is plotted at different times for \( \theta = 30 \) deg and \( \theta = 85 \) deg, respectively. Here, \( \mu_{xz} = (1/LW) \int_0^L \int_0^{h/2} \mu dy dz \) and \( \mu_{x0} \) is the value of \( \mu_{xz} \) at \( x = 0 \). The rest of the parameters are the same as those in Fig. 6. It can be seen that, as expected from the discussion of Fig. 6(a), the variation becomes increasingly complex with increase in \( \theta \). Similarly, the vertical variation of normalized average viscosity, \( \bar{\mu}_{yz} \equiv \mu_{yz} / \mu_{y0} \), where \( \mu_{yz} = (1/LW) \int_0^H \int_0^{l/2} \mu dx dz \), is plotted in Fig. 9 for different times. The value of \( \mu_{yz} \) at \( y = 0 \) is designated by \( \mu_{y0} \). It can be seen that, for higher angles of inclination, a core-annular structure is formed, which was also discussed in Fig. 7. On the other hand, for lower angles of inclination, a two-layer structure is formed as the heavier fluid settles in the lower part of the channel.

Next, we compare the three-dimensional results with those obtained from the two-dimensional simulations. For this, we plot the axial and the vertical variation of \( \bar{\mu}_{xz} \) and \( \bar{\mu}_{yz} \) in Figs. 10 and 11, respectively, for \( \theta = 5 \) deg and \( \theta = 85 \) deg. Here, \( \mu_{y0} = 1/L \int_0^H \mu dy \) and \( \mu_{x0} = 1/L \int_0^{h/2} \mu dx \).

**Fig. 6** Contours of the index function, \( \phi \), at \( t = 20 \) in (a) \( x-y \) plane at \( z = W/2 \) and (b) \( x-z \) plane at \( y = H/2 \). The parameters are \( \text{Re} = 100, \text{At} = 0.2, m = 10, \text{Fr} = 1, \) and \( \text{Ca} = 29.9 \).

**Fig. 7** Contours of the index function, \( \phi \), at \( t = 20 \) in the \( y-z \) plane at \( x = L/2 \). The parameters are \( \text{Re} = 100, \text{At} = 0.2, m = 10, \text{Fr} = 1, \) and \( \text{Ca} = 29.9 \).

penetrates downstream. However, the gravitational force in the negative axial direction increases with increasing \( \theta \), which makes the trailing fingers asymmetrical in the upstream regime. We observe that the fingers have a blunt “nose,” separating the two fluids in these cases. For the largest angle of inclination considered (\( \theta = 85 \) deg), the droplet of fluid 1 is detached from the main finger at \( t = 20 \). As the gravitational force in the negative \( y \) direction increases with \( \theta \), for higher angles of inclination, it is dominated by the effects created due to the imposed pressure gradient, therefore leaving patches of the heavier fluid near the top wall at \( y = H \). This is shown in Fig. 6(b), when one looks from the top in the \( y \) direction.

The contours of \( \phi \) in the \( y-z \) plane at \( x = L/2 \) are shown in Fig. 7 for different angles of inclination. The parameter values are the same as those used to generate Fig. 6. This is a view of a cross section at the midlength of the channel in the \( x \) direction. For smaller angles of inclination, the gravitational force in the negative \( y \) direction is stronger, which tries to settle the heavier fluid in the bottom part of the channel. This can be observed for \( \theta \leq 30 \) deg in Fig. 7. However, as discussed above for higher angles of inclination, the pressure gradient dominates the flow dynamics, creating a core-annular configuration, as shown in Fig. 7 for \( \theta > 60 \) deg.

In Figs. 8(a) and 8(b), the axial variation of normalized average viscosity, \( \bar{\mu}_{xz} \equiv \mu_{xz} / \mu_{x0} \), is plotted at different times for \( \theta = 30 \) deg and \( \theta = 85 \) deg, respectively. Here, \( \mu_{xz} = (1/LW) \int_0^L \int_0^{h/2} \mu dy dz \) and \( \mu_{x0} \) is the value of \( \mu_{xz} \) at \( x = 0 \). The rest of the parameters are the same as those in Fig. 6. It can be seen that, as expected from the discussion of Fig. 6(a), the variation becomes increasingly complex with increase in \( \theta \). Similarly, the vertical variation of normalized average viscosity, \( \bar{\mu}_{yz} \equiv \mu_{yz} / \mu_{y0} \), where \( \mu_{yz} = (1/LW) \int_0^H \int_0^{l/2} \mu dx dz \), is plotted in Fig. 9 for different times. The value of \( \mu_{yz} \) at \( y = 0 \) is designated by \( \mu_{y0} \). It can be seen that, for higher angles of inclination, a core-annular structure is formed, which was also discussed in Fig. 7. On the other hand, for lower angles of inclination, a two-layer structure is formed as the heavier fluid settles in the lower part of the channel.

Next, we compare the three-dimensional results with those obtained from the two-dimensional simulations. For this, we plot the axial and the vertical variation of \( \bar{\mu}_{xz} \) and \( \bar{\mu}_{yz} \) in Figs. 10 and 11, respectively, for \( \theta = 5 \) deg and \( \theta = 85 \) deg. Here, \( \mu_{y0} = 1/L \int_0^H \mu dy \) and \( \mu_{x0} = 1/L \int_0^{h/2} \mu dx \).
\( \mu_{xz} \) are the values of \( \mu_z \) at \( x = 0 \) and \( \mu_z \) at \( y = 0 \), respectively. It can be seen in Fig. 10 that the axial variation of \( \mu_{y} \) is more chaotic in the two-dimensional channel as compared to that in three-dimensional channel (shown in Fig. 8). This is due to the stabilizing effects of the azimuthal walls to the disturbances of smaller wavelength. This behavior is similar to the surface-tension effects in two-dimensional channel \[48\]. Inspection of the vertical variation of \( \mu_{x} \) reveals that the two-dimensional flow remains core annular, even for small angles of inclination. This is in contrast with the phenomena observed in three-dimensional computations (Fig. 9).

Then, we compare the results obtained from the two- and three-dimensional channels by plotting the spatiotemporal evolution of \( \phi \) contours in Figs. 12(a) and 12(b), respectively, for \( Re = 100, m = 20, At = 0.2, Fr = 5, \theta = 30 \) deg, and \( Ca = 29.9 \). It can be observed that 3D effect reduces the short-wave instabilities, which are apparent in the two-dimensional channel at later times (see near the exit of the channel at \( t \geq 24 \) for this set of parameters). This behavior is also discussed above. Close inspection of Fig. 12 also reveals that the velocity of the fingertip is more in the three-dimensional channel than that in the two-dimensional channel. In order to study the effects of viscosity ratio, in Fig. 13, we plot the
spatiotemporal evolution of \( \phi \) contours in the \( x-y \) plane for \( m = 5 \), with the rest of the parameter values the same as those used to generate Fig. 12. It can be seen that, as the viscosity ratio is decreased, the interfacial instabilities reduced significantly. We observed that the instabilities decrease monotonically with decreasing the viscosity ratio for this set of parameters. It can be seen in Fig. 14 that increasing Atwood number decreases the velocity of the fingertip and makes the nose of the finger “blunt”. The latter effect is also associated with the decrease in Froude number as compared to Figs. 12 and 13. We found (not shown here) that the effects of Atwood number, Froude number, and surface-tension parameter are similar to that of two-dimensional channel, and the readers are referred to our previous paper [1] for the detailed parametric study.

5 Summary
Three-dimensional simulations of pressure-driven flow of two immiscible liquids are performed in a square duct using a multiphase lattice Boltzmann approach and compared with the corresponding two-dimensional simulations. A two-phase lattice Boltzmann method (LBM) previously reported by He and co-workers [32] is used to simulate the flow using a three-dimensional, 15-velocity lattice model (D3Q15). We demonstrate that the flow dynamics in a three-dimensional channel is quite different than that in a two-dimensional channel. In particular, screw-type instabilities are found in a three-dimensional channel, whereas saw-tooth–type instabilities are apparent in a 2D channel. We also show that increasing angle of inclination increases the intensity of the instabilities. We observe that the effects of Atwood number, Froude number, and surface tension are qualitatively similar to those in a two-dimensional channel. We conclude that displacement in an inclined three-dimensional channel is a rich problem, which needs further experimental and computational investigations.

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References


