Contrapositive Symmetrization of Fuzzy Implications - Revisited

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Abstract

In (Fuzzy Sets and Systems 69 (1995) 141 - 156) Fodor has discussed the Contrapositive Symmetry of Fuzzy Implications using the Upper Contrapositivisation technique as proposed in Bandler and Kohout (Int. Jl. Man Machine Stud. 12 (1980) 89 - 116). In this work we investigate an alternate technique proposed towards imparting contrapositive symmetry to any given fuzzy implication, viz., Lower Contrapositivisation. First we investigate the conditions under which these two techniques are suitable and enumerate some of their properties. Then we discuss the Lower Contrapositivisation of R-implications and show that this also can be seen as a residuation of a suitable binary operator, along the lines of Fodor. Since the Upper and Lower Contrapositivisation techniques, in general, can be applied to any Implication Operator, we investigate whether they can be written as S-implications for suitable fuzzy disjunctions. Also some sufficient conditions for these fuzzy disjunctions to become t-conorms are given.

Key words: Residuated Implications, Strong Implications, Contrapositive Symmetry, Upper Contrapositivisation, Lower Contrapositivisation.

\textsuperscript{1}This work was done with the research grant that was given by Sri Sathya Sai Institute of Higher Learning to the author.
1 Introduction

In [1] Fodor has discussed the Contrapositive Symmetry of Fuzzy Implication Operators for the three main families, viz., S-, R- and QL-implications. It is well known that the natural negation of R-implications obtained from strict t-norm is not strong, in fact, neither continuous nor strict (see Remark after Theorem 2 in [1]). Towards making this sub-family of R-implications to possess Contrapositive Symmetry with respect to some strong negation \( N \), Fodor [1] has employed the Upper Contrapositivisation technique as proposed in Bandler and Kohout [2], viz.,

\[
x \rightarrow_T y = \max\{J_T(x, y), J_T(N(y), N(x))\}
\]

In this work we investigate an alternate technique proposed towards imparting contrapositive symmetry to a given fuzzy implication, viz., Lower Contrapositivisation technique, again as proposed in [2].

Firstly, in Section 3 we investigate the conditions under which these two techniques are suitable and show that while Upper Contrapositivisation technique is useful when the natural negation of an Implication Operator \( J \), given by \( J(x, 0) \), is less than the strong negation \( N \) considered, the Lower Contrapositivisation technique is useful when \( J(x, 0) \) is greater than the strong negation \( N \) considered.

A t-norm \( T \) and \( J_T \) the R-implication obtained from \( T \) are said to have the residuation principle if they satisfy the following:

\[
T(x, y) \leq z \iff J_T(x, z) \geq y
\]  

(1)

Fodor [1] has proposed a binary operator \( \ast_T \) with which the Upper Contrapositivisation of \( J_T \), denoted \( \rightarrow_T \), has the residuation principle (1), i.e.,

\[
x \ast_T y \leq z \iff x \rightarrow_T z \geq y
\]

Along the same lines, in Section 4, we propose a suitable binary operation \( \ast_t \) such that the Lower Contrapositivisation of the R-implication \( J_T \) has the residuation principle with respect to \( \ast_t \).

Since the Upper and Lower Contrapositivisation techniques, in general, can be applied to any Fuzzy Implication Operator, we investigate, in Section 5, whether they can be written as S-implications for suitable fuzzy disjunctions \( \circ, \diamond \), respectively. Subsequently, we propose some sufficient conditions for these fuzzy disjunctions to become t-conorms.
2 Preliminaries

To make this work self-contained, we briefly mention some of the concepts and results employed in the rest of the work. Also, we denote by $I$ the unit interval $[0, 1]$ in the sequel.

2.1 Negations

**Definition 1** (Fodor and Roubens[6]) A Negation $N$ is a Unary function from $I$ to $I$ such that:

- $N(0) = 1$; $N(1) = 0$;
- $N$ is a non-increasing function.

**Definition 2** (Fodor and Roubens[6]) A negation $N$ if in addition is strict and continuous is called a strict negation.

**Definition 3** (Fodor and Roubens[6]) A strong negation $N$ is a strict negation $N$ that is also involutive, i.e., $N(N(a)) = a$, $\forall a \in I$.

**Theorem 1** (Trillas [11]) $N : I \rightarrow I$ is a strong negation iff there exists a monotone bijection $\phi : I \rightarrow I$, such that $N(x) = \phi^{-1}(1 - \phi(x))$, $\forall x \in I$.

2.2 $T$-norms and $T$-conorms

**Definition 4** (Klement et al.,[7,8]) A t-norm $T$ is a function from $I \times I$ to $I$ such that $\forall a, b, c \in I$,

- $T(a, 1) = a$
- $T(a, b) = T(b, a)$
- $T(a, T(b, c)) = T(T(a, b), c)$
- $T(a, b)$ is monotonic non-decreasing in both the variable

**Definition 5** (Klement et al.,[7,8]) An t-conorm $S$ is a function from $I \times I$ to $I$ such that $\forall a, b, c \in I$,

- $S(a, 0) = a$
- $S(a, b) = S(b, a)$
- $S(a, S(b, c)) = S(S(a, b), c)$
- $S(a, b)$ is monotonic non-decreasing in both the variable.

**Definition 6** (Klement et al., [7,8,9]) A t-norm $T$ is said to be
• Continuous if it is continuous in both the arguments.
• Archimedean if $T(x,x) < x$, $\forall x \in (0,1)$.
• Strict if $T(x,y) > 0$, $\forall x, y \in [0,1]$
• Nilpotent if $T$ is not strict, i.e., $\exists x, y \in (0,1) \ni T(x,y) = 0$, i.e., $T$ has zero-divisors.

**Definition 7** (Klement et al., [7,8,9]) A t-conorm $S$ is said to be

- Continuous if $S$ is continuous in both the arguments.
- Archimedean if $S(x,x) > x$, $\forall x \in (0,1)$.
- Strict if $S(x,y) < 1$, $\forall x,y \in (0,1)$.
- Nilpotent if $S$ is not strict, i.e., $\exists x,y \in (0,1) \ni S(x,y) = 1$.

**Theorem 2** (Fodor, Roubens [6, Thm 1.5]) A continuous t-norm $T$ satisfies $T(x,N(x)) = 0$, $\forall x \in I$ with a strong negation $N$ if $T$ is Nilpotent.

**Theorem 3** (Klement et al., [9, Thm 2.6]) Any nilpotent t-norm $T$ is a conjugate of the Lukasiewicz t-norm, $T_L(x,y) = \max(x+y-1,0)$, i.e., there exists an increasing bijection $\phi$ on $I$ such that

$$T(x,y) = \phi^{-1}(\max\{\phi(x) + \phi(y) - 1, 0\}).$$

### 2.3 Fuzzy Implications

**Definition 8** (Fodor and Roubens [6]) A function $J : I^2 \to I$ is called a Fuzzy Implication if it has the following properties:

1. $J(p,r) \geq J(q,r)$ if $q \geq p$
2. $J(p,r) \geq J(p,s)$ if $r \geq s$
3. $J(0,r) = 1$, $\forall r \in I$
4. $J(p,1) = 1$, $\forall r \in I$.
5. $J(1,0) = 0$

**Definition 9** An $S$-implication $J_S$ is obtained from a t-conorm $S$ and a strong negation $N$ as follows:

$$J_S(a,b) = S(N(a),b), \forall a, b \in I.$$ 

**Definition 10** An $R$-implication $J_T$ is obtained from a (left) continuous t-norm $T$ as its residuation as follows:

$$J_T(a,b) = \text{Sup} \{x \in I : b \geq T(a,x)\}, \forall a, b \in I.$$
Definition 11 A QL-implication is obtained from a t-conorm $S$, t-norm $T$ and a strong negation $N$ as follows:

$$J_{QL}(a,b) = S(N(a), T(a,b)), \forall a, b \in I.$$  \hspace{1cm} (4)

Definition 12 A fuzzy Implication $J$ is said to have

- Contrapositive Symmetry with respect to a strong negation $N$, denoted CPS$(N)$, if

$$J(x, y) = J(N(y), N(x)), \forall x, y \in I$$  \hspace{1cm} (5)

- the Ordering property if

$$y \geq x \iff J(x, y) = 1$$  \hspace{1cm} (6)

- the Neutrality property or is said to be Neutral if

$$J(1, y) = y, \forall y \in I$$  \hspace{1cm} (7)

- the Exchange property if

$$J(x, J(y, z)) \equiv J(y, J(x, z)), \forall x, y, z \in I.$$  \hspace{1cm} (8)

Definition 13 Let $J$ be any fuzzy Implication. The natural negation of $J$, denoted by $\neg$, is given by $J(x, 0) = \neg x, \forall x \in I$. Clearly $\neg(0) = 1$ and $\neg(1) = 0$.

Corollary 14 (Jenei [3, Corollary 2]) Let $T$ be a left-continuous t-norm. Then $J_T$, the R-implication obtained from $T$, has CPS$(N)$ with respect to a strong negation $N$ if and only if the natural negation of $J_T$, namely $J_T(x, 0)$, is involutive. In this case $N(x)$ is equal to $J_T(x, 0)$.

Corollary 15 (Fodor [1, Corollary 2]) Let $T$ be a continuous t-norm. Then $J_T$, the R-implication obtained from $T$, has CPS$(N)$ with respect to a strong negation $N$ if and only if there exists an automorphism $\phi$ of the unit interval such that

$$T(x, y) = \phi^{-1}(\max\{\phi(x) + \phi(y) - 1, 0\})$$

$$N(x) = \phi^{-1}(1 - \phi(x)).$$

In this case $J_T$ is given by

$$J_T(x, y) = \phi^{-1}(\min\{1 - \phi(x) + \phi(y), 1\})$$

Remark 16 While all S-implications possess CPS$(\neg x)$, with respect to their natural negation $\neg x$, (7) and (8), all R-implications possess properties (6),
(7) and (8) (see [12]). Also, from the Corollary 15 and Theorem 3, we have that R-implications $J_T$ obtained from nilpotent t-norms have $\text{CPS}(J_T(x,0))$. In general QL-implications only satisfy (7) and not (6) or (8). Also not all QL-implications have (5) (See Fodor [1, Lemma 1]).

**Lemma 1** Let $J$ be from $I \times I$ to $I$ and $N$ a strong negation. Then

i) $J$ has $\text{CPS}(N)$ and is Neutral $\Rightarrow \neg x = N(x)$.

ii) $\neg x = N(x)$ and $J$ has the Exchange Property $\Rightarrow J$ has $\text{CPS}(N)$ and is Neutral.

iii) $J$ has $\text{CPS}(N)$, the Exchange and Ordering Properties $\Rightarrow \neg x = N(x)$.

**PROOF.**

i) $\neg x = J(x,0) = J(N(0),N(x)) = J(1,N(x)) = N(x)$

ii) $J(N(y),N(x)) = J(N(y),J(x,0)) = J(x,J(N(y),0)) = J(x,N(N(y))) = J(x,y)$, and $J(1,y) = J(N(y),0) = N(N(y)) = y$.

iii) $J$ satisfies (6) and (8) $\Rightarrow J$ satisfies (7) (See Fodor and Roubens [6, Lemma 1.3]. Then using i) we have the result.

**Example 1** That the reverse implication of i) in Lemma 1 does not always hold for all Implications with $J(x,0)$ being a strong negation can be seen from the following theorem in [12]

**Theorem 4** (Trillas and Valverde [12, Theorem 5.2]) Let $J_{QL}$ be a QL-Implication (see Definition 11). $J_{QL}$ has Contrapositive symmetry with respect to $N(x) = \neg x = J(x,0)$ iff $(S,T,N)$ is a Lukasiewicz Triple, i.e., $J_{QL}(x,y) = \max(N(x),y)$.

**Example 2** To see that the reverse implication of ii) in Lemma 1 does not always hold, consider the Upper Contrapositivisation $J^*_G$ [1] of the Goguen’s Implication $J_G$ :

$$J_G(x,y) = \min\left\{1, \frac{y}{x}\right\}, \quad J^*_G(x,y) = \max\left\{\frac{y}{x}, \frac{1-x}{1-y}\right\}$$

$J^*_G$ has both (5) with respect to the strong negation $1-x$, i.e., has $\text{CPS}(1-x)$, (7) (see Proposition 17) and $J^*_G(x,0) = \neg x = 1 - x$ but not (8).

**Example 3** To see that the reverse implication of iii) in Lemma 1 is not true, again for any QL-implication $J_{QL}(x,0) = N(x)$ where $N$ is strong, but not all QL-implications satisfy (8) or have the Ordering Property (6) (See Trillas and Valverde [12]).
Theorem 5 Let $J$ be an Implication Operator and $N$ a strong negation. Then $J$ has $CPS(N)$, the Exchange Property and is Neutral $\iff S(x, y) =_{def} J(N(x), y)$ is a t-conorm.

PROOF. ($\Rightarrow$)
To show that $S(x, y) =_{def} J(N(x), y)$ is a t-conorm, we need to show the following:

1. $S(x, 0) = J(N(x), 0) = \neg(N(x)) = N(N(x)) = x$, from i) above.
2. $x \geq y \Rightarrow N(x) \leq N(y) \Rightarrow J(N(x), .) \geq J(N(y), .) \Rightarrow S(x, .) \geq S(y, .)$ and thus $S$ is non-decreasing in the first place. Since, $J$ is non-decreasing in the second place, so is $S$.
3. $S(x, y) = J(N(x), y) = J(N(y), x) = S(y, x)$, since $J$ has $CPS(N)$. Thus $S$ is commutative.
4. $S(S(x, y), z) = S(y, x), z = S(z, S(y, x)) = J(N(z), J(N(y), x)) = J(N(z), J(N(x), y)) = J(N(x), J(N(z), y)) = J(N(x), J(N(y), z)) = J(N(x), S(y, z)) = S(x, (S(y, z)))$.
Thus $S$ is associative and hence a t-conorm.

($\Leftarrow$)
Let $S(x, y) =_{def} J(N(x), y)$ be a t-conorm.

1. $J(N(y), N(x)) = S(y, N(x)) = S(N(x), y) = J(N(N(x)), y) = J(x, y)$, by the commutativity of $S$ and involution of the strong $N$. Thus $J$ has $CPS(N)$.
2. $J(x, J(y, z)) = S(N(x), J(y, z)) = S(N(x), S(N(y), z)) = S(S(N(x), N(y)), z) = S(S(N(y), N(x)), z) = S(N(y), J(x, z)) = J(y, J(x, z))$, by the associativity and commutativity of $S$. Thus $J$ has the Exchange Property.
3. $J(1, x) = S(N(1), x) = S(0, x) = x$, by the boundary condition of the t-conorm $S$. Thus $J$ is Neutral.

3 Upper Contrapositivisation Vs Lower Contrapositivisation

3.1 Fodor’s Approach : The Upper Contrapositivisation Technique :

In [1], Fodor dealt with the Contrapositive symmetry of Residuated Implications. Let $J_T$ be an R-implication obtained from a t-norm $T$. If $J_T$ is obtained from a nilpotent t-norm, its natural negation $\neg x = J_T(x, 0)$ is a strong negation ( See Fodor [1,Corollary 2] ), while if $J_T$ is obtained from a strict t-norm then, by definition, $\neg x = 0, \forall x \in (0, 1]$ and $\neg x = 1$ if $x = 0$, which is neither strict nor continuous and hence is not strong. Thus, if $T$ is a strict t-norm then $J_T$ does not have contrapositive symmetry with respect to its natural
negation, in the sense of Definition 12 (See Cignoli et al. [5] for a discussion on Contrapositive symmetry with respect to weak negations). When $N$ is a strong Negation, Fodor defines a new implication from $J_T$ as follows,

$$x \xRightarrow{U} y = x \rightarrow_T y = \max\{J_T(x, y), J_T(N(y), N(x))\}$$  \hspace{1cm} (9)

3.2 Suitability of $\xRightarrow{U}$

It can be easily seen that $\xRightarrow{U}$ always has the contrapositive symmetry with respect to the strong negation $N$. The operation $\xRightarrow{U}$ is nothing but the Upper Contrapositivisation of a given fuzzy Implication Operator as proposed by Bandler and Kohout [2]. The following Propositions can be easily verified:

**Proposition 17** If $J$ is a Neutral fuzzy Implication and $N$ is a strong negation such that the natural negation $J(x, 0) = \neg x \leq N(x)$, $\forall x \in I$, then the Upper Contrapositivisation of $J$ with respect to $N$ is such that

i) $x \xRightarrow{U} 0 = N(x)$, $\forall x \in I$.

ii) $1 \xRightarrow{U} y = y$, $\forall y \in I$.

**Proposition 18** If $J$ is a Neutral fuzzy Implication and $N$ is a strong negation such that the natural negation $J(x, 0) = \neg x \geq N(x)$, $\forall x \in I$, then the Upper Contrapositivisation of $J$ with respect to $N$ is such that

i) $x \xRightarrow{U} 0 = \neg x$, $\forall x \in I$.

ii) $1 \xRightarrow{U} y = \neg(N(y))$, $\forall y \in I$.

The above propositions show that only when the natural negation is lesser than the strong negation $N$ considered, does $\xRightarrow{U}$ become Neutral and the natural negation of $\xRightarrow{U}$ is $N$. (Also see Cignoli et al. [5, Lemma 1] where they have proven that $x \xRightarrow{U} 0 = n(x)$ iff $n(x) \geq \neg x$, for a weak negation $N$). Note that the natural negation, $\neg x$, of an R-implication $J_T$ obtained from a strict t-norm is always less than any strong negation $N$. In the case of an R-implication $J_T$ obtained from nilpotent t-norms, if the strong negation $N$ considered is less than its natural negation $\neg x$ then from Proposition 18 we have that Upper Contrapositivisation is not much suitable, in the sense that the natural negation of $\xRightarrow{U}$ is not the strong negation $N$.

**Example 4** Let $T$ be the Lukasiewicz t-norm $T_L(x, y) = \max(x + y - 1, 0)$. Then $T_L$ is a nilpotent t-norm and the R-implication obtained from it is the Lukasiewicz Implication $J_T = \min(1 - x + y, 1)$. The natural negation of $J_T$, is $J_T(x, 0) = 1 - x$, which is a strong negation. From Corollaries 14 and 15 we know that $J_T$ has CPS only with respect to its natural
negation \( \neg x = J_{T_L}(x, 0) = 1 - x \). Thus if we consider the strong negation \( N_1(x) = (1 - \sqrt{x})^2 \), then \( \neg x \geq N_1 \) and from Proposition 18 we have that Upper Contrapositivisation is not much suitable.

We would like to point out that Fodor [1] cites the lack of Neutrality of Lower Contrapositivisation of \( J_T \) (see Definition 19 below), i.e., \( 1 \Rightarrow y \neq y \), for not considering the Lower Contrapositivisation technique. From Proposition 18 we see that the same is true of the Upper Contrapositivisation in case \( \neg x \geq N(x) \).

3.3 Alternate Approach : Lower Contrapositivisation : \( \Rightarrow \)

In this section we discuss the merits of the alternative technique of transforming fuzzy Implications without the Contrapositive Symmetry to ones that possess them with respect to the given strong negation \( N \), viz., Lower Contrapositivisation as proposed in [2].

**Definition 19** Let \( J \) be any fuzzy Implication and \( N \) a strong negation. The Lower Contrapositivisation of \( J \) with respect to \( N \), denoted \( \Rightarrow \), is defined as follows:

\[
x \Rightarrow y = \min\{J(x, y), J(N(y), N(x))\}
\]

(10)

The following is easy to see:

**Theorem 6** Let \( J \) be any fuzzy Implication, \( N \) a strong negation and \( \Rightarrow \) the Lower Contrapositivisation of \( J \) with respect to \( N \). Then

i) \( \Rightarrow \) is a fuzzy Implication as given in Definition 8.

ii) \( \Rightarrow \) has CPS(\( N \)).

3.4 Suitability of \( \Rightarrow \)

In this section we discuss the suitability of \( \Rightarrow \). We present below the counterparts of Propositions 17 and 18 for Lower Contrapositivisation.

**Proposition 20** If \( J \) is a Neutral fuzzy Implication and \( N \) a strong negation such that the natural negation \( J(x, 0) = \neg x \leq N(x) \), \( \forall x \in I \), then the Lower Contrapositivisation of \( J \) with respect to \( N \) is such that

i) \( x \Rightarrow 0 = \neg x \), \( \forall x \in I \).
\[ (\text{i}) \quad 1 \xrightarrow{L} y = \neg(N(y)), \quad \forall y \in I. \]

\text{PROOF.}

\[ (\text{i}) \quad x \xrightarrow{L} 0 = \min\{J(x,0), J(1, N(x))\} = \min(-x, N(x)) = -x, \quad \forall x \in I. \]

\[ (\text{ii}) \quad 1 \xrightarrow{L} y = \min\{J(1,y), J(N(y),0)\} = \min(y, \neg(N(y))) = \neg(N(y)). \]

\text{Proposition 21} If \( J \) is a Neutral fuzzy Implication and \( N \) a strong negation such that the natural negation \( J(x,0) = \neg x \geq N(x) \), \( \forall x \in I \), then the Lower Contrapositivisation of \( J \) with respect to \( N \) is such that

\[ (\text{i}) \quad x \xrightarrow{L} 0 = N(x), \quad \forall x \in I. \]

\[ (\text{ii}) \quad 1 \xrightarrow{L} y = y, \quad \forall y \in I. \]

\text{Example 5} Let us consider the fuzzy Implication \( J_B \) given in Baczynski [10, Example 17]:

\[ J_B(x,y) = \min \left\{ \max \left[ \frac{1}{2}, \min(1-x+y,1) \right], 2 - 2x + 2y \right\}. \] (11)

The following are easy to see:

- \( J_B(0,x) = 1. \)
- \( J_B(x,y) = 1 \iff y \geq x. \)
- The natural negation of \( J_B \) is given by

\[ J_B(x,0) = \neg x = \begin{cases} 1 - x, & \text{if } \frac{1}{2} \geq x \geq 0 \\ \frac{1}{2}, & \text{if } \frac{3}{4} \geq x \geq \frac{1}{2} \\ 2(1-x), & \text{if } 1 \geq x \geq \frac{3}{4} \end{cases} \] (12)

- \( J_B(x,0) \) though continuous is not strict and thus \( J_B \) does not have CPS(\( \neg x \)).
- \( J_B \) does not have the neutrality property (7). For example, \( J_B(1, \frac{1}{4}) = \frac{1}{2} \).

Now let us consider the strong negation \( N_1(x) = (1 - \sqrt{x})^2 \). Then it can be easily verified that \( J_B \) does not have contrapositive symmetry with respect to \( N_1 \), \( \neg x \geq N_1(x) \) and thus Upper Contrapositivisation technique is not suitable here. On the other hand if the strong negation \( N_2(x) = \sqrt{1 - x^2} \) is considered, then again \( J_B \) does not have contrapositive symmetry with respect to \( N_2 \), \( \neg x \leq N_2(x) \) and thus Lower Contrapositivisation technique is not suitable. Interestingly enough, if the strong negation \( N_3(x) = 1 - x \) is considered, then again \( \neg x \geq N_3(x) \) and \( J_B \) does have the contrapositive symmetry with respect to \( N_3 \).
From the above discussion it is clear that when the strong negation $N$ considered is such that $N(x) \geq \neg x = J(x,0)$ then Upper Contrapositivisation is suitable and when $N(x) <= \neg x$ then Lower Contrapositivisation is suitable.

4 Contrapositivisation and the Residuation Principle

4.1 Upper Contrapositivisation as a Residuation of a binary operator

Given a t-norm $T$ and $J_T$ the corresponding R-implication obtained from the t-norm $T$ such that they satisfy the residuation principle (1), Fodor in [1] has defined a binary operation $*^T$ by

$$x *^T y = \min\{T(x,y), N(J_T(y,N(x)))\}. \quad (13)$$

Fodor has shown that $\Rightarrow_U \Rightarrow \Rightarrow_T$ is the fuzzy implication generated by the residuation of $*^T$ (see Fodor [1, Theorem 2(d)]), i.e., $\Rightarrow_T$ and $*^T$ satisfy the residuation principle (1). The $*^T$ operator proposed in [1] is not a t-norm, in general. But $*^T$ in addition to having very attractive properties has also opened avenues for many subsequent research works. Also the Nilpotent Minimum proposed therein has led to some interesting research - Transformations of t-norms called N-annihilation in [3], Characterisation of $R_0$-Implications in [4], study of generalisation of Nilpotent Minimum t-norms in [5].

4.2 Lower Contrapositivisation as a Residuation of a binary operator

It is only natural to ask whether the Lower Contrapositivisation of an R-implication $J_T$ can also be obtained as a residuation of a binary operator. Taking cue from $*^T$ of Fodor we define a binary operator $*_T$ as follows:

$$x *_T y = \max\{T(x,y), N(J_T(y,N(x)))\}. \quad (14)$$

where $J_T$ is the corresponding R-implication obtained from the t-norm $T$. Now the following can be easily shown, along the lines of Theorem 2 in [1]:

**Theorem 7** Suppose that $T$ is a t-norm and $J_T$ the corresponding R-implication such that (1) is true, $N$ is a strong negation such that $N(x) <= J_T(x,0), \forall x \in I$ and operations $\Rightarrow_L$ and $*_T$ are as defined in (10) and (14). Then the following conditions are satisfied:
\[ 1 \ast_t y = y \ast_t 1 = y; \]
\[ x \ast_t 0 = 0 \ast_t x = 0; \]
\[ \ast_t \text{ is non-decreasing in both the variables;} \]
\[ x \ast_t y \leq z \iff x = L \Rightarrow z \geq y \]

Again, \( \ast_t \) is not a t-norm in general. For a (left) continuous t-norm \( T \) the R-implication \( J_T \) has the Ordering Property (6), viz., \( y \geq x \iff J_T(x, y) = 1 \). Thus \( y \leq N(x) \Rightarrow J_T(y, N(x)) = 1 \Rightarrow N[J_T(y, N(x))] = 0 \) and thus \( \ast_t \equiv T \).

Now, the following is easy to see:

**Theorem 8** For a t-norm \( T \) and a strong negation \( N \), if \( T(x, y) \geq N[J_T(y, N(x))] \) for \( y \geq N(x) \) then \( \ast_t \) is a t-norm. In fact \( \ast_t \equiv T \).

As stated earlier, in the case of continuous strict Archimedean t-norms \( T \), every strong negation \( N \) is greater than \( J_T(., 0) \) and thus \( \Rightarrow \) is not suitable. In the case of continuous nilpotent Archimedean t-norms \( T \), we know that \( J_T \) the R-implication obtained from \( T \) is also the S-implication obtained from the corresponding \( N \)-dual t-conorm \( S \) of \( T \), where \( N \equiv \neg = J_T(., 0) \). In which case, we have that \( N[J_T(y, N(x))] = N[J_S(y, N(x))] = N[S(N(y), N(x))] = T(x, y) \). Thus it would be interesting to find t-norms \( T \) such that \( T(x, y) \neq N[J_T(y, N(x))] \), which in the light of Theorem 8, means to leave the realms of continuous Archimedean t-norms and search for such a \( T \) in either the class of continuous non-Archimedean t-norms or left-continuous t-norms (since the Residuation Principle (1) needs to be satisfied).

## 5 Contrapositivisation as Strong Implications

Since Upper and Lower Contrapositivisation techniques can be applied to any general Fuzzy Implication it is only appropriate that we study \( \Rightarrow_U \) and \( \Rightarrow_L \) as strong implications for some fuzzy disjunctions \( \circ \) and \( \diamond \), respectively. An investigation into this forms the rest of this section.

### 5.1 \( \Rightarrow_L \) as an S-implication of a binary operator \( \diamond \)

Taking cue from Lemma 1 iv), we define a binary operator \( \diamond \) as follows:

\[ x \diamond y = \min\{J(N(x), y), J(N(y), x)\}. \quad (15) \]

Note that \( \diamond \) is defined for any Fuzzy Implication Operator, unlike \( \ast_T \) or \( \ast_t \), which were obtained from \( T \) and its residuation \( J_T \). The following are easy to see:
Theorem 9 Let \( J \) be a Neutral fuzzy Implication and \( N \) a strong negation such that the natural negation \( J(x, 0) = \neg x \geq N(x), \forall x \in I. \) Then the Lower Contrapositivisation of \( J \) with respect to \( N \) is such that

\[
\begin{align*}
\text{i)} & \quad x \xrightarrow{L} y = N(x) \circ y, \forall x \in I. \\
\text{ii)} & \quad x \circ 0 = x, \forall x \in I \\
\text{iii)} & \quad x \circ 1 = 1, \forall x \in I \\
\text{iv)} & \quad x \circ y = y \circ x, \forall x \in I
\end{align*}
\]

Thus \( \circ \) has all the properties of a t-conorm except for the associativity. In the following we give some sufficiency conditions for \( \circ \) to be a t-conorm.

Theorem 10 Let \( J \) be any fuzzy Implication such that

1. \( J \) has the Neutrality Property (7)
2. The natural negation of \( J \), \( J(x, 0) = \neg x \geq N(x), \forall x \in I. \)
3. \( \xrightarrow{L} \) has the Exchange Property (8).

Then \( \circ \) is a t-conorm.

PROOF. Since \( \neg x \geq N(x), \forall x \in I \) and \( J \) is Neutral from Proposition 21 we know that \( \xrightarrow{L} \) is also Neutral. That \( \circ \) is a t-conorm now follows straightaway from Lemma 1 iv).

5.2 \( \xrightarrow{U} \) as an S-implication of a binary operator \( \circ \)

Similarly, one can define the following binary operator \( \circ \) and show the following theorems:

\[
x \circ y = \max\{J(N(x), y), J(N(y), x)\}.
\]

(16)

Theorem 11 Let \( J \) be a Neutral fuzzy Implication and \( N \) a strong negation such that the natural negation \( J(x, 0) = \neg x \leq N(x), \forall x \in I. \) Then the Upper Contrapositivisation of \( J \) with respect to \( N \) is such that

\[
\begin{align*}
\text{i)} & \quad x \xrightarrow{U} y = N(x) \circ y, \forall x \in I. \\
\text{ii)} & \quad x \circ 0 = x, \forall x \in I \\
\text{iii)} & \quad x \circ 1 = 1, \forall x \in I \\
\text{iv)} & \quad x \circ y = y \circ x, \forall x \in I
\end{align*}
\]

Theorem 12 Let \( J \) be any fuzzy Implication such that

1. \( J \) has the Neutrality Property (7)
• The natural negation of $J$, $\neg x = J(x, 0) \leq N(x)$, $\forall x \in I$.
• $U \Rightarrow$ has the Exchange Property (8).

Then $\circ$ is a t-conorm.

6 Conclusions

In this work we have investigated the suitability of two techniques proposed in Bandler and Kohout [2] towards imparting contrapositive symmetry to a given fuzzy implication, viz., Upper Contrapositivisation which was studied in detail for R-implications $J_T$ by Fodor [1] and the alternative Lower Contrapositivisation technique. We have shown that while Upper Contrapositivisation technique is useful when the natural negation of an Implication Operator $J$, given by $J(x, 0)$, is less than the strong negation $N$ considered, the Lower Contrapositivisation technique is useful in the other case. Also we have shown that the Lower Contrapositivisation can be seen as the residuation of a suitable binary operator. We have also proposed binary operators (fuzzy disjunctions) such that the Lower and Upper Contrapositivisation can be seen as strong implications obtained from them. Also some sufficient conditions for these fuzzy disjunctions to become t-conorms are given.

References


