(U, N)-Implications and their Characterizations

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Abstract

In this work we characterize (U, N)-implications obtained from disjunctive uninorms U and continuous negations N.

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1 Introduction

(U, N)-implications are some generalizations of (S, N)-implications, where the t-conorm S is replaced by a uninorm U. Recently, some characterizations of (S, N)-implications were given by the authors in [2]. In this work, along similar lines, we investigate and characterize (U, N)-implications obtained from continuous negations N.

After introducing the necessary preliminaries on the basic fuzzy logic operators, we list out some of the most desirable - but relevant to this work - properties of fuzzy implications and investigate their interdependencies. Following this we discuss the class of (U, N)-operators and the properties they satisfy. Finally, based on the above analysis, we derive a characterization for (U, N)-implications generated from continuous negations.

2 Basic Fuzzy Logic Operators

To make this work self-contained, we briefly mention some of the concepts and results employed in the rest of the work.

Definition 1 (see [4, 7])**.** *A decreasing function $N: [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is called*

(i) strict if it is both strictly decreasing and continuous;

(ii) strong if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

It is well-known that if $[a, b]$ and $[c, d]$ are two closed subintervals of $[-\infty, +\infty]$ and $f: [a, b] \to [c, d]$ is a monotone function, then the set of discontinuous points of f is a countable subset of $[a, b]$ (see [9]). In this case we will use the pseudo-inverse $f^{(-1)}: [c, d] \to [a, b]$ of a decreasing and non-constant function f defined by (see [7, Sect. 3.1])

$$f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) > y\}, \qquad y \in [c, d].$$

Lemma 1 ([2], Proposition 28)**.** *If N is a continuous fuzzy negation, then the function $\mathfrak{N}: [0,1] \to [0,1]$ defined by*

$$\mathfrak{N}(x) = \begin{cases} N^{(-1)}(x), & \text{if } x \in (0,1] \\ 1, & \text{if } x = 0 \end{cases} \tag{1}$$

is a strictly decreasing fuzzy negation. Moreover

$$\mathfrak{N}^{(-1)} = N, \tag{2}$$

$$N \circ \mathfrak{N} = id_{[0,1]}, \tag{3}$$

$$\mathfrak{N} \circ N \mid_{Ran(\mathfrak{N})} = id_{Ran(\mathfrak{N})}. \tag{4}$$

Definition 2 (see [5])**.** *An associative, commutative, increasing operation $U: [0,1]^2 \to [0,1]$ is called a uninorm, if there exists an $e \in [0,1]$ (called the the neutral element) such that*

$$U(e, x) = x, \qquad x \in [0,1].$$

Remark 1. *(i) If $e = 0$, then U is a t-conorm and if $e = 1$, then U is a t-norm.*

(ii) It can be easily showed, that the element e corresponding to a uninorm U is unique.

(iii) For any uninorm U we have $U(0, 1) \in \{0, 1\}$.

(iv) A uninorm U such that $U(0, 1) = 0$ is called a conjunctive uninorm and if $U(0, 1) = 1$ it is called a disjunctive uninorm.

Examples of fuzzy negations, uninorms as well as the different classes of uninorms (the classes U_{min}, U_{max}, representable uninorms, idempotent uninorms) can be found in recent literature (see [4, Chap. 1], [7, Sect. 10.2], [3, 5]).

3 Fuzzy Implication Operators

3.1 Definition and Properties

In this work the following equivalent definition proposed by Fodor and Roubens [4] is used.

Definition 3. *A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication operator, or a fuzzy implication, if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, the following conditions:*

$$\text{if } x_1 \le x_2, \text{ then } I(x_1, y) \ge I(x_2, y), \tag{I1}$$
$$\text{if } y_1 \le y_2, \text{ then } I(x, y_1) \le I(x, y_2), \tag{I2}$$
$$I(0,0) = 1, \tag{I3}$$
$$I(1,1) = 1, \tag{I4}$$
$$I(1,0) = 0. \tag{I5}$$

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

Remark 2. *Directly from Definition 3 we see that each fuzzy implication I satisfies the following left and right boundary condition, respectively:*

$$I(0,y) = 1, \qquad y \in [0,1], \tag{LB}$$
$$I(x,1) = 1, \qquad x \in [0,1]. \tag{RB}$$

Therefore, I satisfies also the normality condition:

$$I(0,1) = 1. \tag{NC}$$

Definition 4. *Let I be a fuzzy implication. If for some $\alpha \in [0,1)$ we have $I(1,\alpha) = 0$, then the function N_I^α given by*

$$N_I^\alpha(x) = I(x,\alpha), \qquad x \in [0,1]$$

is called the natural negation of I with respect to α.

It should be noted that for any $I \in \mathcal{FI}$ we have (I5), so for $\alpha = 0$ we have the natural negation $N_I = N_I^0$ of I. Also α should be less than 1, since $I(1,1) = 1$.

In the following we list out some of the desirable properties of fuzzy implications:

Definition 5. *Let I be a fuzzy implication and N a fuzzy negation.*

(i) I is said to have the exchange principle, if

$$I(x, I(y,z)) = I(y, I(x,z)), \tag{EP}$$

for all $x, y, z \in [0,1]$,

(ii) I is said to satisfy the law of left contraposition with respect to N if, for any $x, y \in [0,1]$,

$$I(N(x), y) = I(N(y), x). \tag{L-CP}$$

(iii) I is said to satisfy the law of right contraposition with respect to N, if, for any $x, y \in [0,1]$,

$$I(x, N(y)) = I(y, N(x)). \tag{R-CP}$$

(iv) I is said to satisfy the law of contraposition with respect to N, if, for any $x, y \in [0,1]$,

$$I(x,y) = I(N(y), N(x)). \tag{CP}$$

Lemma 2 ([2], Lemma 17)**.** *Let $I\colon [0,1]^2 \to [0,1]$ be any function and N a continuous fuzzy negation.*

(i) If I satisfies (I1) and R-CP(N), then I satisfies (I2).

(ii) If I satisfies (I2) and R-CP(N), then I satisfies (I1).

Lemma 3. *Let $I\colon [0,1]^2 \to [0,1]$ and N_I^α be a fuzzy negation for an arbitrary but fixed $\alpha \in [0,1)$.*

(i) If I satisfies (I2), then I satisfies (I5).

(ii) Let I have (I2) and (EP). Then I satisfies (I3) if and only if I satisfies (I4).

(iii) If I satisfies (EP), then I satisfies R-CP(N_I^α),

Proof. (i) Since N_I^α is a fuzzy negation and I satisfies (I2) we get $I(1,0) \le I(1,\alpha) = N_I^\alpha(1) = 0$.

(ii) Let I have (I2) and (EP). If I satisfies (I4), then since $N_I^\alpha(0) = 1$ we have $1 = I(1,1) = I(1, N_I^\alpha(0)) = I(1, I(0,\alpha)) = I(0, I(1,\alpha)) = I(0, N_I^\alpha(1)) = I(0,0) = 1$, i.e., I satisfies (I3). The reverse implication can be shown similarly.

(iii) Since I satisfies (EP), we have $I(x, N_I^\alpha(y)) = I(x, I(y,\alpha)) = I(y, I(x,\alpha)) = I(y, N_I^\alpha(x))$, i.e., I has R-CP(N_I^α).

□

Lemma 4. *Let I be any fuzzy implication and N_I^α be a continuous fuzzy negation for an arbitrary but fixed $\alpha \in [0,1)$. If N is a strictly decreasing fuzzy negation such that $N_I^\alpha \circ N = \mathrm{id}_{[0,1]}$ and I satisfies (EP), then I satisfies L-CP(N).*

Proof. By our assumptions we get

$$\begin{aligned} I(N(x), y) &= I(N(x), N_I^\alpha \circ N(y)) \\ &= I(N(x), I(N(y),\alpha)) \\ &= I(N(y), I(N(x),\alpha)) \\ &= I(N(y), N_I^\alpha \circ N(x)) \\ &= I(N(y), x), \end{aligned}$$

for any $x, y \in [0,1]$ □

Remark 3. *Under the assumptions of Lemma 4, we have:*

(i) If N_I^α is a strict negation, then I satisfies L-CP($(N_I^\alpha)^{-1}$).

(ii) If N_I^α is a strong negation, then I satisfies CP(N_I^α).

3.2 (S,N)-Implications and their characterization

In this section, we give a brief introduction to one of the families of fuzzy implications that is very well studied in the fuzzy literature.

Definition 6 (cf. [1, 4, 10])**.** *A function $I\colon [0,1]^2 \to [0,1]$ is called an (S,N)-implication, if there exist a t-conorm S and a fuzzy negation N such that*

$$I(x,y) = S(N(x),y), \qquad x,y \in [0,1]. \tag{5}$$

If N is a strong negation, then I is called a strong implication (shortly S-implication).

The following characterizations of (S,N)-implications are from [2], which is an extension of a result in [10]:

Theorem 1 ([2])**.** *For a function $I\colon [0,1]^2 \to [0,1]$ the following are equivalent:*

(i) I is an (S,N)-implication generated from some t-conorm S and N is a continuous (strict, strong) fuzzy negation.

(ii) I satisfies (I1), (EP), *the function N_I is a continuous (strict, strong) fuzzy negation.*

Moreover, the representation of (S,N)-implication is unique in this case.

In Theorem 1, the property (I1) can be substituted by (I2). Moreover, axioms in above theorem are independent from each other.

4 (U,N)-Operators

A natural generalization of (S,N)-implications in the uninorm framework is to consider a uninorm in the place of a t-conorm.

4.1 Definition and Some Properties

Definition 7. *A function $I\colon [0,1]^2 \to [0,1]$ is called a (U,N)-operator, if there exist a uninorm U and a fuzzy negation N such that*

$$I_{U,N}(x,y) = U(N(x),y), \qquad x,y \in [0,1]. \tag{6}$$

If a (U,N)-operator is generated from U and N, then we will often denote this by $I_{U,N}$.

Proposition 1. *If $I_{U,N}$ is a (U,N)-operator based on some uninorm U and some fuzzy negation N, then*

(i) $I_{U,N}$ satisfies (I1), (I2), (I5), (NC) *and* (EP),

(ii) $N^e_{I_{U,N}} = N$ and $I_{U,N}$ satisfies R-CP(N),

(iii) if N is strict, then $I_{U,N}$ satisfies L-CP(N^{-1}),

(iv) if N is strong, then $I_{U,N}$ satisfies CP(N).

Proof. (i) By the monotonicity of U and N we get that $I_{U,N}$ satisfies (I1) and (I2). Moreover, it can be easily verified that $I_{U,N}$ satisfies (I5) and (NC). Finally, from the associativity and the commutativity of U we have also (EP).

(ii) For any $x \in [0,1]$ we have

$$N^e_{I_{U,N}}(x) = I_{U,N}(x,e) = U(N(x),e) = N(x).$$

Next, since $I_{U,N}$ satisfies (EP), from Lemma 3 (iii) with $\alpha = e$ we have that $I_{U,N}$ satisfies R-CP(N).

(iii) If N is a strict negation, then because of Remark 3 (i) we can deduce, that $I_{U,N}$ satisfies L-CP(N^{-1}).

(iv) If N is a strong negation, then because of Remark 3 (ii) we can deduce, that $I_{U,N}$ satisfies CP(N).

□

If $e = 0$, then U is a t-conorm and $I_{U,N}$, as an (S,N)-implications, is always a fuzzy implication. If $e = 1$, then U is a t-norm and $I_{U,N}$ is not a fuzzy implication, since (I3) is violated. If $e \in (0,1)$, then not for every uninorm U the (U,N)-operator is a fuzzy implication. Next results characterize these (U,N)-operators, which satisfy (I3) and (I4).

Theorem 2. *Let U be a uninorm with the neutral element $e \in (0,1)$. Then the following statements are equivalent:*

(i) The function $I_{U,N}$ as defined in (6) is a fuzzy implication.

(ii) U is a disjunctive uninorm, i.e, $U(0,1) = 1$.

Proof. Let U be a uninorm with the neutral element $e \in (0,1)$.
(i) $\Longrightarrow$ *(ii)* If $I_{U,N}$ as defined in (6) is a fuzzy implication, then from (I3) we have $U(0,1) = U(1,0) = I_{U,N}(0,0) = 1$.
(ii) $\Longrightarrow$ *(i)* Assume that $U(0,1) = 1$. From Proposition 1 it is enough to show only (I3) and (I4):

$$I_{U,N}(0,0) = U(N(0),0) = U(1,0) = U(0,1) = 1,$$
$$I_{U,N}(1,1) = U(N(1),1) = U(0,1) = 1.$$

□

Following the terminology used by Mas *et al.* [8] for QL-implications, only if the (U,N)-operator $I_{U,N}$ is a fuzzy implication we use the term (U,N)-implication.

Lemma 5. *Let $I_{U,N}$ be a (U,N)-implication obtained from a uninorm U with $e \in (0,1)$ as its neutral element and continuous negation N. Let $\alpha \in (0,1)$ be an arbitrary but fixed number. Then the following statements are equivalent:*

(i) $N^{\alpha}_{I_{U,N}} = N$;

(ii) $\alpha = e$.

Proof. Let $e \in (0,1)$ be the neutral element of U and $\alpha \in (0,1)$ be an arbitrary but fixed number.
(i) $\Longrightarrow$ *(ii)* If $\alpha = e$, then $N^{\alpha}_{I_{U,N}}(x) = I_{U,N}(x,\alpha) = I_{U,N}(x,e) = U(N(x),e) = N(x)$ for all $x \in [0,1]$, i.e., $N^{\alpha}_{I_{U,N}} = N$.
(ii) $\Longrightarrow$ *(i)* On the other hand, if $N^{\alpha}_{I_{U,N}} = N$, then since N is continuous there exists an e' such that $e = N(e')$ and $N^{\alpha}_{I_{U,N}}(e') = I_{U,N}(e',\alpha) = U(N(e'),\alpha) = N(e') = e$. But $U(N(e'),\alpha) = U(e,\alpha) = \alpha$, because e is the neutral element of U. Hence $\alpha = e$. □

4.2 Characterizations of (U,N)-Implications

We start our presentation with following result.

Proposition 2. *Let I be a fuzzy implication and N any fuzzy negation. Let us define a binary operation U_I on $[0,1]$ as follows:*

$$U_{I,N}(x,y) = I(N(x),y), \qquad x,y \in [0,1]. \tag{7}$$

Then for all $x,y \in [0,1]$, we have

(i) $U_{I,N}(x,1) = U_I(1,x) = 1$, *in particular* $U_{I,N}(0,1) = 1$,

(ii) $U_{I,N}$ *is increasing in both the variables,*

(iii) $U_{I,N}$ *is commutative if and only if I has L-CP(N).*

In addition, if I has L-CP(N), then

(iv) $U_{I,N}$ *is associative if and only if I satisfies the exchange property* (EP).

(v) an arbitrary $\alpha \in (0,1)$ *is the neutral element of* $U_{I,N}$ *if and only if* $N^{\alpha}_I \circ N = id_{[0,1]}$.

Proof. (i) $U_{I,N}(x,1) = I(N(x),1) = 1$, by the boundary condition (RB) on I. Also, $U_I(1,x) = I(N(1),x) = I(0,x) = 1$ again by (LB) of I.

(ii) That $U_{I,N}$ is increasing in both the variables is a direct consequence of the monotonicity of I and N.

(iii) If $U_{I,N}$ is commutative, then $I(N(x),y) = U_{I,N}(x,y) = U_{I,N}(y,x) = I(N(y),x)$, i.e., I satisfies L-CP(N). The reverse implication can be obtained by retracing the above steps.

(iv) If I satisfies (EP), then

$$\begin{aligned} U_{I,N}(x,U_I(y,z)) &= I(N(x),I(N(y),z)) \\ &= I(N(x),I(N(z),y)) \\ &= I(N(z),I(N(x),y)) \\ &= I(N[I(N(x),y)],z) \\ &= I(N[U_I(x,y)],z) \\ &= U_I(U_I(x,y),z). \end{aligned}$$

On the other hand, if $U_{I,N}$ is associative, then

$$\begin{aligned} I(x,I(y,z)) &= U_{I,N}(N(x),U_{I,N}(N(y),z)) \\ &= U_{I,N}(U_{I,N}(N(x),N(y)),z)) \\ &= U_{I,N}(U_{I,N}(N(y),N(x)),z)) \\ &= U_{I,N}(N(y),U_{I,N}(N(x),z)) \\ &= I(y,I(x,z)). \end{aligned}$$

(v) Let $\alpha \in (0,1)$ be arbitrary fixed. If α is the neutral element of $U_{I,N}$, then, for any $x \in [0,1]$, we have $x = U_{I,N}(x,\alpha) = I(N(x),\alpha) = N^{\alpha}_I(N(x))$. Conversely, if $N^{\alpha}_I \circ N = id_{[0,1]}$, then, for any $x \in [0,1]$ we get $U_{I,N}(\alpha,x) = U_{I,N}(x,\alpha) = I(N(x),\alpha) = N^{\alpha}_I(N(x)) = x$ and α is the neutral element of $U_{I,N}$.

□

If N^{α}_I is a continuous fuzzy negation for an arbitrary but fixed $\alpha \in (0,1)$, then by Lemma 1 and previous results we can consider the modified pseudo-inverse $\mathfrak{N}^{\alpha}_I$ given by

$$\mathfrak{N}^{\alpha}_I(x) = \begin{cases} \left(N^{\alpha}_I\right)^{(-1)}(x), & \text{if } x \in (0,1] \\ 1, & \text{if } x = 0 \end{cases} \tag{8}$$

as the potential candidate for the fuzzy negation N in (7). Hence from Lemma 4 with $N = \mathfrak{N}^{\alpha}_I$ we obtain the following result.

Corollary 1 (cf. [2], Corollary 29)**.** *If a fuzzy implication I satisfies* (EP) *and N^{α}_I, the natural negation of I with respect to an arbitrary but fixed $\alpha \in (0,1)$, is a continuous fuzzy negation, then I satisfies* (L-CP) *with $\mathfrak{N}^{\alpha}_I$ from* (8).

Hence, if a fuzzy implication I satisfies (EP) and N^{α}_I is a *continuous fuzzy negation* for some $\alpha \in (0,1)$, then we conclude, that the formula (7) can be considered for the modified pseudo-inverse of the natural negation of I.

Corollary 2. *If $I \in \mathcal{FI}$ satisfies* (EP) *and N_I^α is a continuous fuzzy negation with respect to an arbitrary but fixed $\alpha \in (0,1)$, then the function U_I defined by*

$$U_I(x,y) = I(\mathfrak{N}_I^\alpha(x), y), \qquad x,y \in [0,1] \tag{9}$$

is a disjunctive uninorm with neutral element α, where $\mathfrak{N}_I$ is as defined in (8).

Theorem 3. *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an (U,N)-operator generated from some disjunctive uninorm U with neutral element $e \in (0,1)$ and some continuous fuzzy negation N.

(ii) I is an (U,N)-implication generated from some uninorm U with neutral element $e \in (0,1)$ and some continuous fuzzy negation N.

(iii) I satisfies (I1), (I3), (EP) *and the function N_I^e is a continuous negation for some $e \in (0,1)$.*

Moreover, the representation (6) *of (U,N)-implication is unique in this case.*

Proof. That (i) is equivalent to (ii) follows immediately from Theorem 2.

(ii) $\Longrightarrow$ *(iii)* Assume, that I is an (U,N)-implication based on a uninorm U with neutral element $e \in (0,1)$ and a continuous negation N. Since every (U,N)-implication is a fuzzy implication, I satisfies (I1) and (I3). Moreover, by Proposition 1 it satisfies (EP) and $N_I^e = N$. In particular N_I^e is continuous.

(iii) $\Longrightarrow$ *(ii)* Firstly see, that from Lemma 3 (iii) it follows that I satisfies (R-CP) with respect to the continuous N_I^e. Next, Lemma 2 (i) implies that I satisfies (I2). Once again from Lemma 3 (i) and (ii) we have that I satisfies (I3), (I4) and (I5), and hence $I \in \mathcal{FI}$. Further, by virtue of Lemmas 1 and 4 the implication I satisfies L-CP($\mathfrak{N}_I^e$). Because of Corollary 2 the function U_I defined by (9) is a disjunctive uninorm with the neutral element e.
We will show that $I_{U_I,N_I^e} = I$. Fix arbitrarily $x,y \in [0,1]$. If $x \in \operatorname{Ran}(\mathfrak{N}_I^e)$, then by (4) we have

$$\begin{aligned} I_{U_I,N_I^e}(x,y) &= U_I(N_I^e(x),y) \\ &= I(\mathfrak{N}_I^e \circ N_I^e(x), y) = I(x,y). \end{aligned}$$

If $x \notin \operatorname{Ran}(\mathfrak{N}_I^e)$, then from the continuity of N_I^e there exists $x_0 \in \operatorname{Ran}(\mathfrak{N}_I^e)$ such that $N_I^e(x) = N_I^e(x_0)$. Firstly see, that $I(x,y) = I(x_0,y)$ for all $y \in [0,1]$. Indeed, let us fix arbitrarily $y \in [0,1]$. From the continuity of N_I^e there exists $y' \in [0,1]$ such that $N_I^e(y') = y$, so

$$\begin{aligned} I(x,y) &= I(x, N_I^e(y')) = I(y', N_I^e(x)) \\ &= I(y', N_I^e(x_0)) = I(x_0, N_I^e(y')) = I(x_0,y). \end{aligned}$$

From the above fact we get

$$\begin{aligned} I_{U_I,N_I^e}(x,y) &= U_I(N_I^e(x),y) \\ &= U_I(N_I^e(x_0),y) = I(x_0,y) = I(x,y), \end{aligned}$$

so I is an (U,N)-implication.
Finally, assume that there exist two continuous fuzzy negations N_1, N_2 and two uninorms U_1, U_2 with neutral elements $e, e' \in (0,1)$, respectively, such that $I(x,y) = U_1(N_1(x),y) = U_2(N_2(x),y)$ for all $x,y \in [0,1]$. Fix arbitrarily $x_0, y_0 \in [0,1]$. Firstly observe that from Proposition 1 we get $N_1 = N_2 = N_I^e = N_I^{e'}$. By virtue of Lemma 5 we get, that $e' = e$. Now, since N_I^e is a continuous negation there exists $x_1 \in [0,1]$ such that $N_I^e(x_1) = x_0$. Thus $U_1(x_0,y_0) = U_1(N_I^e(x_1),y_0) = U_2(N_I^e(x_1),y_0) = U_2(x_0,y_0)$, i.e., $U_1 = U_2$. Hence N and U are uniquely determined. In fact $U = U_I$ defined by (9). □

In above theorem the property (I1) can be substituted by (I2) and the property (I3) can be substituted by I4. Moreover, the above axioms are independent from each other.

Now, the following result easily follows:

Theorem 4. *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an (U,N)-implication generated from some disjunctive uninorm U with neutral element $e \in (0,1)$ and some strict (strong) fuzzy negation N.

(ii) I satisfies (I1), (I3), (EP) *and the function N_I^e is a strict (strong) negation.*

Once again, the representations of the (U,N)-implications described above are unique and the presented axioms are independent from each other. It is interesting, that using similar methods as in this section we are able to obtain the following characterization of (U,N)-operators.

Theorem 5. *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an (U,N)-operator generated from some uninorm U with neutral element $e \in (0,1)$ and some continuous fuzzy negation N.

(ii) I satisfies (I1), (EP) *and the function N_I^e is a continuous negation for some $e \in (0,1)$.*

5 Some Concluding Remarks

In this work, we characterize (U,N)-implications obtained from disjunctive uninorms U and continuous negations N. Toward this end, we have investigated some desirable algebraic properties of fuzzy implication operators and obtained some characterization results. It should be noted, that (U, N)-implications are closely related with e-implications investigated in [6], whose representation is still unknown.

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