MATH 215 HW # 5

PLEASE TURN IN THE SOLUTIONS TO THE FOLLOWING PROBLEMS:

SECTION 3.4 # 25, 28

SECTION 3.5 # 1, 3, 6, 8, 10, 13, 14, 16, 17, 30

MATH 215 HW #5 SOLUTIONS

SECTION 3.4 # 25 $t^2 y'' + 3t y' + y = 0$ $y_1 = t^{-1}$ IS A SOLUTION. FIND y_2.

WE LET $y = t^{-1} v$. SO $y' = -t^{-2} v + t^{-1} v'$, $y'' = 2t^{-3} v - 2t^{-2} v' + t^{-1} v''$.

SUBSTITUTE TO GET $t^2 \left(t^{-1} v'' - 2t^{-2} v' + 2t^{-3} v \right) + 3t\left(-t^{-2} v + t^{-1} v'\right) + t^{-1} v = 0.$

THIS SIMPLIFIES TO $t v'' + v'(-2+3) + v\left(2/t - 3/t + 1/t\right) = 0 \Rightarrow v'' + \frac{v'}{t} = 0.$

LET $w = v' \longrightarrow w' + \frac{1}{t} w = 0 \longrightarrow t w' + w = (tw)' = 0$. SO $tw = c$.

THUS $w = c/t = v' \longrightarrow v = c \ln t \longrightarrow y = t^{-1} v$ IS A SOLUTION.

SO let $c = 1$, $y_2 = \ln t / t$ IS THE OTHER SOLUTION FOR $y_1 = 1/t$.

SECTION 3.4 #28 $(t-1) y'' - t y' + y = 0$ $y_1 = e^t$ IS A SOLUTION. FIND y_2.

WE PUT $y = e^t v$ SO $y' = e^t v + e^t v'$, $y'' = v'' e^t + 2v' e^t + e^t v$.

THUS UPON SUBSTITUTING INTO $(t-1) y'' - t y' + y = 0$ WE GET e^t CANCELS AND

$$(t-1)\left[v'' + 2v' + v \right] - t\left(v + v'\right) + v = 0.$$

SO $$(t-1) v'' + v'\left[2(t-1) - t\right] + v\left[(t-1) - t + 1\right] = 0.$$

THUS $$v'' + v'\left[2 - \frac{t}{t-1}\right] = 0 \longrightarrow v'' + v'\left(\frac{t-2}{t-1}\right) = 0.$$

THUS $$v'' + v'\left(\frac{(t-1) - 1}{(t-1)}\right) = 0 \longrightarrow v'' + \left(1 - \frac{1}{t-1}\right) v' = 0.$$

LET $w = v'$ SO THAT $w' + \left(1 - \frac{1}{t-1}\right) w = 0$ INTEGRATING FACTOR $\phi = \exp\left(\int^t \left(1 - \frac{1}{t-1}\right) dt\right)$

THUS $\phi = e^{t - \ln(t-1)} = \frac{1}{(t-1)} e^t \longrightarrow \left(\frac{e^t}{(t-1)} w\right)' = 0$. THUS $w = c(t-1)e^{-t}$.

WE SET $c = 1$ WITHOUT LOSS OF GENERALITY. THEN $v' = w = (t-1)e^{-t}$.

THEN $v = \int^t (t-1) e^{-t} dt = -te^{-t}$ BY integration by parts

HENCE $y_2 = e^t v \longrightarrow y_2 = (-te^{-t}) e^t = -t.$

SO $y_2 = ct$ IS A SOLUTION FOR ANY c.

SECTION 3.5 #1

$$y'' - 2y' - 3y = 3e^{2t}$$

HOMOG. PROB. LET $y = e^{\lambda t} \longrightarrow \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0 \longrightarrow \lambda = -1, 3.$

THUS $y_1 = e^{-t}, \quad y_2 = e^{3t}.$

PARTICULAR SOLUTION SINCE $\alpha = 2 \neq -1, 3$ THEN $y_p = Ae^{2t}.$

SO $y_p' = 2Ae^{2t} \quad y_p'' = 4Ae^{2t} \longrightarrow (4A - 2(2A) - 3A)e^{2t} = 3e^{2t}.$

SO $-3A = 3 \longrightarrow A = -1 \longrightarrow y_p = -e^{2t}.$

THUS $y = c_1 e^{-t} + c_2 e^{3t} - e^{2t}$ " GEN. SOLUTION.

SECTION 3.5 # 3

$$y'' - 2y' - 3y = -3te^{-t}.$$

HOMOG. PROB. LET $y = e^{rt} \longrightarrow r^2 - 2r - 3 = (r-3)(r+1) = 0 \longrightarrow r = 3, -1.$

THUS $y_1 = e^{-t}, \quad y_2 = e^{3t}.$

PARTICULAR SOLUTION SINCE $\alpha = -1$ IS ONE OF TWO ROOTS $r = -1, 3$

THEN $y_p = t(At + B)e^{-t} = (At^2 + Bt)e^{-t}$

SO $y_p' = (2At + B)e^{-t} - (At^2 + Bt)e^{-t}$

$y_p'' = 2Ae^{-t} - 2(2At + B)e^{-t} + (At^2 + Bt)e^{-t}$

SO $y'' - 2y' - 3y = -3te^{-t} \longrightarrow 2A - 4At - 2B + At^2 + Bt - 4At - 2B$

$+ 2At^2 + 2Bt - 3At^2 - 3Bt = -3t.$

SO $t(-4A - 4A + B + 2B - 3B) + (-2B - 2B + 2A) = -3t$

$\longrightarrow -8A = -3 \rightarrow A = 3/8 \quad 4B = 2A \rightarrow B = 3/16$

SO $y = c_1 e^{-t} + c_2 e^{3t} + (3/8\, t^2 + 3t/16)e^{-t}.$

SECTION 3.5 # 6

$$y'' + 2y' + y = 2e^{-t}.$$

HOMOG. PROBLEM $\quad y = e^{\lambda t} \rightarrow \lambda^2 + 2\lambda + 1 = 0 \rightarrow (\lambda+1)^2 = 0$ so $\lambda = -1$ Repeated.

THUS $\quad y_1 = e^{-t}, \quad y_2 = te^{-t}.$

PARTICULAR SOLUTION $\quad$ SINCE $\alpha = -1$ IS A REPEATED ROOT OF CHARACTERISTIC POLYNOMIAL,

THEN $\quad y_p = At^2e^{-t}$

SO $\quad y_p' = 2Ate^{-t} - At^2e^{-t}$

$\quad y_p'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}.$

THUS $\quad y'' + 2y' + y = 2e^{-t}$ BECOMES $\quad 2A - 4At + At^2 + 2(2At - At^2) + At^2 = 2$

THUS $\quad 2A = 2 \rightarrow A = 1$

SO $\quad y_p = t^2e^{-t}.$

gen. SOL'N is $\quad y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}.$

SECTION 3.5 # 8 $\qquad y'' + y = 3\sin(2t) + t\cos(2t)$

HOMOG. PROB. $\quad$ let $y = e^{\lambda t} \rightarrow \lambda^2 + 1 = 0$ so $\lambda = \pm i.$

THUS $\quad y_1 = \cos t, \quad y_2 = \sin t$ ARE SOLUTIONS.

PARTICULAR SOLUTIONS $\quad$ LET $\quad y_{p_1}'' + y_{p_1} = 3\sin(2t) \qquad y_{p_1} = \mathrm{IM}(\hat{y}_{p_1})$

WHERE $\quad \hat{y}_{p_1}'' + \hat{y}_{p_1} = 3e^{2it}.\quad$ LET $\quad \hat{y}_{p_1} = Ae^{2it} \qquad \hat{y}_{p_1}' = A2ie^{2it}$

THEN $\quad \hat{y}_{p_1}'' = -4Ae^{2it}$ SO THAT $\quad -4A + A = 3 \rightarrow A = -1.$

THUS $\quad y_{p_1} = \mathrm{IM}(-e^{2it}) = -\sin(2t)$

NOW LET $\hat{y}_{p_2}'' + \hat{y}_{p_2} = t e^{2it}$ $\qquad y_{p_2} = \text{RE}(\hat{y}_{p_2})$.

THEN $\hat{y}_{p_2} = (At+B)e^{2it}$ IS FORM OF PART. SOL'N.

NOW $\hat{y}_{p_2}' = (Ae^{2it} + (At+B)2ie^{2it})$

$$\hat{y}_{p_2}'' = 2iAe^{2it} + (2Ai\,e^{2it} + 2i(2i)(At+B)e^{2it}).$$

SO SUBSTITUTING:

$$4Ai - 4(At+B) + At + B = t$$

$$\rightarrow -3At + 4Ai - 3B = t$$

$$\rightarrow -3A = 1, \quad 4Ai - 3B = 0$$

SO $A = -1/3$ AND $B = \frac{4Ai}{3} = -\frac{4i}{9}$.

THUS $\hat{y}_{p_2} = \left(-\frac{t}{3} - \frac{4i}{9}\right)e^{2it}$

$$y_{p_2} = \text{RE}\left(-\left(\frac{t}{3} + \frac{4i}{9}\right)(\cos 2t + i\sin 2t)\right)$$

$$\rightarrow y_{p_2} = -\frac{t}{3}\cos(2t) + \frac{4}{9}\sin(2t)$$

THUS GEN. SOL'N IS $y = C_1 y_1 + C_1 y_2 + y_{p_1} + y_{p_2}$

$$y = C_1\cos t + C_2\sin t - \sin(2t) - \frac{t}{3}\cos(2t) + \frac{4}{9}\sin(2t)$$

SECTION 3.5 # 10

$$u'' + w_0^2 u = \cos(w_0 t) = RE(e^{iw_0 t})$$

HOMOG. PROBLEM LET $u = e^{\lambda t} \rightarrow \lambda^2 + w_0^2 = 0, \quad \lambda_\pm = \pm i w_0.$

THUS $y_1 = \cos(w_0 t) \qquad y_2 = \sin(w_0 t).$

PARTICULAR SOLUTION SINCE $\alpha = i w_0 = \lambda_+$ (IS ONE OF TWO ROOTS)

THEN $\tilde{u}'' + w_0^2 \tilde{u} = e^{i w_0 t} \longrightarrow \hat{u}_p = A t e^{i w_0 t}$

AND $u_p = RE(\tilde{u}_p)$. THUS,
$$\tilde{u}_p' = A e^{i w_0 t} + A i w_0 t e^{i w_0 t}$$
$$\tilde{u}_p'' = -A w_0^2 t e^{i w_0 t} + 2 A i w_0 e^{i w_0 t}$$

SO $\tilde{u}_p'' + w_0^2 \tilde{u}_p = e^{i w_0 t} \longrightarrow -A w_0^2 t + 2 A i w_0 + A w_0^2 t = 1.$

THUS $2 A i w_0 = 1 \longrightarrow A = \dfrac{1}{2 i w_0} = -\dfrac{i}{2 w_0}$

SO $\hat{u}_p = \dfrac{-i}{2w_0} t e^{i w_0 t} \qquad u_p = RE\left(\dfrac{-it}{2w_0} (\cos w_0 t + i \sin w_0 t) \right)$

THUS $u_p = \dfrac{t}{2w_0} \sin(w_0 t)$

GEN. SOLUTION IS $u = C_1 \cos(w_0 t) + C_2 \sin w_0 t + \dfrac{t}{2w_0} \sin(w_0 t)$

SECTION 3.5 # 13

$$y'' + y' - 2y = 2t \qquad y(0) = 0, \quad y'(0) = 1$$

HOMOG. PROBLEM $\quad y = e^{\lambda t} \rightarrow \lambda^2 + \lambda - 2\lambda = (\lambda+2)(\lambda-1) = 0 \rightarrow \lambda = 1, -2.$

THUS $\quad y_1 = e^t, \ y_2 = e^{-2t}.$

PARTICULAR SOLUTION $\quad y_p = At + B. \quad y_p' = A, \quad y_p'' = 0.$

THUS $\quad A - 2(At + B) = 2t \rightarrow -2A = 2, \quad -2B + A = 0$

SO $\quad A = -1, \ B = -1/2 \rightarrow y_p = -t - 1/2.$

THUS GEN. SOLUTION IS $\quad y = c_1 e^{+t} + c_2 e^{-2t} - t - 1/2.$

NOW

$$\left.\begin{array}{l} y(0) = 0 \rightarrow \quad c_1 + c_2 - 1/2 = 0 \\ y'(0) = 1 \rightarrow \quad c_1 - 2c_2 - 1 = 1 \end{array}\right\} \rightarrow c_1 = 1, \ c_2 = -1/2$$

THUS $\quad y = e^t - \dfrac{e^{-2t}}{2} - t - 1/2$

SECTION 3.5 # 14

$$y'' + 4y = t^2 + 3e^t \qquad y(0) = 0, \quad y'(0) = 2.$$

HOMOG. PROBLEM $\quad$ PUT $y = e^{\lambda t} \rightarrow \lambda^2 + 4 = 0 \rightarrow \lambda = \pm 2i$

THUS $\quad y_1 = \cos(2t), \ y_2 = \sin(2t)$

PARTICULAR SOLUTIONS $\quad y_{p_1}'' + 4y_{p_1} = t^2.$

SO $\quad y_{p_1} = At^2 + Bt + C \qquad y_{p_1}' = 2At + B, \quad y_{p_1}'' = 2A$

THUS $\quad 2A + 4At^2 + 4Bt + 4C = t^2$

$\rightarrow 4A = 1, \quad 4B = 0, \quad 2A + 4C = 0 \rightarrow A = 1/4, \ B = 0, \ C = -\frac{1}{8}$

THUS $y_{p_1} = t^2/4 - 1/8$.

NOW LET y_{p_2} SATISFY $y_{p_2}'' + 4y_{p_2} = 3e^t$.

THEN $y_{p_2} = Ae^t \rightarrow y_{p_2}' = Ae^t \quad y_{p_2}'' = Ae^t$.

THUS, $Ae^t + 4Ae^t = 3e^t \rightarrow A = 3/5$.

SO $y_{p_2} = 3/5\, e^t$.

GEN SOLUTION IS $y = C_1 \cos(2t) + C_2 \sin(2t) + t^2/4 - 1/8 + 3/5\, e^t$.

NOW $y(0) = 0 \rightarrow C_1 - 1/8 + 3/5 = 0 \longrightarrow C_1 = -19/40$

$y'(0) = 2 \longrightarrow 2C_2 + 3/5 = 2 \longrightarrow C_2 = 7/10$

SO $$y = -\frac{19}{40}\cos(2t) + \frac{7\sin(2t)}{10} + \frac{t^2}{4} - \frac{1}{8} + \frac{3}{5}e^t.$$

SECTION 3.5 # 16 $\quad y'' - 2y' - 3y = 3te^{2t} \quad y(0) = 1,\ y'(0) = 0$.

HOMOG. PROBLEM $\quad y = e^{\lambda t} \rightarrow \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0 \rightarrow \lambda = -1, 3$.

THUS $y_1 = e^{-t},\ y_2 = e^{3t}$.

PARTICULAR SOLUTION $\quad y_p = (At+B)e^{2t},\ y_p' = Ae^{2t} + 2(At+B)e^{2t}$

$$y_p'' = 2Ae^{2t} + 2Ae^{2t} + 4(At+B)e^{2t} = e^{2t}(4At + 4A + 4B).$$

SUBSTITUTE INTO $y'' - 2y' - 3y = 3te^{2t}$

$$\longrightarrow 4At + 4A + 4B - 2(A + 2B + 2At) - 3At - 3B = 3t.$$

THUS $-3At = +3t$ AND $4A + 4B - 2A - 4B - 3B = 0 \rightarrow 2A = 3B$

SO $A = -1,\ B = -2/3 \rightarrow y_p = -(t + 2/3)e^{2t}$.

GEN. SOLUTION IS

$$y = C_1 e^{-t} + C_2 e^{3t} - (t + 2/3) e^{2t}$$

$$\left.\begin{aligned} y(0) = 1 &\longrightarrow \quad C_1 + C_2 - 2/3 = 1 \\ y'(0) = 0 &\longrightarrow \quad -C_1 + 3C_2 - 4/3 - 1 = 0 \end{aligned}\right\} \longrightarrow \quad C_1 = 2/3, \quad C_2 = 1.$$

THUS $\quad y = 2/3\, e^{-t} + e^{3t} - (t + 2/3) e^{2t}.$

SECTION 3.5 # 17 $\quad y'' + 4y = 3\sin(2t) \qquad y(0) = 2, \ y'(0) = -1.$

HOMOG PROBLEM $\quad y = e^{\lambda x} \longrightarrow \lambda^2 + 4 = 0 \quad$ SO $\quad \lambda_{\pm} = \pm 2i$

THUS $\quad y_1 = \cos(2t), \quad y_2 = \sin(2t).$

PARTICULAR SOLUTION $\quad \hat{y}'' + 4\hat{y} = 3e^{2it} \qquad$ SINCE $\quad \alpha = 2i = \lambda_+$

THEN $\quad \hat{y}_p = Ate^{2it} \quad$ AND $\quad y_p = \mathrm{IM}(\hat{y}_p).$

NOW $\quad \hat{y}_p' = 2Aite^{2it} + Ae^{2it}, \quad \hat{y}_p'' = 4Aie^{2it} - 4Ate^{2it}.$

SUBSTITUTE INTO $\quad \hat{y}'' + 4\hat{y} = 3e^{2it} \rightarrow 4Ai - 4At + 4At = 3.$

THUS $\quad A = 3/4i \longrightarrow A = -\frac{3i}{4} \Longrightarrow y_p = \mathrm{IM}\left(-\frac{3i}{4} te^{2it}\right).$

SO $\quad y_p = \mathrm{IM}\left(-\frac{3i}{4} t\left(\cos(2t) + i\sin(2t)\right)\right) = -\frac{3t}{4}\cos(2t)$

SO GEN. SOLUTION $\quad y = C_1\cos(2t) + C_2\sin(2t) - \frac{3t}{4}\cos(2t)$

NOW $\quad y' = -2C_1\sin(2t) + 2C_2\cos(2t) - \frac{3}{4}\cos(2t) + \frac{3}{2}t\sin(2t).$

$y(0) = 2 \rightarrow C_1 = 2 \quad ; \quad y'(0) = -1 \rightarrow 2C_2 - 3/4 = -1 \rightarrow C_2 = -1/8.$

THUS $\quad y = 2\cos(2t) - \frac{1}{8}\sin(2t) - \frac{3t}{4}\cos(2t)$

SECTION 3.5 # 30

$$y'' + 2y' + 5y = \begin{cases} 1 & 0 \le t \le \pi/2 \\ 0 & t > \pi/2 \end{cases} \qquad y(0) = y'(0) = 0.$$

HOMOGENEOUS PROBLEM $\quad y = e^{\lambda t} \rightarrow \lambda^2 + 2\lambda + 5 = (\lambda^2 + 2\lambda + 1) + 4 = 0.$

SO $\quad (\lambda + 1)^2 = -4 \rightarrow \lambda = -1 \pm 2i.$

THUS $\quad y_1 = e^{-t}\cos(2t) \qquad y_2 = e^{-t}\sin(2t).$

PARTICULAR SOLUTION $\quad y'' + 2y' + 5y = 1 \rightarrow y_p = 1/5.$

SO

$$(*) \quad y = \begin{cases} c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + 1/5 & 0 \le t \le \pi/2 \\ d_1 e^{-t}\cos(2t) + d_2 e^{-t}\sin(2t) & t \ge \pi/2. \end{cases}$$

MUST SET $\quad y(0) = y'(0) = 0 \quad$ AND $\quad y, y'$ CONTINUOUS AT $t = \pi/2$.

$\longrightarrow$ 4 equations FOR the 4 UNKNOWNS c_1, c_2, d_1, d_2.

NOW

$$y' = \begin{cases} -2c_1 e^{-t}\sin(2t) - c_1 e^{-t}\cos(2t) - c_2 e^{-t}\sin(2t) + 2c_2 e^{-t}\cos(2t) & 0 \le t \le \pi/2 \\ -2d_1 e^{-t}\sin(2t) - d_1 e^{-t}\cos(2t) - d_2 e^{-t}\sin(2t) + 2d_2 e^{-t}\cos(2t) & t \ge \pi/2 \end{cases}$$

NOW $\quad y(0) = 0 \longrightarrow c_1 + 1/5 = 0 \rightarrow \quad c_1 = -1/5$

$y'(0) = 0 \longrightarrow -c_1 + 2c_2 = 0 \qquad c_2 = c_1/2 = -1/10.$

y CONTINUOUS AT $t = \pi/2 \longrightarrow -c_1 e^{-\pi/2} + 1/5 = -d_1 e^{-\pi/2}$

y' CONTINUOUS AT $t = \pi/2 \rightarrow +c_1 e^{-\pi/2} - 2c_2 e^{-\pi/2} = d_1 e^{-\pi/2} - 2d_2 e^{-\pi/2}.$

THUS $\quad d_1 = c_1 - 1/5\, e^{\pi/2} = -1/5 - 1/5\, e^{\pi/2} = -\frac{1}{5}(1 + e^{\pi/2})$

AND $\quad d_2 = d_1/2 + c_2 - c_1/2 = d_1/2 = -\frac{1}{10}(1 + e^{\pi/2})$

WITH c_1, c_2, d_1, d_2 KNOWN WE OBTAIN y FROM $(*)$.