MATH 215 HW #10

SECTION 7.8 # 1, 2, 7 (ONLY DO PART C OF THESE QUESTIONS)

SECTION 7.9 # 1, 4, 8, 11

IN ADDITION PLEASE DO THE TWO EXTRA PROBLEMS

PROBLEM 1 CONSIDER THE LINEAR SYSTEM

$$\underline{x}' = \begin{pmatrix} \alpha & 1 \\ -2 & -3 \end{pmatrix} \underline{x} + \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$ WHERE α IS A PARAMETER.

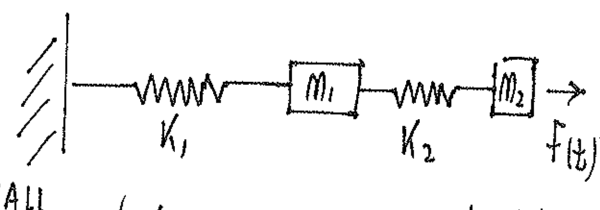
(i) FIND THE PARTICULAR SOLUTION $\underline{x}_p$. (ANSWER IS IN TERMS OF α)

(ii) FIND THE RANGE OF THE PARAMETER α FOR WHICH

$\underline{x}(t) \to \underline{x}_p$ AS $t \to \infty$ FOR ANY INITIAL CONDITION $\underline{x}(0)$.

PROBLEM 2 CONSIDER THE COUPLED MASS SPRING SYSTEM MODELED BY

$$(*) \begin{cases} m_1 x'' = -k_1 x + k_2 (y - x) \\ m_2 y'' = -k_2 (y - x) + f(t) \end{cases}$$



(K_1, K_2 SPRING CONSTANTS)

(i) IF $m_1 = m_2 = 1$, $K_1 = 5$ AND $K_2 = 6$ AND $f(t) = 0$, FIND THE GENERAL SOLUTION TO $(*)$

(ii) IF $f(t) = \sin(wt)$ FOR WHAT VALUES OF w WILL RESONANCE OCCUR? (TAKE $w > 0$).

(iii) FIND A PARTICULAR SOLUTION FOR $(*)$ WHEN $f(w) = \sin(wt)$ IN THE FORM $\underline{x}(t) = \underline{c} \sin(wt)$ AND PLOT $|\underline{c}(w)|$ VERSUS w. (HERE $|\;|$ MEANS THE LENGTH OF THE VECTOR)

SECTION 7.8 #1 $\quad \underline{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \underline{x}$

NOW $\det(A-\lambda I) = 0 \to (3-\lambda)(-1-\lambda)+4=0 \to \lambda^2 - 2\lambda + 1 = 0.$

THUS $\lambda_1 = 1$ IS A REPEATED ROOT

NOW $(A-I)\underline{v}_1 = 0 \to \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \underline{v}_1 = 0 \to \begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix} \underline{v}_1 = \underline{0} \quad \underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

THUS $\underline{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t$ IS A SOLUTION.

NOW PUT FOR THE SECOND SOLUTION $\underline{x} = \underline{v}_1 t e^t + \underline{\xi} e^t$

THEN $\underline{v}_1 e^t + \underline{v}_1 t e^t + \underline{\xi} e^t = A(\underline{v}_1 t e^t + \underline{\xi} e^t) = t e^t A \underline{v}_1 + e^t A \underline{\xi}$

$$\underline{v}_1 e^t + \underline{v}_1 t e^t + \underline{\xi} e^t = t e^t \lambda_1 \underline{v}_1 + e^t A \underline{\xi}$$

SO WITH $\lambda_1 = 1 \to (A-I)\underline{\xi} = \underline{v}_1 \to \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

SO $\begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \to \begin{array}{l} \xi_1 - 2\xi_2 = 1 \\ \xi_2 = \xi_2 \end{array}$

SO $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \xi_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

THUS $\underline{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^t + \left[\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \xi_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) e^t \right]$ IS SECOND SOLUTION.

WLOG WE PUT $\xi_2 = 0$ SO THAT GENERAL SOLUTION IS

$$\underline{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right]$$

SECTION 7.8 #2 $\quad \underline{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \underline{x}$

NOW $\det(A-\lambda I) = 0 \to (4-\lambda)(-4-\lambda)+16=0 \to \lambda^2 = 0$ SO $\lambda_1 = 0.$

NOW $(A-\lambda_1 I)\underline{v}_1 = 0 \to \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

WE THEN PUT $\underline{X} = V_1 t + \underline{\xi}$ TO FIND SECOND SOLUTION AND

SO $\underline{X}' = V_1 \rightarrow \underline{V}_1 = A(\underline{V}_1 t + \underline{\xi}) = t A \underline{V}_1 + A\underline{\xi}$

NOW $A\underline{V}_1 = 0 \rightarrow A\underline{\xi} = \underline{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

SO $\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

NOW $(*)\begin{cases} 4\xi_1 - 2\xi_2 = 1 \\ \xi_2 = \xi_2 \end{cases} \rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + \xi_2 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

THUS SET $\xi_2 = 0$ WITHOUT LOSS OF GENERALITY,

$$\underline{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$$

SO gen. SOL'N IS $\underline{X} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right]$

REMARK: IN SOLVING $(*)$ WE CAN TAKE ALSO $\xi_1 = 0$ AND $\xi_2 = -1/2$ OR ANY OTHER CHOICE THAT SATISFIES $(*)$.

SO $\underline{X} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \right]$ ALSO WORKS.

<u>SECTION 7.8 #7</u> $\underline{X}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \underline{X}$ $\underline{X}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

NOW $\det(A - \lambda I) = 0 \longrightarrow (1-\lambda)(-7-\lambda) + 16 = 0 \longrightarrow \lambda^2 + 6\lambda + 9 = 0 \rightarrow (\lambda+3)^2 = 0$

SO $\lambda_1 = -3$. THEN, $(A + 3I)\underline{V}_1 = \underline{0} \longrightarrow \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \underline{V}_1 = 0 \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \underline{V}_1 = 0$

THUS $\underline{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. NOW PUT $\underline{X} = \underline{V}_1 t e^{-3t} + \underline{\xi} e^{-3t}$ SO THAT

$$(A + 3I)\underline{\xi} = \underline{V}_1 \longrightarrow \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \underline{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -4 \\ 0 & 0 \end{pmatrix} \underline{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

THUS $4\xi_1 - 4\xi_2 = 1 \rightarrow \xi_1 = \xi_2 + 1/4$

$\xi_2 = \xi_2 \qquad \xi_2 = \xi_2$

HENCE $\underline{\xi} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + \xi_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we can set $\xi_2 = 0$ WITHOUT LOSS OF GENERALITY.

HENCE, $\underline{X} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$

THUS GEN. SOL'N IS $\underline{X} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + C_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + e^{-3t} \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right].$

NOW $\underline{X}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$

SO $2 = C_1 \qquad 3 = C_1 + C_2/4 \rightarrow C_2 = 4.$

SO $\underline{X} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + 4 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + e^{-3t} \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right]$

$\underline{X} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + e^{-3t} \begin{pmatrix} 2+1 \\ 2 \end{pmatrix}$

SO $\underline{X} = \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix} e^{-3t}$

SECTION 7.9 # 1

$$\underline{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \underline{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

SOLUTION FOR THE HOMOGENEOUS PROBLEM

$\det(A - \lambda I) = 0 \rightarrow (2-\lambda)(-2-\lambda) + 3 = 0 \rightarrow \lambda^2 - 1 = 0$

THUS $\lambda = \pm 1$ SO $\lambda_1 = 1, \ \lambda_2 = -1.$

NOW $(A-\lambda_1 I)\,\underline{v}_1 = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \underline{v}_1 = \underline{0} \rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A-\lambda_2 I)\,\underline{v}_2 = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \underline{v}_2 = \underline{0} \rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

SO $S = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ IS EIGENVECTOR MATRIX AND $S^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$ $\qquad AS = S\Lambda$

NOW $\underline{x}' = S\Lambda S^{-1}\underline{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

LET $\underline{y} = S^{-1}\underline{x}$ OR $\underline{x} = S\underline{y}$ SO $\underline{y}' = \Lambda y + S^{-1}\begin{pmatrix} e^t \\ t \end{pmatrix} = \Lambda y + \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t \\ t \end{pmatrix}$

THEN, $y_1' = y_1 + \frac{1}{2}(3e^t - t) \rightarrow y_1' - y_1 = \frac{3}{2}e^t - \frac{t}{2}$

$y_2' = -y_2 + \frac{1}{2}(-e^t + t) \rightarrow y_2' + y_2 = \frac{t}{2} - \frac{e^{2t}}{2}$

SO $(y_1 e^{-t})' = \frac{3}{2} - \frac{1}{2}te^{-t} \rightarrow y_1 e^{-t} = \frac{3}{2}t - \frac{1}{2}\left[-te^{-t}\ e^{-t}\right] + C_1$

$(y_2 e^t)' = \frac{t}{2}e^t - \frac{1}{2}e^{2t} \rightarrow y_2 e^t = \frac{1}{2}\left(te^t - \frac{1}{2}e^{2t}\right) + C_2$
$-e^t$

THUS $y_1 = \frac{3}{2}te^t + \frac{t}{2} + \frac{1}{2} + C_1 e^t, \quad y_2 = \frac{1}{2}t - \frac{1}{4}e^t + C_2 e^{-t} - \frac{1}{2}$

NOW $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = S\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rightarrow \begin{aligned} x_1 &= y_1 + y_2 \\ x_2 &= y_1 + 3y_2 \end{aligned}$

SO $x_1 = y_1 + y_2 = \frac{3}{2}te^t + \frac{t}{2} + \frac{1}{2} + C_1 e^t + \frac{1}{2}t - \frac{1}{4}e^t - \frac{1}{2} + C_2 e^{-t}$

$x_2 = y_1 + 3y_2 = \frac{3}{2}te^t + \frac{t}{2} + \frac{1}{2} + C_1 e^t + \frac{3t}{2} - \frac{3}{4}e^t + 3C_2 e^{-t} - \frac{3}{2}$

THUS $\underline{x} = \frac{3}{2}te^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{t}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^t + C_1 e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$+ C_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

SO $\underline{x} = \frac{3}{2}te^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^t + C_1 e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

SECTION 7.9 # 4

$$\underline{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \underline{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

NOW $\det(A - \lambda I) = 0 \rightarrow (1-\lambda)(-2-\lambda) - 4 = 0 \rightarrow \lambda^2 + \lambda - 6 = 0$

THUS $(\lambda + 3)(\lambda - 2) = 0 \rightarrow \lambda = 2, -3$. $\lambda_1 = 2,\ \lambda_2 = -3$

- $(A - 2I)\underline{v}_1 = \underline{0} \rightarrow \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \underline{v}_1 = \underline{0} \rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \underline{x}_1 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$
- $(A + 3I)\underline{v}_2 = \underline{0} \rightarrow \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \underline{v}_2 = \underline{0} \rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \rightarrow \underline{x}_2 = c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$.

NOW WRITE $\underline{x}' = A\underline{x} + e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{t}\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

- LET $\underline{x}_{p_1}$ SOLVE $\underline{x}' - A\underline{x} = e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \underline{x} = e^{-2t}\underline{\xi}$.

SO $-2\underline{\xi} - A\underline{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow (A - 2I)\underline{\xi} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

SO $\begin{pmatrix} -1 & 1 \\ 4 & 0 \end{pmatrix}\underline{\xi} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 4 \end{pmatrix}\underline{\xi} = +\begin{pmatrix} -1 \\ -4 \end{pmatrix}$

SO $\xi_2 = -1,\ \xi_1 = 0 \qquad \underline{\xi} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

- LET $\underline{x}_{p_2}$ SOLVE $\underline{x}' - A\underline{x} = e^{t}\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ $\quad$ LET $\underline{x} = e^{t}\underline{\xi}$

THEN $(A - I)\underline{\xi} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}\underline{\xi} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

THUS $\xi_2 = 0$ AND $4\xi_1 = 2 \rightarrow \xi_1 = 1/2 \rightarrow \xi = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

THE GEN. SOLUTION IS $\underline{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{-2t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}e^{t}$

SECTION 7.9 # 8

$$\underline{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \underline{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

(i) WE WILL USE UNDETERMINED COEFFICIENTS FOR PARTICULAR SOLUTION.

HOMOGENEOUS PROBLEM

$$\det(A - \lambda I) = 0 \rightarrow (2-\lambda)(-2-\lambda) + 3 = 0 \qquad \lambda^2 - 1 = 0 \quad \text{so} \quad \lambda_1 = 1,\ \lambda_2 = -1$$

- $(A - \lambda_1 I)\underline{v}_1 = 0 \rightarrow \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \underline{v}_1 = \underline{0} \rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $(A - \lambda_2 I)\underline{v}_2 = \underline{0} \rightarrow \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \underline{v}_2 = \underline{0} \rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

NOW FOR PARTICULAR SOLUTION LET $x_p = \underline{\xi}\, t e^t + \underline{\eta}\, e^t$ $\qquad \underline{x}' = A\underline{x} \Longrightarrow$

so $x_p' = \underline{\xi} e^t + \underline{\xi} t e^t + \underline{\eta} e^t = t e^t A\underline{\xi} + e^t A\underline{\eta} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$

$$\underline{\xi} + \underline{\xi} t + \underline{\eta} = t A\underline{\xi} + A\underline{\eta} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

NOW PUT $A\underline{\xi} = \underline{\xi} \rightarrow \underline{\xi} = \mu \underline{v}_1 \qquad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mu$ = ANYTHING.

THEN $\qquad A\underline{\eta} - \underline{\eta} = \underline{\xi} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

SO $\qquad (A - I)\underline{\eta} = \underline{\xi} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

NOW $\qquad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \underline{\eta} = \begin{pmatrix} \mu - 1 \\ \mu + 1 \end{pmatrix}$

NOW ROW REDUCE $\qquad \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \underline{\eta} = \begin{pmatrix} \mu - 1 \\ \mu + 1 - 3\mu + 3 \end{pmatrix} = \begin{pmatrix} \mu - 1 \\ -2\mu + 4 \end{pmatrix}$

Set $\mu = 2 \longrightarrow \begin{array}{l} \eta_1 - \eta_2 = 1 \\ \eta_2 = \eta_2 \end{array} \longrightarrow \underline{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

WITHOUT LOSS OF generality set $\eta_2 = 0$.

THEN $\quad \underset{\sim}{X}_p = 2\begin{pmatrix}1\\1\end{pmatrix} t e^{t} + \begin{pmatrix}1\\0\end{pmatrix} e^{t}$

NOW ADDING HOMOGENEOUS SOLUTION, THE GEN. SOLUTION IS

$$\underset{\sim}{X} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t} + 2\begin{pmatrix}1\\1\end{pmatrix}te^{t} + \begin{pmatrix}1\\0\end{pmatrix}e^{t}.$$

SECTION 7.9 # 11

$$\underset{\sim}{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

SOLUTION CONSIDER THE HOMOGENEOUS SYSTEM

$$\det(A - \lambda I) = 0 \rightarrow (2-\lambda)(-2-\lambda) + 5 = 0 \rightarrow \lambda^2 + 1 = 0 \quad \text{so } \lambda_\pm = \pm i.$$

NOW $(A - \lambda_+ I)\underset{\sim}{v} = 0 \rightarrow \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \begin{pmatrix} 2-i & -5 \\ 0 & 0 \end{pmatrix} \underset{\sim}{v} = \underset{\sim}{0}$

NOW $(2-i)v_1 = 5v_2$, $v_2 = v_2$ $\longrightarrow$ LET $v_1 = 5$, $v_2 = 2-i$ $\quad \underset{\sim}{v} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

NOW $\underset{\sim}{x}_R = \mathrm{RE}\left[\left[\begin{pmatrix} 5 \\ 2 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right](\cos t + i\sin t)\right] = \begin{pmatrix} 5 \\ 2 \end{pmatrix}\cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t$

$\underset{\sim}{x}_I = \mathrm{IM}\left[\left[\begin{pmatrix} 5 \\ 2 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right](\cos t + i\sin t)\right] = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 5 \\ 2 \end{pmatrix}\sin t$

THUS $\underset{\sim}{x}_R = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} \qquad \underset{\sim}{x}_I = \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

THE SOLUTION TO THE HOMOGENEOUS PROBLEM IS

$$\underset{\sim}{x} = C_1 \underset{\sim}{x}_R + C_2 x_I.$$

PARTICULAR SOLUTION WE CONSIDER $\tilde{x}$ THAT SATISFIES

$$\underset{\sim}{\tilde{x}}' = A\underset{\sim}{\tilde{x}} + e^{it}\underset{\sim}{b} \qquad \underset{\sim}{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad x = \mathrm{RE}(\hat{x})$$

NOW SINCE i IS ONE OF THE TWO ROOTS $\lambda_\pm$, WE MUST GUESS

$$\underset{\sim}{\tilde{x}} = t\underset{\sim}{v}e^{it} + \underset{\sim}{s}e^{it}$$

THEN $\underset{\sim}{\hat{x}}' = \underset{\sim}{v}e^{it} + it\underset{\sim}{v}e^{it} + i\underset{\sim}{s}e^{it}$

$$it\underset{\sim}{v}e^{it} + \underset{\sim}{v}e^{it} + i\underset{\sim}{s}e^{it} = A(t\underset{\sim}{v}e^{it} + \underset{\sim}{s}e^{it}) + e^{it}\underset{\sim}{b}$$

THEN, $\quad it\underline{v}e^{it} + \underline{v}e^{it} + i\underline{\varsigma}e^{it} = te^{it}A\underline{v} + e^{it}A\underline{\varsigma} + e^{it}\underline{b}$

THEN $\quad it\,\underline{v} + \underline{v} + i\underline{\varsigma} = tA\underline{v} + A\underline{\varsigma} + \underline{b}$

NOW LET $(A - iI)\underline{v} = 0 \qquad \underline{v} = \mu\begin{pmatrix} 5 \\ 2-i \end{pmatrix}$ where μ IS A SCALAR.

THEN, $\quad \underline{v} + i\underline{\varsigma} = A\underline{\varsigma} + \underline{b}$

SO $\quad (A - iI)\underline{\varsigma} = \underline{v} - \underline{b} = \mu\begin{pmatrix} 5 \\ 2-i \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

SO $\quad \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\underline{\varsigma} = \begin{pmatrix} 5\mu \\ \mu(2-i)-1 \end{pmatrix}$

NOW WRITE $-\dfrac{(\text{ROW }1)}{(2-i)} + \text{ROW }2.$

THEN $\quad \begin{pmatrix} 2-i & -5 \\ 0 & 0 \end{pmatrix}\underline{\varsigma} = \begin{pmatrix} 5\mu \\ \mu(2-i) - 1 - 5\mu/(2-i) \end{pmatrix}$

WE MUST SET $\quad \mu(2-i) - 1 = \dfrac{5\mu}{2-i} = \dfrac{5\mu(2+i)}{5} = \mu(2+i)$

SO $\quad 2\mu - i\mu - 1 = 2\mu + i\mu$

$\rightarrow \quad 2i\mu = -1 \quad$ SO $\quad 2\mu = \dfrac{-1}{i} = i.$

THUS $\quad \mu = i/2.$

NOW WE MUST SOLVE

$\begin{pmatrix} 2-i & -5 \\ 0 & 0 \end{pmatrix}\underline{\varsigma} = \begin{pmatrix} 5\mu \\ 0 \end{pmatrix} \qquad (2-i)\varsigma_1 - 5\varsigma_2 = 5\mu = \dfrac{5i}{2}.$

LET $\varsigma_1 = 2+i$

THEN $\quad 5 - 5\varsigma_2 = 5i/2$

$\varsigma_2 = (5 - 5i/2)/5$

THIS GIVES $\quad \underset{\sim}{\zeta} = \begin{pmatrix} 2+i \\ 1-i/2 \end{pmatrix}$

NOW $\quad \hat{\underset{\sim}{X}} = \frac{i}{2} \begin{pmatrix} 5 \\ 2-i \end{pmatrix} t e^{it} + \begin{pmatrix} 2+i \\ 1-i/2 \end{pmatrix} e^{it}$

NOW
$$\underset{\sim}{X_P} = RE(\hat{X}) = RE\left[\frac{i}{2}\left[\begin{pmatrix} 5 \\ 2 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] t\,(\cos t + i\sin t) \right]$$
$$+ RE\left[\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \right] (\cos t + i \sin t) \right]$$

NOW
$$\underset{\sim}{X_P} = -\frac{1}{2}\begin{pmatrix} 5 \\ 2 \end{pmatrix} t\sin t + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} t\cos t + \begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \sin t$$

$$\underset{\sim}{X_P} = \begin{pmatrix} -5/2 \sin t \\ -\sin t + \frac{1}{2}\cos t \end{pmatrix} t + \begin{pmatrix} 2\cos t - \sin t \\ \cos t + \frac{1}{2}\sin t \end{pmatrix}$$

THIS YIELDS THE GENERAL SOLUTION

$$\underset{\sim}{X} = C_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + C_2 \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} + \begin{pmatrix} -5/2 \sin t \\ \frac{1}{2}\cos t - \sin t \end{pmatrix} t$$
$$+ \begin{pmatrix} 2\cos t - \sin t \\ \cos t + \frac{1}{2}\sin t \end{pmatrix}$$

PROBLEM 1

$$\underline{x}' = A\underline{x} + \begin{pmatrix} 5 \\ 10 \end{pmatrix} \qquad A = \begin{pmatrix} \alpha & 1 \\ -2 & -3 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} -3 & -1 \\ 2 & \alpha \end{pmatrix} \frac{1}{(2-3\alpha)}$$

(i) NOW LET $\underline{x}_p = \underline{\xi}$ SO THAT $A\underline{\xi} + \underline{b} = 0$.

THEN $$\underline{\xi} = -A^{-1}\underline{b} = \frac{1}{(3\alpha-2)} \begin{pmatrix} -3 & -1 \\ 2 & \alpha \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \frac{1}{(3\alpha-2)} \begin{pmatrix} -25 \\ 10+10\alpha \end{pmatrix}$$

HENCE $$\underline{x} = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} + \underline{\xi} \qquad \underline{\xi} = \frac{1}{(3\alpha-2)} \begin{pmatrix} -25 \\ 10+10\alpha \end{pmatrix}$$

THIS FORMULA IS VALID PROVIDED THAT $\alpha \neq 2/3$, i.e. WE NEED $\det A \neq 0$.

(ii) FOR $\mathrm{Re}(\lambda_1) < 0$, $\mathrm{Re}(\lambda_2) < 0$ WE NEED

$$\operatorname{trace} A < 0 \qquad \det A > 0$$

$$\rightarrow -3 + \alpha < 0 \qquad -3\alpha + 2 > 0$$

$$\rightarrow \alpha < 3 \quad , \quad \alpha < 2/3$$

THESE ARE BOTH SATISFIED WHEN $\alpha < 2/3$.

NOW SUPPOSE $\alpha = 2/3$ THEN WHAT IS THE PARTICULAR SOLUTION?

WE WRITE

$$\underline{x}_p = \underline{v}_1 t + \underline{\xi} \quad \rightarrow \quad \underline{v}_1 = (A\underline{v}_1)t + A\underline{\xi} + \underline{b}.$$

NOW SET $A\underline{v}_1 = 0 \rightarrow \underline{v}_1 = \mu \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ μ IS ANY SCALAR.

THUS $$A\underline{\xi} + \underline{b} = \mu \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad \rightarrow \quad A\underline{\xi} = \mu \begin{pmatrix} -3 \\ 2 \end{pmatrix} - \underline{b}$$

SO $$\begin{pmatrix} 2/3 & 1 \\ -2 & -3 \end{pmatrix} \underline{\xi} = \begin{pmatrix} -5 - 3\mu \\ -10 + 2\mu \end{pmatrix}$$

NOW USE ROW REDUCTION

$$\begin{pmatrix} 2/3 & 1 \\ 0 & 0 \end{pmatrix} \underline{\xi} = \begin{pmatrix} -5-3\mu \\ -25-7\mu \end{pmatrix}$$

NOW MUST SET $\mu = -25/7$ TO SOLVE THE SYSTEM.

HENCE $\quad 2/3\, \xi_1 + \xi_2 = -5 + 75/7 = 40/7$

SET $\xi_1 = 0$ SO $\xi_2 = 40/7$. (TO FIND ANY SOLUTION)

THEN, $\quad \underline{X}_p = -25/7 \begin{pmatrix} -3 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ 40/7 \end{pmatrix}$

SO $\quad \underline{X}_p = \begin{pmatrix} 75/7 \\ -50/7 \end{pmatrix} t + \begin{pmatrix} 0 \\ 40/7 \end{pmatrix} \quad (*)$

THUS IF $\alpha = 2/3$ THE PARTICULAR SOLUTION IS GIVEN IN $(*)$

WE NOTICE THAT FROM

$$\begin{aligned} 2/3\, \xi_1 + \xi_2 &= 40/7 \\ \xi_2 &= \xi_2 \end{aligned} \longrightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \xi_2 + \begin{pmatrix} 60/7 \\ 0 \end{pmatrix}$$

THUS EQUIVALENTLY, THE PARTICULAR SOLUTION IS

$$\underline{X}_p = \begin{pmatrix} 75/7 \\ -50/7 \end{pmatrix} t + \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \xi_2 + \begin{pmatrix} 60/7 \\ 0 \end{pmatrix} \quad \text{FOR ANY } \xi_2.$$

PROBLEM 2

(i) IF $m_1 = m_2 = 1$, $k_1 = 5$ AND $k_2 = 6$ AND $f(t) = 0$, THEN

$$\underline{x}'' = \begin{pmatrix} -11 & 6 \\ 6 & -6 \end{pmatrix} \underline{x} \qquad \text{PUT} \quad \underline{x} = e^{\sqrt{\lambda} t} \underline{v} \Rightarrow \underline{v} \lambda I = A \underline{v}.$$

NOW $\det(A - \lambda I) = 0 \rightarrow (-11-\lambda)(-6-\lambda) - 36 = 0$ SO $\lambda^2 + 17\lambda + 30 = 0$

SO $(\lambda + 15)(\lambda + 2) = 0 \rightarrow \lambda_1 = -15,\ \lambda_2 = -2$.

NOW IF WE PUT $\underline{x} = e^{\sqrt{\lambda} t} \underline{v} \rightarrow \underline{x}_1 = e^{i w_1 t} \underline{v}_1,\ \underline{x}_1 = e^{i w_2 t} \underline{v}_2$

ARE THE TWO COMPLEX SOLUTIONS.

IF $\lambda_1 = -15 \rightarrow (A - \lambda_1 I) \underline{v}_1 = \underline{0} \rightarrow \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix} \underline{v}_1 = \underline{0} \rightarrow \underline{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

IF $\lambda_2 = -2 \rightarrow (A - \lambda_2 I) \underline{v}_2 = \underline{0} \rightarrow \begin{pmatrix} -9 & 6 \\ 6 & -4 \end{pmatrix} \underline{v}_2 = \underline{0} \rightarrow \underline{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

THUS $\underline{x}(t) = a_1 \underline{v}_1 \cos(\sqrt{15}\, t) + b_1 \underline{v}_1 \sin(\sqrt{15}\, t) + a_2 \underline{v}_2 \cos(\sqrt{2}\, t) + b_2 \underline{v}_2 \sin(\sqrt{2}\, t)$

IS THE GENERAL SOLUTION, WITH a_1, a_2, b_1, b_2 ARBITRARY CONSTANTS.

(ii) RESONANCE WILL OCCUR IF $f(t) = \sin(wt)$ AND $w = \sqrt{15}$ OR $w = \sqrt{2}$.

NOW $\underline{x}'' = A \underline{x} + \underline{f}_0 \sin(wt) \qquad \underline{f}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

NOW PUT $\underline{x} = \underline{\Gamma} \sin(wt)$ SO THAT

$$-w^2 \underline{\Gamma} = A \underline{\Gamma} + \underline{f}_0 \longrightarrow (A + I w^2) \underline{\Gamma} = -\underline{f}_0$$

SO IF $w^2 \neq -\lambda$ (i.e. IS NOT ONE OF THE Eigenvalues) THEN WE CAN FIND $\underline{\Gamma}$.

$$\begin{pmatrix} -11 + w^2 & 6 \\ 6 & -6 + w^2 \end{pmatrix} \underline{\Gamma} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

THUS $$\underset{\sim}{\Gamma} = -\begin{pmatrix} -6+w^2 & -6 \\ -6 & -11+w^2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{[(w^2-6)(w^2-11)-36]}$$

THUS $$\underset{\sim}{\Gamma} = \begin{pmatrix} 6 \\ 11-w^2 \end{pmatrix} \frac{1}{[(w^2-15)(w^2-2)]}$$ AND $$\underset{\sim}{X} = \underset{\sim}{\Gamma} \sin(wt)$$

IS THE PARTICULAR SOLUTION.

NOW $$|\underset{\sim}{\Gamma}| = \frac{[36 + (11-w^2)^2]^{1/2}}{|w^2-15||w^2-2|}$$