HOMEWORK # 9 MATH 215

PLEASE TURN IN THE SOLUTION TO

SECTION 6.6 p. 350 - 352 10, 13, 17

SECTION 7.5 p. 398 - 401 3, 5, 7, 16, 29

SECTION 7.6 1, 3, 5, 9

MATH 215 HW # 9

SECTION 6.6 # 10

NOW $F(s) = \dfrac{1}{(s+1)^2}\,\dfrac{1}{s^2+4} = G(s)H(s)$

LET $G(s) = \dfrac{1}{(s+1)^2} \longrightarrow g(t) = \mathcal{L}^{-1}[G(s)] = te^{-t}.$

$H(s) = \dfrac{1}{2}\left(\dfrac{2}{s^2+4}\right) \longrightarrow h(t) = \dfrac{1}{2}\sin(2t)$

THUS $f(t) = \mathcal{L}^{-1}[F(s)] = \int_0^t g(t-\tau)h(\tau)\,d\tau = \int_0^t (t-\tau)e^{-(t-\tau)}\dfrac{1}{2}\sin(2\tau)\,d\tau.$

SECTION 6.6 # 13

SOLVE $y'' + w^2 y = g(t) \qquad y(0)=0,\ y'(0)=1.$

NOW $s^2\bar{Y} - s\,y(0) - y'(0) + w^2\bar{Y} = G(s) \qquad G(s) = \mathcal{L}(g(t)).$

THEN $(s^2+w^2)\bar{Y} = 1 + G(s)$

$$\bar{Y} = \frac{1}{w}\,\frac{w}{s^2+w^2} + \frac{1}{w}\,\frac{w}{s^2+w^2}\,G(s)$$

WE WRITE $\bar{Y}(s) = \dfrac{1}{w}\,\dfrac{w}{s^2+w^2} + \mathcal{F}(s)\,G(s) \qquad \mathcal{F}(s) = \dfrac{1}{w}\,\dfrac{w}{s^2+w^2}.$

NOW $f(t) = \mathcal{L}^{-1}[\mathcal{F}(s)] = \dfrac{1}{w}\sin(wt)$, $g(t) = \mathcal{L}^{-1}[G(s)].$

THUS $y(t) = \dfrac{1}{w}\sin(wt) + \int_0^t \dfrac{1}{w}\sin(w(t-\tau))\,g(\tau)\,d\tau$

— CONVOLUTION THEOREM —

SECTION 6.6 # 17

$$y'' + 4y' + 4y = g(t), \quad y(0) = 2, \ y'(0) = -3.$$

NOW $$s^2 \bar{Y} - s\,y(0) - y'(0) + 4[s\bar{Y} - y(0)] + 4\bar{Y} = G(s)$$

SO $$(s^2 + 4s + 4)\bar{Y} = 2s - 3 + 8 + G(s)$$

SO $$\bar{Y} = \frac{2s+5}{(s+2)^2} + \frac{G(s)}{(s+2)^2}.$$

SO $$\bar{Y} = \frac{2(s+2)+1}{(s+2)^2} + F(s)\,G(s)$$ WHERE $$F(s) = \frac{1}{(s+2)^2}.$$

NOW $$\bar{Y} = \frac{2}{(s+2)} + \frac{1}{(s+2)^2} + F(s)\,G(s)$$

NOW $$f(t) = \mathcal{L}^{-1}[F(s)] = t e^{-2t}.$$

THUS $$y(t) = \mathcal{L}^{-1}[\bar{Y}(s)] = 2e^{-2t} + te^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)} g(\tau)\,d\tau.$$

SECTION 7.5 # 3

$\underset{\sim}{x}' = A\underset{\sim}{x} \qquad A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \qquad$ PUT $\underset{\sim}{x} = e^{\lambda t}\underset{\sim}{v}$.

a) $\quad (A - \lambda I)\underset{\sim}{v} = 0 \qquad \det(A - \lambda I) = 0 \longrightarrow (2-\lambda)(-2-\lambda) + 3 = 0.$

so $\lambda^2 - 1 = 0$ so $\lambda = \pm 1. \qquad \lambda_1 = 1, \ \lambda_2 = -1.$

- NOW $(A - I)\underset{\sim}{v}_1 = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \underset{\sim}{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \underset{\sim}{v}_1 = \underset{\sim}{0}$

so $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- NOW $(A + I)\underset{\sim}{v}_2 = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \underset{\sim}{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \underset{\sim}{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

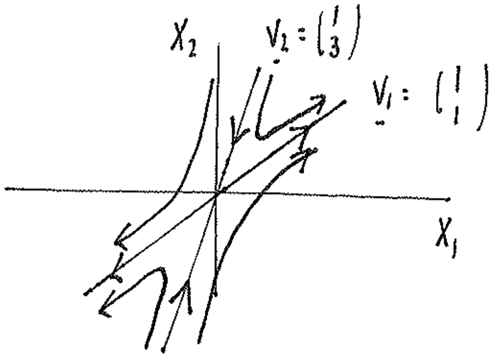
so $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

THUS BY SUPERPOSITION

$$\underset{\sim}{x} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

HENCE $\quad \underset{\sim}{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$ general solution.

b)



AS $t \to \infty$ WE CONCLUDE THAT

$\underset{\sim}{x} \sim c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$ AS $t \to \infty$

WHEN $c_1 \neq 0$.

SECTION 7.5 # 5

$$\underset{\sim}{x}' = A\underset{\sim}{x} \qquad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

a) PUT $\underset{\sim}{x} = e^{\lambda t}\underset{\sim}{v} \rightarrow (A - \lambda I)\underset{\sim}{v} = 0 \qquad \det\left(A - \lambda I\right) = 0.$

THUS $(-2-\lambda)(-2-\lambda) - 1 = 0 \rightarrow \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1) = 0.$

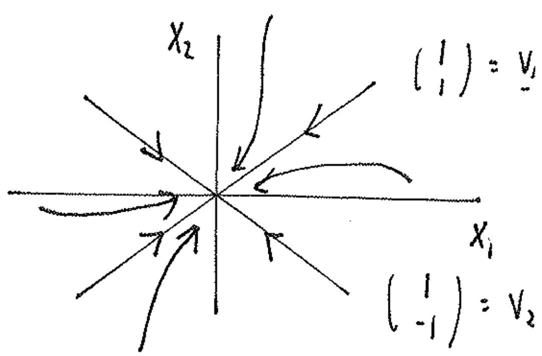
THUS $\lambda_1 = -1, \quad \lambda_2 = -3.$

- $(A - \lambda_1 I)\underset{\sim}{v}_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\underset{\sim}{v}_1 = \underset{\sim}{0} \Rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\underset{\sim}{v}_1 = \underset{\sim}{0} \rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $(A - \lambda_2 I)\underset{\sim}{v}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\underset{\sim}{v}_2 = \underset{\sim}{0} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\underset{\sim}{v}_2 = \underset{\sim}{0} \rightarrow \underset{\sim}{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

THUS BY SUPERPOSITION $\quad \underset{\sim}{x} = c_1 e^{\lambda_1 t}\underset{\sim}{v}_1 + c_2 e^{\lambda_2 t}\underset{\sim}{v}_2.$

HENCE $\quad \underset{\sim}{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t}$

b)



NOW AS $t \rightarrow \infty$ THEN

$$\underset{\sim}{x} \sim c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} \rightarrow \underset{\sim}{0}$$

UNLESS $c_1 = 0.$

SECTION 7.5 # 7

$$\underset{\sim}{x}' = A\underset{\sim}{x} \qquad A = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$$

a) PUT $\underset{\sim}{x} = e^{\lambda t}\underset{\sim}{v}. \rightarrow (A - \lambda I)\underset{\sim}{v} = 0 \qquad \det(A - \lambda I) = 0.$

SO $(4-\lambda)(-6-\lambda) + 24 = 0 \rightarrow \lambda^2 + 2\lambda = 0 \rightarrow \lambda_1 = 0 \text{ OR } \lambda_2 = -2.$

- $(A-\lambda_1 I)\,v_1 = 0 \quad \longrightarrow \quad \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} v_1 = 0 \quad \longrightarrow \quad \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix} v_1 = 0$

  THUS $\quad v_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

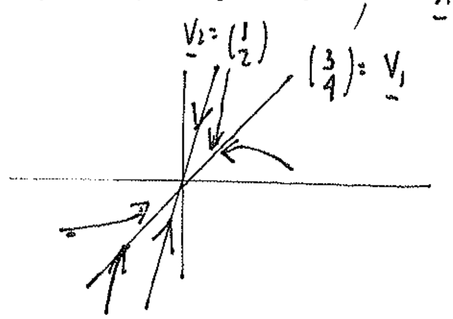
- $(A-\lambda_2 I)\,v_2 = 0 \quad \longrightarrow \quad \begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix} v_2 = 0 \quad \longrightarrow \quad \begin{pmatrix} 6 & -3 \\ 0 & 0 \end{pmatrix} v_2 = 0$

  THUS $\quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

HENCE $\quad X = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 .$

HENCE $\quad X = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$

b) NOW AS $t \to \infty$, $\quad X \to c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ UNLESS $c_1 = 0$.



SECTION 7.5 # 16

SOLVE $\quad X' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} X \qquad X(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

SOLUTION $\quad$ PUT $X = e^{\lambda t} v$. THEN $(A - \lambda I) v = 0$

SO $\quad \det(A - \lambda I) = 0 \quad \to \quad (-2-\lambda)(+4-\lambda) + 5 = 0 \quad \to \quad \lambda^2 - 2\lambda - 3 = 0$

SO $\quad (\lambda - 3)(\lambda + 1) = 0 \quad \to \quad \lambda_1 = 3, \ \lambda_2 = -1$

- $(A-\lambda_1 I)\,v_1 = 0 \quad \longrightarrow \quad \begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} v_1 = 0 \quad$ SO $\quad v_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$
- $(A-\lambda_2 I)\,v_2 = 0 \quad \longrightarrow \quad \begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} v_2 = 0 \quad$ SO $\quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

THE SOLUTION IS

$$\underline{X} = C_1 \underline{V}_1 e^{\lambda_1 t} + C_2 \underline{V}_2 e^{\lambda_2 t}$$

HENCE $\underline{X} = C_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$

NOW $\underline{X}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \longrightarrow C_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

SO $\begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ RECALL $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{-1}{4} \begin{pmatrix} 1 & -1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 1-3 \\ -5+3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

THUS $C_1 = C_2 = 1/2 \longrightarrow \underline{X} = \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$.

SECTION 7.5 # 29

$a y'' + b y' + c y = 0 \qquad a \neq 0, \quad a, b, c$ CONSTANTS.

a) LET $X_1 = y, \quad X_2 = y'$. THEN $X_1' = X_2$

$$a X_2' + b X_2 + c X_1 = 0.$$

THUS $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$

b) NOW $\det(A - \lambda I) = 0$ YIELDS

$$(-\lambda)(-\lambda - b/a) + c/a = 0 \longrightarrow \lambda^2 + \lambda\, b/a + c/a = 0$$

SO THAT $a\lambda^2 + b\lambda + c = 0$.

THIS IS SAME AS CHARACTERISTIC POLYNOMIAL ASSOCIATED WITH putting $y = e^{\lambda t}$ INTO $a y'' + b y' + c y = 0$.

SECTION 7.6 # 1

$$\underline{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \underline{x} \qquad A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

a) PUT $\underline{x} = e^{\lambda t} \underline{v}$. THEN $(A - \lambda I)\underline{v} = 0$ SO $\det(A - \lambda I) = 0$

YIELDS $(3-\lambda)(-1-\lambda) + 8 = 0 \rightarrow \lambda^2 - 2\lambda + 5 = 0 \rightarrow \lambda^2 - 2\lambda + 1 = -4$.

THUS $\lambda_{\pm} = 1 \pm 2i$

• $(A - \lambda_+ I)\underline{v} = \underline{0} \rightarrow \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \underline{v} = \underline{0} \rightarrow \begin{pmatrix} 1-i & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \underline{0}$

THUS $(1-i) v_1 = v_2$, $v_2 = v_2$ $\longrightarrow$ let $v_1 = 1$, THEN $v_2 = (1-i)$

SO $\underline{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ IS A COMPLEX EIGENVECTOR.

DEFINE $\underline{x}_R = RE(\underline{v} e^{\lambda_+ t}) = RE\left(\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] e^t (\cos 2t + i \sin 2t)\right)$

$\underline{x}_I = IM(\underline{v} e^{\lambda_+ t}) = IM\left(\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] e^t (\cos 2t + i \sin 2t)\right)$

THEN $\underline{x}_R = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \cos(2t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \sin(2t) = e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$

$\underline{x}_I = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \sin(2t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \cos(2t) = e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$

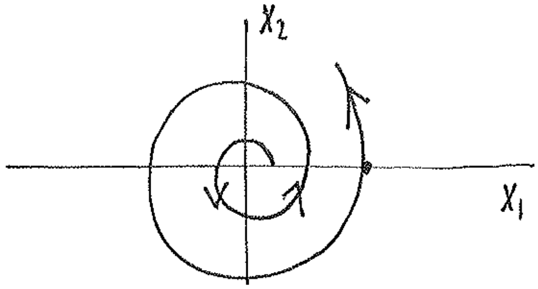
THE GENERAL SOLUTION IS

$$\underline{x} = c_1 \underline{x}_R + c_2 \underline{x}_I = c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

b) WE HAVE AN <u>UNSTABLE</u> SPIRAL POINT.

NOTICE WHEN $x_2 = 0$, $x_1 > 0$ THEN $x_2' = 4x_1 - x_2 = 4x_1 > 0$.

COUNTERCLOCKWISE ROTATION



SECTION 7.6 # 3

$$\underset{\sim}{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \underset{\sim}{x}$$

a) LET $\underset{\sim}{x} = e^{\lambda t}\underset{\sim}{v} \rightarrow (A - \lambda I)\underset{\sim}{v} = 0$ $\quad \det(A - \lambda I) = 0 \rightarrow (2-\lambda)(-2-\lambda) + 5 = 0.$

SO $\lambda^2 + 1 = 0$ SO $\lambda_{\pm} = \pm i.$

• $(A - \lambda_+ I)\underset{\sim}{v} = 0 \rightarrow \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\underset{\sim}{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2-i & -5 \\ 0 & 0 \end{pmatrix}\underset{\sim}{v} = \underset{\sim}{0}$

SO $(2-i)v_1 = 5v_2$
$v_2 = v_2$

LET $v_1 = 2+i \quad v_2 = 1$
WHICH SOLVES THIS SYSTEM.

HENCE $\underset{\sim}{v} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ IS A COMPLEX eigenvector.

DEFINE $\underset{\sim}{x}_R = RE\left[\underset{\sim}{v} e^{\lambda_+ t}\right] = RE\left[\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)(\cos t + i \sin t)\right]$

$\underset{\sim}{x}_I = IM\left[\underset{\sim}{v} e^{\lambda_+ t}\right] = IM\left[\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)(\cos t + i \sin t)\right]$

THEN $\underset{\sim}{x}_R = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$

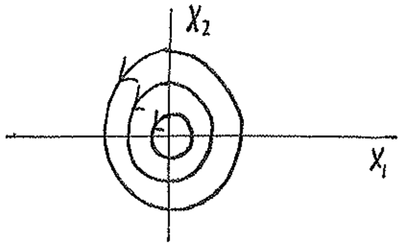
$\underset{\sim}{x}_I = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t = \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$

THE GENERAL SOLUTION IS $\underset{\sim}{x} = C_1 \underset{\sim}{x}_R + C_2 \underset{\sim}{x}_I = C_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + C_2\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$

THIS IS A center

b) LET $X_1 > 0$ AND $X_2 = 0$.

THEN $X_2' = X_1 - 2X_2 = X_1 > 0$ COUNTERCLOCKWISE



SECTION 7.6 # 5

$X' = A\underline{X}$ $\quad A = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$ $\quad \det(A - \lambda I) = 0 \rightarrow (1-\lambda)(-3-\lambda) + 5 = 0$

SO $\lambda^2 + 2\lambda + 2 = 0$.

a) SO $\lambda^2 + 2\lambda + 1 = -1 \longrightarrow \lambda_\pm + 1 = \pm i$ SO $\lambda_+ = -1 + i$.

NOW $(A - \lambda_+ I)\underline{V} = 0 \longrightarrow \begin{pmatrix} 1-(-1+i) & -1 \\ 5 & -3-(-1+i) \end{pmatrix}\underline{V} = \begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}\underline{V} = \underline{0}$.

THUS $\begin{pmatrix} 2-i & -1 \\ 0 & 0 \end{pmatrix}\underline{V} = 0$. $\longrightarrow (2-i)V_1 = V_2$, $V_1 = V_1$ LET $V_1 = 1$, $V_2 = -i + 2$

THUS $\underline{V} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$ IS A COMPLEX EIGENVECTOR.

NOW $\underline{X_R} = RE\left(\underline{V}e^{\lambda_+ t}\right) = RE\left(\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)e^{-t}(\cos t + i\sin t)\right)$

$\underline{X_I} = IM\left(\underline{V}e^{\lambda_+ t}\right) = IM\left(\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)e^{-t}(\cos t + i\sin t)\right)$

SO $\underline{X_R} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t}\cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t}\sin t = e^{-t}\begin{pmatrix} \cos t \\ \sin t + 2\cos t \end{pmatrix}$

$\underline{X_I} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin t\, e^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t}\cos t = e^{-t}\begin{pmatrix} \sin t \\ -\cos t + 2\sin t \end{pmatrix}$

THE GENERAL SOLUTION IS $\underline{X} = C_1\underline{X_R} + C_2\underline{X_I}$ SO THAT

$\underline{X} = C_1 e^{-t}\begin{pmatrix} \cos t \\ \sin t + 2\cos t \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$ STABLE SPIRAL POINT

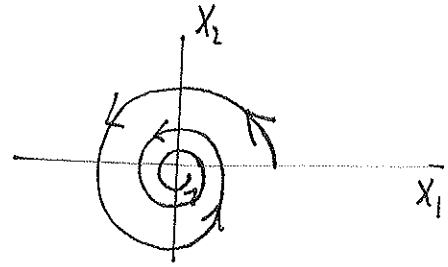
b) WE HAVE A STABLE SPIRAL.

TO DETERMINE DIRECTION OF ROTATION WE LET $X_1 > 0$ AND $X_2 = 0$.

THEN WE CALCULATE AT THIS POINT THAT

$$X_2' = 5X_1 - 3X_2 = 5X_1 > 0$$

$\longrightarrow$ COUNTERCLOCKWISE ROTATION



SECTION 7.6 # 9

SOLVE $\underline{X}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \underline{X}$ WITH $\underline{X}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\det(A - \lambda I) = 0 \longrightarrow (1-\lambda)(-3-\lambda) + 5 = 0 \longrightarrow \lambda^2 + 2\lambda + 2 = 0$

SO $\lambda^2 + 2\lambda + 1 = -1 \longrightarrow (\lambda+1)^2 = -1 \qquad \lambda_{\pm} = -1 \pm i$

- NOW $(A - \lambda_+ I)\underline{v} = 0 \rightarrow \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \underline{v} = \underline{0} \rightarrow \begin{pmatrix} 2-i & -5 \\ 0 & 0 \end{pmatrix} \underline{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

SO $(2-i)V_1 = 5V_2 \qquad$ let $V_1 = 5$

$V_2 = V_2 \qquad\qquad V_2 = 2-i$

SO $\underline{v} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$ IS A COMPLEX eigenvector

LET $\underline{X_R} = RE\left(\underline{v}\, e^{\lambda_+ t}\right) = RE\left[\left(\begin{pmatrix} 5 \\ 2 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) e^{-t}(\cos t + i \sin t)\right]$

$\underline{X_I} = IM\left(\underline{v}\, e^{\lambda_+ t}\right) = IM\left[\left(\begin{pmatrix} 5 \\ 2 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) e^{-t}(\cos t + i \sin t)\right]$

THEN $\underline{X_R} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{-t} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \sin t = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} e^{-t}$

$\underline{X_I} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{-t} \sin t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \cos t = \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} e^{-t}$

THE GENERAL SOLUTION IS $\underline{X} = C_1 \underline{X_R} + C_2 \underline{X_I}$.

NOW $\quad \underline{X} = C_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} e^{-t}$

NOW $\quad \underline{X}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

so $\quad 5C_1 = 1 \qquad C_1 = 1/5$

$2C_1 - C_2 = 1 \qquad C_2 = -3/5$

so $\quad \underline{X} = \frac{1}{5} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} e^{-t} - \frac{3}{5} \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} e^{-t} = \begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix} e^{-t}$

so $\quad \underline{X} \to 0$ AS $t \to \infty$.