MATH 215 HW 3

PLEASE TURN IN THE SOLUTIONS TO THE FOLLOWING PROBLEMS

SECTION 2.4 PAGE 76-77 : # 26, 23

SECTION 2.5 PAGE 88-93: # 3, 4, 9, 16, 17a), 18, 20

SECTION 2.9 PAGE 132-133 # 1, 22, 26.

MATH 215 HW #3

SECTION 2.4 #23

a) $y' - 2y = 0$. DEFINE $L(y) \equiv y' - 2y$.

THEN $L(e^{2t}) = (e^{2t})' - 2(e^{2t}) = 2e^{2t} - 2e^{2t} = 0$.

THUS $L(e^{2t}) = 0 \rightarrow y = e^{2t} = \phi$ IS A SOLUTION.

NOW LET $\phi(t) = e^{2t}$. SHOW $y = c\phi(t)$ IS A SOLUTION.

WE CHECK $L[c\phi(t)] = (c\phi)' - 2(c\phi) = c[\phi' - 2\phi] = 0. \checkmark$

b) NOW SHOW THAT $y = \phi(t) = 1/t$ SOLVES $y' + y^2 = 0$.

WE CALCULATE $(1/t)' + 1/t^2 = -1/t^2 + 1/t^2 = 0$.

THUS $y = \phi(t) = 1/t$ SOLVES $y' + y^2 = 0$.

NOW LET $y = c/t$. WE CALCULATE

$$(c/t)' + (c/t)^2 = -c/t^2 + c^2/t^2 = 0 \quad \text{ONLY IF } c = 0 \text{ OR } c = 1.$$

THUS $y = c/t$ IS <u>NOT</u> A SOLUTION UNLESS $c = 0$, $c = 1$.

SECTION 2.4 #26

$y' + p(t)y = g(t)$. $(*)$

WE LET $y(t) = cy_1(t) + y_2(t)$. THEN

$$(cy_1 + y_2)' + p(cy_1 + y_2) = g(t) \longrightarrow c(y_1' + py_1) + y_2' + py_2 = g(t).$$

THUS LET y_1 SATISFY $y_1' + py_1 = 0$ (HOMOGENEOUS PROBLEM)

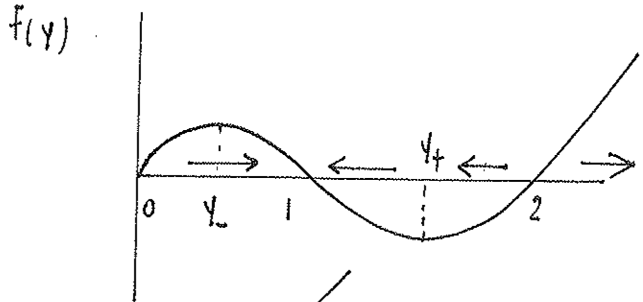
AND y_2 SATISFY $y_2' + py_2 = g$ (FULL LINEAR PROBLEM)

THEN $y = cy_1 + y_2$ SATISFIES $(*)$

SECTION 2.5 # 3

$$\frac{dy}{dt} = y(y-1)(y-2) = f(y), \quad y(0) = y_0 \qquad \frac{d^2y}{dt^2} = f'(y)\frac{dy}{dt} = f(y)\,f'(y)$$

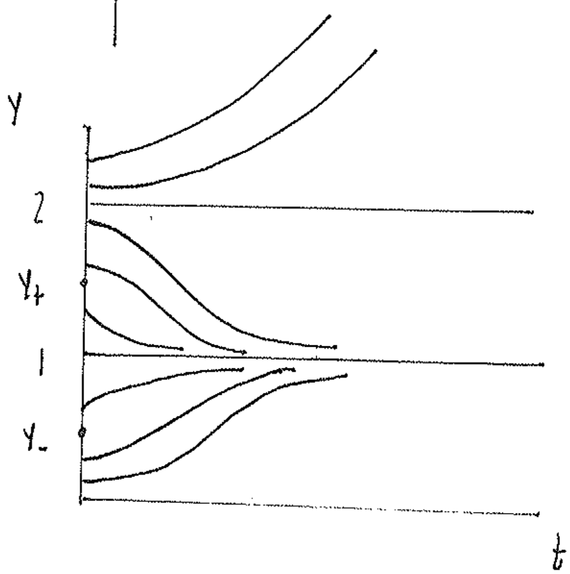
EQ. POINTS ARE AT $y_e = 0$, $y_e = 1$, $y_e = 2$.



$y_e = 0$ is unstable

$y_e = 2$ is unstable

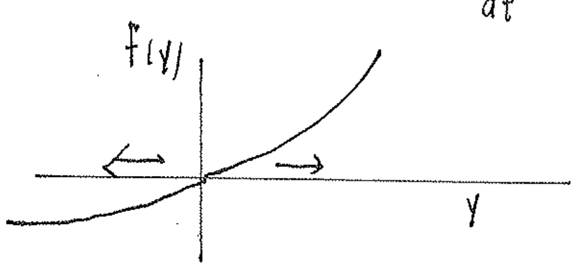
$y_e = 1$ is stable



y_0	$dy/dt = f(y)$	$d^2y/dt^2 = f(y)\,f'(y)$
$0 < y < y_-$	>0	$(>0)(>0) = (>0)$
$y_- < y < 1$	>0	$(>0)(<0) = (<0)$
$1 < y < y_+$	<0	$(<0)(<0) = (>0)$
$y_+ < y < 2$	<0	$(<0)(>0) = (<0)$
$y > 2$	>0	$(>0)(>0) = (>0)$

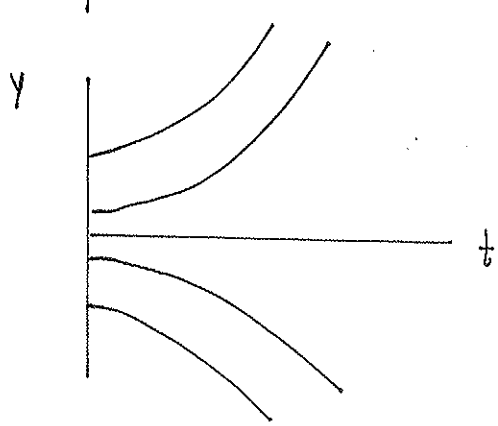
SECTION 2.5 # 4

$$\frac{dy}{dt} = e^y - 1 = f(y) \qquad \frac{d^2y}{dt^2} = f(y)\,f'(y)$$



$y_e = 0$ is EQ. POINT

$y_e = 0$ is UNSTABLE.

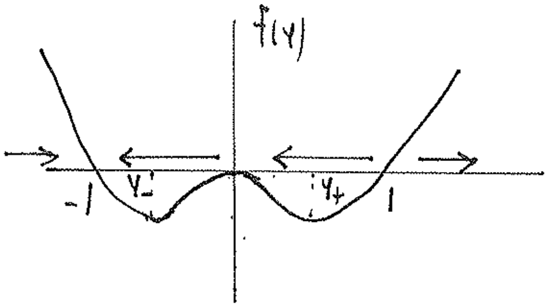


y_0	$y' = f(y)$	$y'' = f(y)\,f'(y)$
$-\infty < y_0 < 0$	<0	$(<0)(>0) = (<0)$
$0 < y_0 < \infty$	>0	$(>0)(>0) = (>0)$

SECTION 2.5 # 9

$$y' = y^2(y^2-1) = y^2(y-1)(y+1) = f(y) \qquad d^2y/dt^2 = f(y)\,f'(y)$$

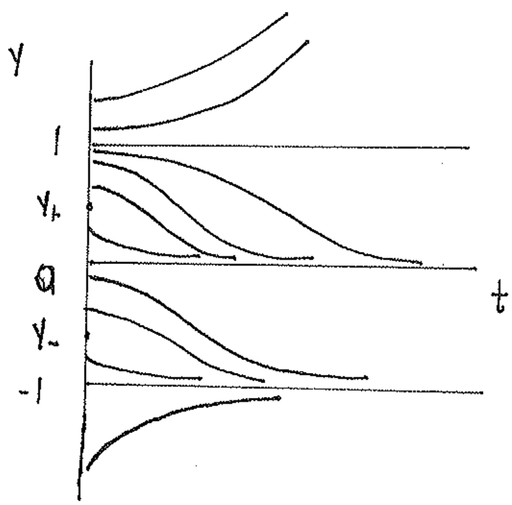
$y_e = 0, 1, -1$ ARE EQUILIBRIA



$y_e = 0$ IS SEMI-STABLE

$y_e =$ UNSTABLE

$y_e = -1$ IS STABLE.



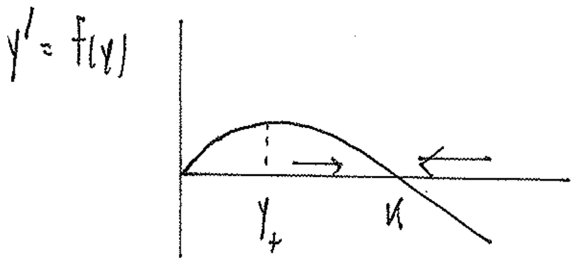
y_0	$dy/dt = f(y)$	$d^2y/dt^2 = f(y)f'(y)$
$1 < y_0 < \infty$	(>0)	$(>0)(>0) = (>0)$
$y_+ < y < 1$	(<0)	$(<0)(>0) = <0$
$0 < y < y_+$	(<0)	$(<0)(<0) = (>0)$
$y_- < y < 0$	(<0)	$(<0)(>0) = (<0)$
$-1 < y < y_-$	(<0)	$(<0)(<0) = (>0)$
$y < -1$	(>0)	$(>0)(<0) = (<0)$

SECTION 2.5 # 16

$$dy/dt = r\,y \ln(K/y) = f(y), \quad r > 0, \ K > 0.$$

a) NOW EQUILIBRIA ARE $y_e = 0$, $y_e = K$

$$d^2y/dt^2 = f(y)\,f'(y).$$



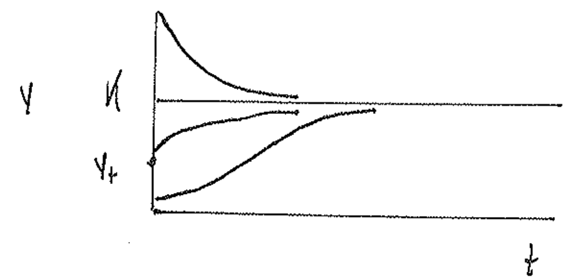
$$\lim_{y \to 0} r\,y \ln(K/y) = 0.$$

$y_e = 0$ IS UNSTABLE

$y_e = K$ IS STABLE.

b)

y_0	$y' = f(y)$	$y'' = f(y)\, f'(y)$
$0 < y_0 < y_t$	>0	$(>0)(>0) = (>0)$ concave up
$y_t < y_0 < K$	>0	$(>0)(<0) = (<0)$ concave down
$y_0 > K$	(<0)	$(<0)(<0) = (>0)$ concave up.



c) NOW LET $\left(\dfrac{dy}{dt}\right)_{GOM} = r\, y \ln\left(K/y\right)$ (GOMPERTZ)

$\left(\dfrac{dy}{dt}\right)_{LOG} = r\, y\left(1 - y/K\right)$ LOGISTIC

MUST SHOW FOR $0 < y \leq K \longrightarrow$ $\left(dy/dt\right)_{GOM} \geq \left(dy/dt\right)_{LOG}$ (TO BE SHOWN)

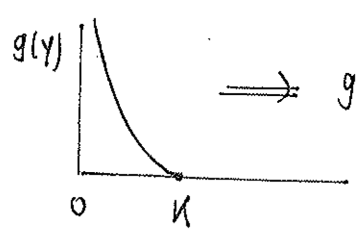
WE CALCULATE

(*) $\left(dy/dt\right)_{GOM} - \left(dy/dt\right)_{LOG} = r\, y\left[\ln\left(K/y\right) - \left(1 - y/K\right)\right]$

MUST SHOW $g(y) = \ln\left(K/y\right) - 1 + y/K \geq 0$ ON $0 \leq y \leq K$.

NOTICE $g(K) = 0$, $\lim\limits_{y\to 0} g(y) = +\infty$. $g'(y) = -\dfrac{1}{y} + \dfrac{1}{K} < 0$

ON $0 < y < K$. THUS WE HAVE



$\Longrightarrow g(y) > 0$ ON $0 < y < K$.

THEN (*) YIELDS $\left(dy/dt\right)_{GOM} - \left(dy/dt\right)_{LOG} = r\, y\, g(y) > 0$

ON $0 < y < K$.

SECTION 2.5 # 17

$$dy/dt = ry \ln(K/y) \qquad r>0,\ K>0.$$

$$y(0) = y_0.$$

a) LET $u = \ln(y/K)$ THEN $y/K = e^u$ SO $y = Ke^u$.

THUS $y' = Ku'e^u = rKe^u \ln\left(e^{-u}\right) = -rKe^u(u)$

THUS $Ku'e^u = -rKe^u u \rightarrow u' = -ru.$

HENCE $u = ce^{-rt}$.

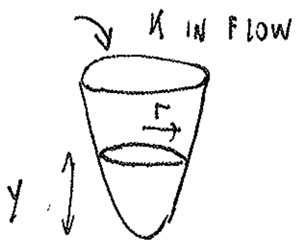
THUS $y = Ke^{ce^{-rt}}$

NOW $y(0) = y_0 \rightarrow y_0 = Ke^{c} \rightarrow c = \ln(y_0/K)$

$\rightarrow y = Ke^{\ln(y_0/K)e^{-rt}}$

SECTION 2.5 # 18

a)



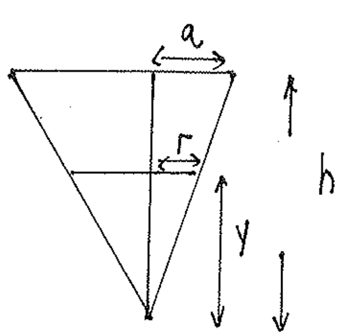
$$\frac{dV}{dt} = K - \alpha A \qquad (*)$$

$\downarrow$ EVAPORATION

where A = SURFACE AREA OF CONE.

NOW $V = \frac{\pi}{3}r^2 y \qquad A = \pi r^2.$

SIDE VIEW



NOW $a/r = h/y$ SIMILAR TRIANGLES.

NOW $y = h/a\, r$

SO $V = \frac{\pi}{3}\frac{h}{a}r^3 \rightarrow r = \left(\frac{3aV}{\pi h}\right)^{1/3}$

THUS $A = \pi r^2 \rightarrow A = \pi\left(\frac{3aV}{\pi h}\right)^{2/3} \rightarrow \frac{dV}{dt} = K - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3}V^{2/3}.$

b) EQUILIBRIUM DEPTH IS FROM

$$dV/dt = f(V) = K - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3}$$

OBTAINED BY SETTING $f(V_e) = 0$.

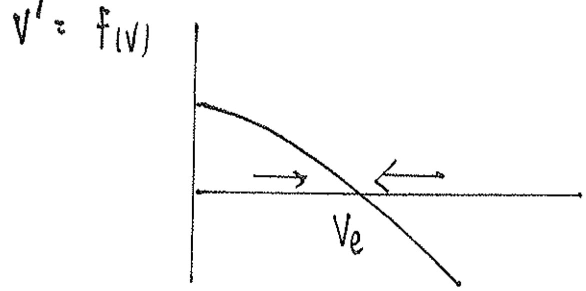
$$K = \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3} V_e^{2/3} \qquad V_e = \frac{K^{3/2}}{(\alpha\pi)^{3/2}\left(3a/\pi h\right)}$$

BUT $V_e = \frac{\pi}{3} r_e^2 y_e$ WITH $r_e = \frac{a}{h} y_e$

$$V_e = \frac{\pi}{3}\frac{a^2}{h^2} y_e^3 = \frac{K^{3/2}}{(\alpha\pi)^{3/2}}\frac{\pi h}{3a}.$$

THUS $$y_e^3 = \frac{K^{3/2}}{(\alpha\pi)^{3/2}}\frac{h^3}{a^3}.$$

HENCE $$y_e = \sqrt{\frac{K}{\alpha\pi}}\, h/a.$$



THIS EQUILIBRIUM IS STABLE

c) WE NEED $y_e < h$ FOR NO OVERFLOW

THUS $$\sqrt{\frac{K}{\alpha\pi}}\frac{h}{a} < h \longrightarrow a > \sqrt{\frac{K}{\alpha\pi}}$$ FOR NO OVERFLOW

OR EQUIVALENTLY

$K/\alpha < \pi a^2 \rightarrow$ NO OVERFLOW.

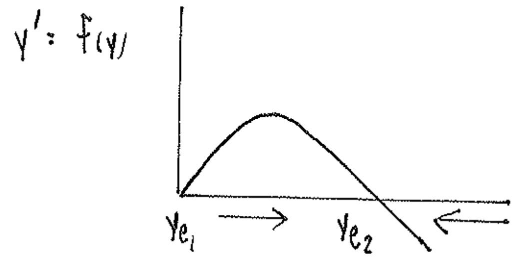
SECTION 2.5 # 20

$$dy/dt = r\left(1 - y/K\right)y - Ey = f(y). \qquad E>0,\ r>0,\ K>0.$$

a) CONSIDER $E < r$

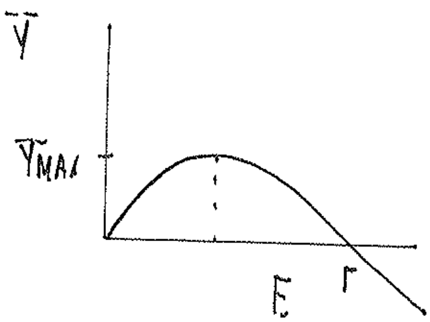
WE WRITE $\dfrac{dy}{dt} = (r-E)y - \dfrac{ry^2}{K} = f(y)$

EQUILIBRIA ARE $y_{e_1} = 0$, $y_{e_2} = \dfrac{K}{r}(r-E) > 0.$



b) y_{e_1} IS UNSTABLE, y_{e_2} IS STABLE (see arrows above)

c) BY DEFINITION $Y = E\,y_{e_2} = \dfrac{EK}{r}(r-E)$ $\qquad Y$ = yield



Y_{MAX} OCCURS AT $E = r/2$

SO $Y_{MAX} = \dfrac{EK}{r}(r-E)\Big|_{E=r/2}$

$$Y_{MAX} = \frac{r}{2}\left(\frac{K}{r}\right)\left(r/2\right) = Kr/4.$$

d) $Y_{MAX} = Kr/4$ IS MAXIMUM SUSTAINABLE YIELD (SEE c))

SECTION 2.9 # 1

$y' = (x^3 - 2y)/x$

so $y' + \frac{2}{x} y = x^2$ LINEAR ODE

INTEGRATING FACTOR $\phi(x) = e^{\int^x 2/x \, dx} = e^{2 \ln x} = x^2$

so $(x^2 y)' = x^4 \longrightarrow x^2 y = \frac{1}{5} x^5 + C.$

THUS $y = \frac{1}{5} x^3 + \frac{C}{x^2}$

SECTION 2.9 # 22

$\frac{dy}{dx} = \frac{x^2 - 1}{y^2 + 1}, \quad y(-1) = 1.$

SEPARABLE $\longrightarrow (y^2 + 1)\, dy = (x^2 - 1)\, dx$

so $y^3/3 + y = x^3/3 - x + C.$

NOW $y(-1) = 1 \longrightarrow 1/3 + 1 = -1/3 + 1 + C \rightarrow 4/3 - 2/3 = C.$

THUS $C = 2/3.$

so $y^3/3 + y = x^3/3 - x + 2/3 \rightarrow y^3 + 3y = x^3 - 3x + 2.$

SECTION 2.9 # 26

$x y' = y + x e^{y/x}$

so $y' = \frac{y}{x} + e^{y/x} = f(y/x).$

LET $v = y/x$. THEN $y = xv \longrightarrow y' = xv' + v$

so $xv' + v = v + e^v \longrightarrow xv' = e^v.$ $\quad x \frac{dv}{dx} = e^v$

THUS $e^{-v} dv = dx/x \longrightarrow -e^{-v} = \ln|x| + C.$

THUS $e^{-y/x} + \ln|x| = -C.$