PROBLEM 1: (10 Points). Solve the initial value problem

$$y'' + 2y' + 2y = 0\,, \quad y(\pi/4) = 2\,, \quad y'(\pi/4) = -2\,.$$

PROBLEM 2: (15 Points). The problem

$$x^2y'' + 3xy' + y = 0$$

has a solution of the form $y = x^p$ for some value of p. Find p and then determine the general solution using reduction of order.

PROBLEM 3: (25 Points). Find the solution to

$$y'' + py' + y = 1 + e^{-x}\,, \qquad y(0) = y'(0) = 0$$

Account for all values of p with p real and $p \geq 0$ (There are several cases to consider). Calculate $\lim_{x\to\infty} y(x)$ in each case. Draw a plot of y versus x when $p = 0$.

PROBLEM 1: (15 Points). Find the general solution to

$$y' - \frac{p}{x}y = x\,, \qquad \text{for } x \geq 0\,.$$

Here p is a constant. For what values of p is the solution undefined at $x = 0$?

PROBLEM 2: (15 Points). Solve the initial value problem

$$\frac{dy}{dx} = \frac{x(y-1)^2}{x-1}\,, \quad y(2) = 2\,.$$

What is the solution when the initial condition is changed to $y(2) = 1$?

PROBLEM 4: (10 Points). Consider the differential equation

$$\frac{dy}{dt} = -y(y^4 - 4y^2 + 4)\,.$$

Calculate $\lim_{t\to\infty} y(t)$ for each of the three initial conditions: $y(0) = 2$, $y(0) = -1/2$ and $y(0) = -4$.

PROBLEM 3: (15 Points). Water flows into a conical water tank of radius a and depth h at a constant rate k (with $k > 0$). Water is lost by evaporation at a rate proportional to the area of the exposed surface (see the picture below).

i) Derive an ODE for the depth $y(t)$ of water in the tank.

ii) Find the equilibrium level of water in the tank.

iii) The water will not overflow out of the tank when $k < k_c$, where k_c is some critical value. Calculate k_c explicitly.