MATH 215, SECTION 202, MIDTERM II: MARCH 19 2012 (H. DIXIT)

Closed Book and Notes. 50 Minutes. Total 50 points

Problem 1: [10 points] Verify that $y_1(t) = (1 + t)$ and $y_2(t) = e^t$ are solutions of the homogeneous part of the following differential equation,

$$ty'' - (1 + t)y' + y = t^2 e^{2t}, \qquad t > 0,$$

and using variation of parameters, find the particular solution of the inhomogenous equation.

Problem 2: [10 points] Consider the following differential equation with discontinuous forcing:

$$y'' + 2y' + 8y = \begin{cases} e^{2t} & 0 < t < 1, \\ 0 & t \geq 1, \end{cases} \quad ; \quad y(0) = y'(0) = 0,$$

i) Calculate the solution $y(t)$ using Laplace transforms. [8 points]

ii) Obtain $\lim_{t\to\infty} y(t)$ and show your work. [2 points]

Problem 3: [20 points] Consider the following differential equation for a spring-mass-damper system with damping rate $\gamma \geq 2$:

$$y'' + \gamma y' + y = \delta(t - 1), \quad y(0) = y'(0) = 0$$

i) Using Laplace transforms, find the solution $y(t)$ assuming $\gamma > 2$. [10 points]

ii) Solve the differential again with $\gamma = 2$. [5 points]

iii) Show that you can recover the solution of (ii) by using (i) as $\gamma \to 2$. (Hint: Will require l'Hospital's rule). [5 points]

Problem 4: [10 points] Consider the following second order differential equation for $y(t)$ with $t > 0$:

$$y'' - 9y = 1 + e^{-2t}.$$

i) Obtain the general solution. [6 points]

ii) If $y(0) = \alpha, y'(0) = 0$, then determine the value of α such that $\lim_{t\to\infty} y'(t) = 0$. (Hint: You can use the general solution obtained in (i) or use a Laplace Transform approach). [4 points]

SOLUTIONS TO MIDTERM II

(1) $y_1(t) = 1+t$

$y_2(t) = e^t$

$$ty'' - (1+t)\,y' + y = t^2 e^{2t} \qquad t > 0$$

(A) $y_1(t) = 1+t$

If y_1 is a solution of homogeneous part, then

$L[y_1] = 0$ where

$$L[y] = y'' - \frac{(1+t)}{t}\,y' + \frac{y}{t}$$

$y_1' = 1$

$y_1'' = 0$

$$\Rightarrow L[y_1] = 0 - \frac{1+t}{t} \times 1 + \frac{1+t}{t} = 0.$$

$\Rightarrow$ $y_1(t)$ is a solution of homogenous part.

Similarly; $y_2' = e^t$

$y_2'' = e^t$

$$\Rightarrow L[y_2] = e^t - \left(\frac{1+t}{t}\right) e^t + \frac{e^t}{t}$$

$$= e^t \left[\frac{t-1-t+1}{t}\right] = 0$$

$\Rightarrow$ $y_2(t)$ is also a solution of homogeneous part.

Wronskian of $y_1(t)$ & $y_2(t)$:

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = te^t \neq 0$$

for $t > 0$.

Particular Solution: From Theorem (3.6.1):

$$y_p(t) = -y_1(t)\int^t \frac{y_2(s)\, g(s)}{W[y_1, y_2](s)}\, ds + y_2(t)\int^t \frac{y_1(s)\, g(s)}{W[y_1, y_2](s)}\, ds$$

where $g(t)$ is the inhomogeneous part of

$$y'' + py' + q\, y = g(t)$$

$$\Rightarrow \quad g(t) = te^{2t}$$

$$\Rightarrow \quad y_p(t) = -y_1(t)\int^t \frac{e^s \cdot se^{2s}}{se^s}\, ds + y_2(t)\int^t \frac{(1+s)\, se^{2s}}{se^s}\, ds$$

$$= -y_1(t)\int^t e^{2s}\, ds + y_2(t)\int^t (1+s)\, e^s\, ds$$

$$= -y_1(t)\, \frac{e^{2s}}{2}\Big]^t + y_2(t)\left\{(1+s)\, e^s\Big]^t - \int^t 1 \cdot e^s\, ds\right\}$$

$$= -y_1(t)\, \frac{e^{2t}}{2} + y_2(t)\left\{(1+t)\, e^t - e^t\right\}$$

$$= -y_1(t)\, \frac{e^{2t}}{2} + y_2(t)\, te^t$$

$$= -(1+t)\frac{e^{2t}}{2} + e^t \cdot te^t$$

$$\Rightarrow \quad y_p(t) = e^{2t}\left[t - \frac{t+1}{2}\right]$$

$$\boxed{y_p(t) = \left(\frac{t-1}{2}\right)e^{2t}}$$

<u>CHECK!</u> If $y_p(t)$ is a solution, then

$$L[y_p] = te^{2t}$$

$$y_p' = \left(\frac{t-1}{2}\right)2e^{2t} + e^{2t}\cdot\frac{1}{2} = (t-1)e^{2t} + \frac{1}{2}e^{2t}$$

$$y_p'' = (t-1)2e^{2t} + e^{2t} + \frac{1}{2}\cdot 2e^{2t} = 2(t-1)e^{2t} + 2e^{2t}$$

$$\Rightarrow \quad L[y_p] = y_p'' - \left(\frac{1+t}{t}\right)y_p' + \frac{y_p}{t}$$

$$= 2(t-1)e^{2t} + 2e^{2t} - \left(\frac{1+t}{t}\right)\left[(t-1)e^{2t} + \frac{1}{2}e^{2t}\right] + \left(\frac{t-1}{2t}\right)e^{2t}$$

$$= 2te^{2t} - \underbrace{\left(\frac{t^2-1}{t}\right)e^{2t}} - \underbrace{\frac{t+1}{2t}e^{2t} + \frac{t-1}{2t}e^{2t}}$$

$$= \underbrace{2te^{2t} - te^{2t}} + \frac{1}{t}e^{2t} + \left(\frac{-1}{t}\right)e^{2t}$$

$$= te^{2t}$$

$$= g(t)$$

2 $\quad y'' + 2y' + 8y = \begin{cases} e^{2t} & 0 < t < 1 \\ 0 & t \geq 1 \end{cases}$

$y(0) = y'(0) = 0$

A) Let $g(t) = \begin{cases} e^{2t} & 0 < t < 1 \\ 0 & t \geq 1 \end{cases}$

$$\mathcal{L}\{g(t)\} = \int_0^\infty g(t) e^{-st}\,dt = \int_0^1 e^{2t} e^{-st}\,dt + \int_1^\infty 0 \cdot e^{-st}\,dt$$

$$= \int_0^1 e^{-(s-2)t}\,dt = \left.\frac{e^{-(s-2)t}}{-(s-2)}\right]_0^1$$

$$= \frac{-1}{s-2}\left[e^{-(s-2)} - 1\right] = \frac{1}{s-2} - \frac{e^{-(s-2)}}{s-2}$$

Alternate way!

$g(t) = e^{2t} - e^{2(t-1)} e^{2} u_1(t)$

CHECK!
$t < 1$; $u_1(t) = 0$
$\Rightarrow g(t) = e^{2t}$
$t \geq 1$; $g(t) = e^{2t} - e^{2} e^{2(t-1)}$
$= e^{2t} - e^{2} e^{2t} e^{-2}$
$= 0$

$$\therefore \mathcal{L}\{g(t)\} = \mathcal{L}\{e^{2t}\} - e^{2}\mathcal{L}\{u_1(t)\, e^{2(t-1)}\}$$

$\hookrightarrow \mathcal{L}\{u_c(t) f(t-c)\}$
$= e^{-cs} F(s)$

Here $c = 1$; $f(t) = e^{2t}$
$\Rightarrow f(t-1) = e^{2(t-1)}$
$\Rightarrow F(s) = \frac{1}{s-2}$

$$\Rightarrow \mathcal{L}\{g(t)\} = \frac{1}{s-2} - e^{2} e^{-s} \frac{1}{s-2}$$

Taking Laplace transform of the differential equation,

$$[s^2 Y(s) - s\,y(0) - y'(0)] + 2[s\,Y(s) - y(0)] + 8\,Y(s) = \mathcal{L}\{g(t)\}$$

$\because$ $y(0) = y'(0) = 0$, we get

$$(s^2 + 2s + 8)\,Y(s) = \frac{1}{s-2} - e^2 e^{-s}\frac{1}{s-2}$$

$$\Rightarrow \quad Y(s) = \frac{1}{(s-2)(s^2+2s+8)} - e^2 e^{-s}\frac{1}{(s-2)(s^2+2s+8)}$$

$$= H(s) - e^2 e^{-s} H(s)$$

where $H(s) = \dfrac{1}{(s-2)(s^2+2s+8)} = \dfrac{A}{s-2} + \dfrac{Bs+C}{s^2+2s+8}$

$$A = \frac{1}{16}, \quad B = \frac{-1}{16}, \quad C = -\frac{4}{16}$$

$$\Rightarrow \quad H(s) = \frac{1}{16}\left[\frac{1}{s-2} - \frac{s+4}{s^2+2s+8}\right]$$

$$= \frac{1}{16}\left[\frac{1}{s-2} - \frac{s+4}{(s+1)^2 + (\sqrt{7})^2}\right] \qquad [s+4 = (s+1)+3]$$

$$= \frac{1}{16}\left[\frac{1}{s-2} - \frac{s+1}{(s+1)^2 + (\sqrt{7})^2} - \frac{3}{\sqrt{7}}\frac{\sqrt{7}}{(s+1)^2+(\sqrt{7})^2}\right]$$

$$\Rightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{16}\left[e^{2t} - e^{-t}\cos(\sqrt{7}\,t) - \frac{3}{\sqrt{7}}e^{-t}\sin(\sqrt{7}\,t)\right]$$

$$Y(s) = H(s) - e^{2} e^{-s} H(s)$$

$$\therefore \; y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{H(s)\} - e^{2}\, \mathcal{L}^{-1}\{e^{-s} H(s)\}$$

$$\mathcal{L}^{-1}\{e^{-s} H(s)\} = u_1(t)\, h(t-1)$$ (FORMULA 13, Table 6.2.1)

$$\Rightarrow \quad y(t) = h(t) - e^{2} u_1(t)\, h(t-1)$$

where $h(t) = \frac{1}{16}\left\{ e^{2t} - e^{-t}\cos(\sqrt{7}\,t) - \frac{3}{\sqrt{7}} e^{-t}\sin(\sqrt{7}\,t)\right\}$

As $t \to \infty$: The e^{-t} terms vanish as $t \to \infty$.

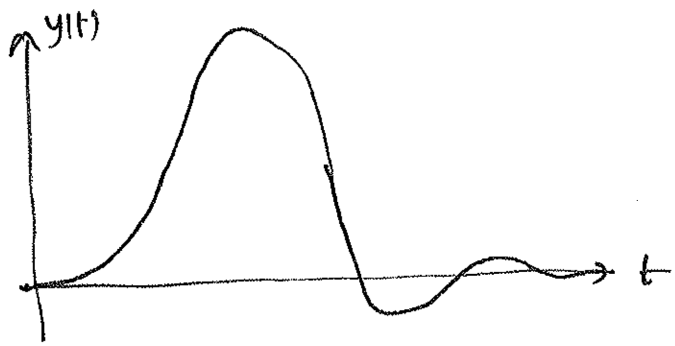
$$\therefore \lim_{t\to\infty} h(t) \approx \lim_{t\to\infty} \frac{1}{16} e^{2t}$$

$$\therefore \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[h(t) - e^{2} u_1(t)\, h(t-1)\right]$$

$$\approx \lim_{t\to\infty}\left[\frac{1}{16}e^{2t} - e^{2}\cdot 1 \cdot \frac{1}{16} e^{2(t-1)}\right]$$ (Since $u_1(t) = 1$ as $t \to \infty$)

$$= \lim_{t\to\infty} \frac{1}{16}\left(e^{2t} - e^{2t}\right) = 0$$

PLOT OF $y(t)$:



(7)

(3) $y'' + \gamma y' + y = \delta(t-1)$

$y(0) = y'(0) = 0$

Ans) There are three cases in this problem. Though only $\gamma \geq 2$ is part of the midterm, we'll look at all the three cases here.

(i) Taking Laplace Transform of the equation, we get

$$\left[s^2 Y(s) - s y(0) - y'(0)\right] + \gamma\left[s Y(s) - y(0)\right] + Y(s) = e^{-s}$$

where $Y(s) = \mathcal{L}\{y(t)\}$

and $\mathcal{L}\{\delta(t-1)\} = e^{-s}$

Since $y(0) = y'(0) = 0$, we get

$$(s^2 + \gamma s + 1)\, Y(s) = e^{-s}$$

$$\boxed{Y(s) = \frac{e^{-s}}{s^2 + \gamma s + 1}}$$

Case I: $\underline{\gamma \leq 2}$:

$$s^2 + \gamma s + 1 = s + 2 \cdot \frac{\gamma}{2} \cdot s + \left(\frac{\gamma}{2}\right)^2 + 1 - \left(\frac{\gamma}{2}\right)^2 = \left(s + \frac{\gamma}{2}\right)^2 + \left(1 - \frac{\gamma^2}{4}\right)$$

$$Y(s) = \frac{e^{-s}}{\left(s + \frac{\gamma}{2}\right)^2 + \left(1 - \frac{\gamma^2}{4}\right)}$$

Multiplying and dividing by $\sqrt{1 - \frac{\gamma^2}{4}}$, we get

$$Y(s) = \frac{1}{\sqrt{1 - \frac{\gamma^2}{4}}} \; \frac{\sqrt{1 - \frac{\gamma^2}{4}}\; e^{-s}}{\left(s + \frac{\gamma}{2}\right)^2 + \left(\sqrt{1 - \frac{\gamma^2}{4}}\right)^2}$$

$$\boxed{Y(s) = \frac{1}{\sqrt{1 - \gamma^2/4}}\, e^{-s} F(s)}$$

where $F(s) = \dfrac{\sqrt{1-r^2/4}}{\left(s+\frac{r}{2}\right)^2+\left(\sqrt{1-r^2/4}\right)^2}$ $(c=1)$

Using formula (13), we get

$$y(t) = \frac{1}{\sqrt{1-\frac{r^2}{4}}}\, u_1(t)\, f(t-1)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-\frac{rt}{2}} \sin\left[\sqrt{1-\frac{r^2}{4}}\, t\right]$

<u>case II</u>: <u>$r = 2$</u>:

The differential equation becomes

$y'' + 2y' + y = \delta(t-1)$ with $y(0) = y'(0) = 0$

Taking Laplace Transform, we get

$(s^2+2s+1)\, Y(s) = e^{-s} \Rightarrow Y(s) = \dfrac{e^{-s}}{(s^2+2s+1)}$

$$\Rightarrow Y(s) = \frac{e^{-s}}{(s+1)^2} = e^{-s} H(s)$$

$\therefore y(t) = u_1(t)\, h(t-1)$ where $h(t) = \mathcal{L}^{-1}\{H(s)\}$ [Using formula (13)]

$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\{F(s+1)\}$

where $F(s) = \dfrac{1}{s^2} \Rightarrow F(s+1) = \dfrac{1}{(s+1)^2}$

from formula (14); $\mathcal{L}^{-1}\{F(s-c)\} = e^{ct} f(t)$

$\therefore \mathcal{L}^{-1}\{F(s+1)\} = e^{-t} f(t)$ where

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$ (Using formula (3)).

$\therefore \mathcal{L}^{-1}\{F(s+1)\} = e^{-t} t = h(t)$

$$\therefore \quad y(t) = u_1(t)\, h(t-1)$$

$$\Rightarrow \quad y(t) = u_1(t)\, e^{-(t-1)}\,(t-1)$$

<u>Case III</u>: $\underline{\gamma > 2}$:

Now $\quad s^2 + \gamma s + 1 = (s-a)(s-b) \quad$ where

$$a = \frac{-\gamma + \sqrt{\gamma^2 - 4}}{2} \; ; \; b = \frac{-\gamma - \sqrt{\gamma^2-4}}{2}$$

$$\therefore \quad Y(s) = \frac{e^{-s}}{(s-a)(s-b)} = e^{-s}\left[\frac{A}{s-a} + \frac{B}{s-b}\right] \; ; \; A = -B = \frac{1}{\sqrt{\gamma^2 - 4}}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s-a}\right\} = \mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} \quad \text{where } c = 1, \; F(s) = \frac{1}{s-a}$$

$$= u_1(t)\, f(t-1) \quad \text{where } f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$= \mathcal{L}^{-1}\{G(s-a)\}$$

$$= e^{at} g(t)$$

$$\text{where } g(t) = \mathcal{L}^{-1}\{G(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\Rightarrow f(t) = e^{at}$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s-a}\right\} = u_1(t)\, e^{a(t-1)}$$

$$\text{and } \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s-b}\right\} = u_1(t)\, e^{b(t-1)}$$

$$\therefore y(t) = \mathcal{L}^{-1}\{Y(s)\} = A\, u_1(t)\, e^{a(t-1)} + B\, u_1(t)\, e^{b(t-1)}$$

$$= \frac{u_1(t)}{\sqrt{\gamma^2-4}}\left[e^{a(t-1)} - e^{b(t-1)}\right]$$

LIMITING CASE OF $\gamma \to 2$:-

FROM $\gamma < 2$ to $\gamma = 2$:-

for $\gamma < 2$; $y(t) = \dfrac{1}{\sqrt{1-\frac{\gamma^2}{4}}} u_1(t) f(t-1)$

where $f(t) = e^{-\frac{\gamma t}{2}} \sin\left[\sqrt{1-\frac{\gamma^2}{4}}\, t\right]$

$$\therefore y(t) = \frac{1}{\sqrt{1-\frac{\gamma^2}{4}}} u_1(t)\, e^{-\frac{\gamma}{2}(t-1)} \sin\left[\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)\right]$$

As $\gamma \to 2$ from below ; $\sqrt{1-\frac{\gamma^2}{4}} \to 0$

$\Rightarrow \sin\left[\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)\right] \to 0$

Using L'Hospital's rule (differentiating numerator and denominator with respect to γ),

$$\lim_{\gamma\to 2} y(t) = \lim_{\gamma\to 2} \frac{u_1(t)\left\{e^{-\frac{\gamma}{2}(t-1)}\left[-\frac{(t-1)}{2}\right]\sin\left[\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)\right] + e^{-\frac{\gamma}{2}(t-1)}\cos\left[\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)\right] \times (t-1)\frac{d}{d\gamma}\left(\sqrt{1-\frac{\gamma^2}{4}}\right)\right\}}{\frac{d}{d\gamma}\left(\sqrt{1-\frac{\gamma^2}{4}}\right)}$$

As $\gamma \to 2$; the first term in the numerator vanishes.

$$\therefore \lim_{\gamma\to 2} y(t) = \lim_{\gamma\to 2} \frac{u_1(t)\left\{0 + e^{-\frac{\gamma}{2}(t-1)} \times 1 \times (t-1)\,\frac{d}{d\gamma}\left(\sqrt{1-\frac{\gamma^2}{4}}\right)\right\}}{\frac{d}{d\gamma}\left(\sqrt{1-\frac{\gamma^2}{4}}\right)}$$

$= u_1(t)\, e^{-(t-1)}(t-1)$: Same as the result obtained with $\gamma = 2$

FROM $\gamma > 2$ to $\gamma = 2$:-

for $\gamma > 2$: $y(t) = \dfrac{u_1(t)}{\sqrt{\gamma^2-4}}\left[e^{a(t-1)} - e^{b(t-1)}\right]$

$$\Rightarrow y(t) = \frac{u_1(t)}{\sqrt{\gamma^2-4}}\left\{e^{\left(\frac{-\gamma+\sqrt{\gamma^2-4}}{2}\right)(t-1)} - e^{\left(\frac{-\gamma-\sqrt{\gamma^2-4}}{2}\right)(t-1)}\right\}$$

$$= \frac{u_1(t)\, e^{-\frac{\gamma}{2}(t-1)}}{\sqrt{\gamma^2-4}}\left\{e^{\frac{1}{2}\sqrt{\gamma^2-4}\,(t-1)} - e^{-\frac{1}{2}\sqrt{\gamma^2-4}\,(t-1)}\right\}$$

Differentiating numerator and denominator with respect to γ, we get

$$y(t) = \lim_{\gamma\to 2} \frac{u_1(t)\, e^{-\frac{\gamma}{2}(t-1)}\left\{\frac{1}{2}(t-1)e^{\frac{1}{2}\sqrt{\gamma^2-4}(t-1)}\frac{d}{d\gamma}\left(\sqrt{\gamma^2-4}\right) + \frac{1}{2}(t-1)\, e^{-\frac{1}{2}\sqrt{\gamma^2-4}\,(t-1)}\frac{d}{d\gamma}\left(\sqrt{\gamma^2-4}\right)\right\} + u_1(t)\, e^{-\frac{\gamma}{2}(t-1)}\left\{-\frac{1}{2}(t-1)\right\}\left\{e^{\frac{1}{2}\sqrt{\gamma^2-4}(t-1)} - e^{-\frac{1}{2}\sqrt{\gamma^2-4}\,(t-1)}\right\}}{\frac{d}{d\gamma}\left(\sqrt{\gamma^2-4}\right)}$$

The second term vanishes as $\gamma \to 2$.

$$\therefore y(t) = \lim_{\gamma\to 2} \frac{u_1(t)\, e^{-\frac{\gamma}{2}(t-1)} \cdot \frac{1}{2}(t-1)\left\{e^0 \cdot \frac{d}{d\gamma}\sqrt{\gamma^2-4} + e^0 \cdot \frac{d}{d\gamma}\sqrt{\gamma^2-4}\right\}}{\frac{d}{d\gamma}\sqrt{\gamma^2-4}}$$

$$= u_1(t)\, e^{-(t-1)}(t-1)$$: Same as the result obtained with $\gamma = 2$

(4) $y'' - 9y = 1 + e^{-2t}$; $t > 0$

(i) This is an inhomogeneous equation with constant coefficients.

<u>Homogeneous part</u>:- $y'' - 9y = 0$

Let $y = e^{rt}$ $\Rightarrow$ $r^2 - 9 = 0$

$\Rightarrow$ $r = \pm 3$

$\therefore$ $y_H = c_1 e^{3t} + c_2 e^{-3t}$ (Homogeneous Solution)

<u>Particular Solution</u>:- Let $y_p = A + Be^{-2t}$

$y_p' = -2Be^{-2t}$

$y_p'' = 4Be^{-2t}$

$\therefore$ $4Be^{-2t} - 9(A + Be^{-2t}) = 1 + e^{-2t}$

$\Rightarrow$ $-9A = 1$ $\Rightarrow$ $A = -1/9$

$4B - 9B = 1$ $\Rightarrow$ $B = -1/5$

$\therefore$ $y_p = \frac{-1}{9} - \frac{1}{5}e^{-2t}$

$\therefore$ $\boxed{y(t) = y_H + y_p = c_1 e^{3t} + c_2 e^{-3t} - \frac{1}{9} - \frac{1}{5}e^{-2t}}$ (*)

(ii) $y(0) = \alpha$

$y'(0) = 0$

Differentiating the solution (*), we get

$$y'(t) = 3c_1 e^{3t} - 3c_2 e^{-3t} + \frac{2}{5} e^{-2t}$$

$\underline{y(0) = \alpha}$: $\quad \alpha = c_1 + c_2 - \frac{1}{9} - \frac{1}{5}$

$\underline{y'(0) = 0}$: $\quad 0 = 3c_1 - 3c_2 + \frac{2}{5}$

$$\Rightarrow \quad c_1 + c_2 = \alpha + \frac{14}{45}$$
$$3c_1 - 3c_2 = -\frac{2}{5}$$

Solving for c_1 & c_2, we get

$$6c_1 = 3\alpha + \frac{24}{45}$$

$$\Rightarrow c_1 = \frac{\alpha}{2} + \frac{4}{45}$$

$$\Rightarrow c_2 = \frac{\alpha}{2} + \frac{10}{45}$$

As $t \to \infty$; $\quad e^{-3t} \to 0$
$e^{-2t} \to 0$
$e^{3t} \to \infty$.

$\therefore$ For $y'(t) \to 0$, we require $c_1 = 0$

$\therefore \quad \frac{\alpha}{2} + \frac{4}{45} = 0 \quad \Rightarrow \boxed{\alpha = \frac{-8}{45}}$

(Note that c_2 is not necessary and need not be evaluated).

(4) Using Laplace Transforms:- (REPEAT)

$$y'' - 9y = 1 + e^{-2t}$$

$$y(0) = \alpha$$

$$y'(0) = 0$$

Taking Laplace transform of the equation, we get

$$\left[s^2 Y(s) - s y(0) - y'(0)\right] + 9\, Y(s) = \mathcal{L}\{1\} + \mathcal{L}\{e^{-2t}\}$$

$$\Rightarrow (s^2 - 9)\, Y(s) - s\alpha = \frac{1}{s} + \frac{1}{s+2}$$

$$\Rightarrow (s^2 - 9)\, Y(s) = s\alpha + \frac{2s+2}{s(s+2)}$$

$$\therefore Y(s) = \frac{s\alpha}{s^2 - 9} + \frac{2s+2}{s(s+2)(s^2-9)}$$

$$\Rightarrow Y(s) = \frac{s\alpha \cdot s(s+2) + 2s+2}{s(s+2)\cdot(s^2-9)}$$

Since $s^2 - 9 = (s+3)(s-3)$, we get

$$Y(s) = \frac{s^2\alpha(s+2) + 2s+2}{s(s+2)(s-3)(s+3)} = \frac{P(s)}{Q(s)}$$

Using partial fractions, $Y(s)$ will take the form

$$Y(s) = \frac{A}{s+3} + \frac{B}{s-3} + \frac{C}{s} + \frac{D}{s+2}$$

Therefore, the solution $y(t)$ will be of the form

$$y(t) = Ae^{-3t} + Be^{3t} + C + De^{-2t}$$

$$\therefore \; y'(t) = -3Ae^{-3t} + 3Be^{3t} - 2De^{-2t}$$

$\therefore$ As $t \to \infty$; $\quad y'(t) \to 3Be^{3t}$

for $y'(t) \to 0$ as $t \to \infty$, we require $\boxed{B = 0}$

Multiplying $Y(s)$ by $(s-3)$, we get

$$(s-3)Y(s) = (s-3)\frac{P(s)}{Q(s)} = (s-3)\frac{A}{(s+3)} + B + (s-3)\frac{C}{s} + (s-3)\frac{D}{s+2}$$

$$\therefore \lim_{s\to 3} (s-3)\frac{P(s)}{Q(s)} = B$$

But $Q(s) = s(s+2)(s-3)(s+2)$

$\therefore \; Q(3) = 0$

$$\therefore \lim_{s\to 3} (s-3)\frac{P(s)}{Q(s)} = \lim_{s\to 3} (s-3)\frac{P(s)}{Q(s) - Q(3)} = B$$

We have subtracted a zero which makes no difference.

$$\Rightarrow \lim_{s\to 3} \frac{P(s)}{\underbrace{\left[\frac{Q(s) - Q(3)}{(s-3)}\right]}_{\text{same as } Q'(3)}} = B$$

$$\therefore \quad B = \frac{P(3)}{Q'(3)}$$

for $B = 0$, we want $P(3) = 0$

But $P(s) = s^2\alpha(s+2) + 2s+2$

$$\therefore \quad P(3) = 9\alpha(5) + 6 + 2 = 0$$

$$\Rightarrow \quad 45\alpha + 8 = 0$$

$$\Rightarrow \quad \boxed{\alpha = \frac{-8}{45}}$$

This is the same result we got using our previous approach.