PROBLEM 3: (20 Points)

By using Laplace transforms, find the solution to the initial value problem

$$y'' - 2y' + y = \begin{cases} 0 & 0 \leq t < 1 \\ t & 1 \leq t < 2 \\ 0 & 2 \leq t < \infty \end{cases} \qquad y(0) = 0\,, \quad y'(0) = 1\,.$$

PROBLEM 3

USING LAPLACE TRANSFORMS SOLVE

$$y'' - 2y' + y = \begin{cases} 0 & 0 \le t < 1 \\ t & 1 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases} \qquad y(0) = 0 \quad y'(0) = 1$$

$$y'' - 2y' + y = t\,u_1(t) - t\,u_2(t) = [(t-1)+1]\,u_1(t) - [(t-2)+2]\,u_2(t)$$

TAKE LAPLACE TRANSFORMS

$$(s^2 - 2s + 1)Y(s) = 1 + \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} - \left(\frac{1}{s^2} + \frac{2}{s}\right)e^{-2s}$$

$$(1) \qquad Y(s) = \frac{1}{(s-1)^2} + \frac{(1+s)}{s^2(s-1)^2}e^{-s} - \frac{(1+2s)}{s^2(s-1)^2}e^{-2s}$$

LET $\quad H_1(s) = \dfrac{(1+s)}{s^2(s-1)^2} \qquad H_2(s) = \dfrac{(1+2s)}{s^2(s-1)^2}$

EVALUATE $\quad h_1(t) = \mathcal{L}^{-1}[H_1(s)]$

$$\frac{(1+s)}{s^2(s-1)^2} = \frac{As+B}{s^2} + \frac{C(s-1)+D}{(s-1)^2} = \frac{(As+B)(s-1)^2 + (C(s-1)+D)s^2}{s^2(s-1)^2}$$

$$1 + s = (As+B)(s-1)^2 + (C(s-1)+D)s^2$$

let $s = 0$: $B = 1$

$s = 1$: $D = 2$

EQUATE POWERS OF s^3 : $\quad 0 = A + C \qquad A + C = 0$

" " " s^2 : $\quad 0 = B - 2A + D - C \qquad 2A + C = 3$

$\Rightarrow A = 3 \quad C = -3$

$$\frac{1+s}{s^2(s-1)^2} = \frac{3}{s} + \frac{1}{s^2} - \frac{3}{(s-1)} + \frac{2}{(s-1)^2}$$

$$(2) \qquad h_1(t) = \mathcal{L}^{-1}[H_1(s)] = 3 + t - 3e^t + 2te^t$$

EVALUATE $h_2(t) = \mathcal{L}^{-1}[H_2(s)]$

$$\frac{(1+2s)}{s^2(s-1)^2} = \frac{As+B}{s^2} + \frac{C(s-1)+D}{(s-1)^2}$$

$$1+2s = (As+B)(s-1)^2 + \big(C(s-1)+D\big)s^2$$

LET $s=0$: $B=1$

$s=1$: $D=3$

EQUATE POWERS OF s^3 : $0 = A + C$ $\quad A+C=0 \quad A=4 \quad C=-4$

" " " s^2 : $0 = B-2A+D-C \quad 2A+C = B+D = 4$

SO $$H_2(s) = \frac{4}{s} + \frac{1}{s^2} - \frac{4}{(s-1)} + \frac{3}{(s-1)^2}$$

(3) $$h_2(t) = 4 + t - 4e^t + 3te^t$$

FROM (1) WE HAVE $$Y(s) = \frac{1}{(s-1)^2} + H_1(s)e^{-s} - H_2(s)e^{-2s}$$

SO $$y(t) = te^t + u_1(t)\,h_1(t-1) - u_2(t)\,h_2(t-2)$$

FROM (2) AND (3) :

$$\boxed{\begin{aligned} y(t) = te^t &+ u_1(t)\left[t+2+(2t-5)e^{t-1}\right] \\ &- u_2(t)\left[t+1+(2t-7)e^{t-2}\right]\end{aligned}}$$

<u>PROBLEM 1:</u> (25) Points). Calculate the steady-state response $y_p(t)$ for

$$y'' + 0.1y' + 4y = F_0 \sin(\omega t)\,, \tag{$*$}$$

in the form $y_p(t) = R\sin(\omega t - \phi)$ for some R and ϕ to be determined. Here F_0 and ω are positive constants.

(i) Plot the amplitude R as a function of ω. Where is the maxiumum value of R obtained?

(ii) Plot the steady-state solution y_p versus t when $\omega = 2$.

MATH 215 QUIZ #4

PROBLEM 1 $$y'' + 0.1y' + 4y = F_0 \sin(wt)$$

now we get $$\tilde{y}'' + 0.1\hat{y}' + 4\hat{y} = F_0 e^{iwt}$$

now let $\hat{y} = A e^{iwt}$.

THEN, $$-Aw^2 + 0.1Awi + 4A = F_0$$

$$A\left[(4-w^2) + 0.1iw\right] = F_0$$

$$A = \frac{F_0}{(4-w^2) + 0.1w}$$

$$\hat{y} = \frac{F_0}{[4-w^2] + 0.1wi} e^{iwt}$$

take imaginary part: $$y_p = \mathrm{IM}(\hat{y}) = \mathrm{IM}\left[\frac{F_0}{[4-w^2]^2 + .01w^2}\left((4-w^2) - .1wi\right)e^{iwt}\right]$$

$$y_p = \frac{F_0}{[4-w^2]^2 + .01w^2}\, \mathrm{IM}\left[\left((4-w^2) - .1iw\right)\left(\cos wt + i \sin wt\right)\right]$$

$$y_p = \frac{F_0}{[4-w^2]^2 + .01w^2}\left[(4-w^2)\sin wt - .1w\cos wt\right]$$

$$\longrightarrow \quad y_p = \frac{F_0}{\Delta^{1/2}}\left[\frac{(4-w^2)}{\Delta^{1/2}}\sin(wt) - \frac{.1w}{\Delta^{1/2}}\cos(wt)\right]$$

$$\Delta = [4-w^2]^2 + .01w^2$$

now let $$\cos\phi = \frac{(4-w^2)}{\Delta^{1/2}} \qquad \sin\phi = \frac{.1w}{\Delta^{1/2}}.$$

THEN $$y_p = \frac{F_0}{\Delta^{1/2}}\left[\sin(wt)\cos\phi - \cos(wt)\sin\phi\right]$$

$$\Longrightarrow \quad y_p = \frac{F_0}{\Delta^{1/2}}\sin(wt - \phi) \qquad \Delta = [(4-w^2)]^2 + .01w^2$$

THUS $R = \dfrac{F_0}{\left[(4-w^2)^2 + .01\,w^2\right]^{1/2}}$ 15/15 is the amplitude $R = R(w)$.

i) PLOT $R = R(w)$: $R(0) = F_0/4$ AND $R(w) \sim F_0/w^2$ AS $w \to \infty$.

~~NOW R has a maximum where $\Delta = (4-w^2)^2 + .01\,w^2$ has a minimum.~~

$$\frac{d\Delta}{dw} = 2(4-w^2)(-2w) + .02\,w = 0 \quad \text{when} \quad w = 0 \ \text{AND}$$

$$2(4-w^2) = +.01$$
$$4 - w^2 = .005$$
$$w^2 = 4 - .005$$
$$w = \sqrt{4-.005} \approx 2.$$

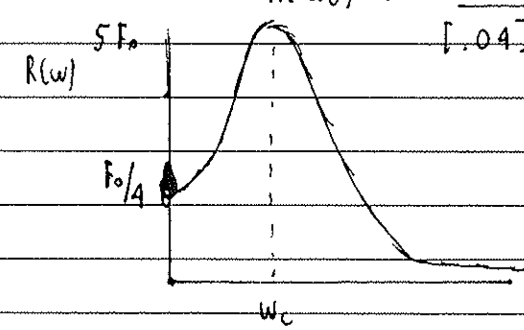
NOTICE: if there is no damping resonance occurs when $W = 2$. Here there is a small damping coefficient. Thus it is not surprising that the maximum value of $R(w)$ occurs near $W = 2$.

let $w_c = \sqrt{4-.005}$. Then $(4 - w_c^2) = .005$

HENCE $R(w_c) = \dfrac{F_0}{\left[(.005)^2 + .01\,w_c^2\right]^{1/2}} = \dfrac{F_0}{\left[(.005)^2 + .01(4-.005)\right]}$

6/6

$$R(w_c) \approx \frac{F_0}{[.04]^{1/2}} \approx F_0/[.2] = 5F_0.$$



Notice a big response near $w = w_c$

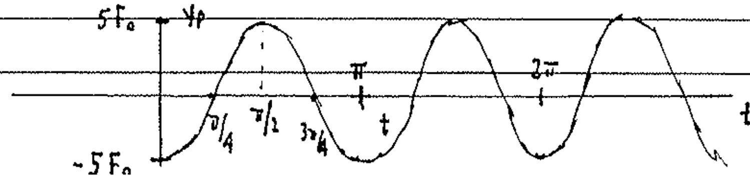
$(w_c \approx 2)$

ii) at $w = w_c \approx 2$ $\quad F_0/\Delta^{1/2} \approx 5F_0$. HENCE $y_p = 5F_0 \sin(wt - \phi)$

NOW WHEN $w = 2$ $\quad \cos\phi = 0 \quad \sin\phi \approx 1 \to \phi \approx \pi/2$.

HENCE $y_p \approx 5F_0 \sin(2t - \pi/2) = -5F_0\cos(2t)$

4/4



[20] **4.** Consider the following differential equation modeling a mass-spring system:

$$y'' + cy' + y = \cos(\omega t)\,, \quad y(0) = 1\,, \quad y'(0) = 0\,,$$

where $c \geq 0$.

(i) Calculate the steady-state response of the system (i.e. the particular solution) in the form $y = R\cos(\omega t - \phi)$ for some R and ϕ to be determined.

(ii) Plot R versus ω for $c = 0$, $c = .01$ and $c = 5$. What is the physical significance of the different shapes of these curves?

(iii) Solve the initial value problem for the case where $c = 0$ and $\omega = 1$.

Continued on page 3

$$\cancel{T = \frac{6\pi}{5\alpha H^2} H^{5/2} = \frac{6\pi}{5\alpha} H^{1/2}.}$$

PROBLEM 4

$$y'' + cy' + y = \cos(\omega t)$$

$$y'' + cy' + y = e^{i\omega t}$$

$$y = e^{i\omega t} A \qquad -\omega^2 A + ci\omega A + A = 1$$

$$A = \frac{1}{(1-\omega^2) + ci\omega}$$

$$A = \frac{(1-\omega^2) - ci\omega}{(1-\omega^2)^2 + c^2\omega^2}$$

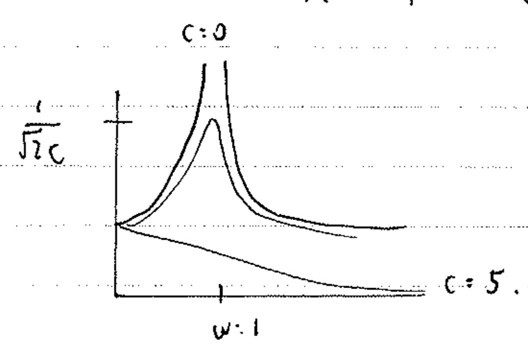
$$y_p = \frac{(1-\omega^2)}{(1-\omega^2)^2 + c^2\omega^2}\cos\omega t + \frac{c\omega}{(1-\omega^2)^2 + c^2\omega^2}\sin\omega t$$

$$y_p = \frac{1}{[(1-\omega^2)^2 + c^2\omega^2]^{1/2}}\cos(\omega t - \phi)$$

$$\cos\phi = (1-\omega^2)/\Delta$$

$$\sin\phi = c\omega/\Delta$$

$$R(\omega) = \frac{1}{[(1-\omega^2)^2 + c^2\omega^2]^{1/2}}$$



$x = \omega^2$

$$(1-x)^2 + c^2 x$$

$$-2(1-x) + 2c^2 = 0$$

$$\cancel{(1+2c^2)\,x = -2x}$$

$$\cancel{2c^2} \qquad 2x = 2 - 2c^2$$

$$(1-x) = c^2 \qquad x = 1 - c^2$$

$$\frac{1}{(2c^2)^{1/2}} = \frac{1}{\sqrt{2}\,c}$$

If $c=0$ then let y_p have the form

$$y = te^{it}$$

$$y' = e^{it} + ite^{it}$$

$$y'' = -te^{it} + 2ie^{it}$$

$$A\,2ie^{it} = 1 \qquad A = -\frac{i}{2}$$

$$y_p = \frac{t}{2}\sin t.$$

$$y = c_1\cos t + c_2\sin t + \frac{t}{2}\sin t$$

$$y(0) = 1 \rightarrow c_1 = 1$$

$$y'(0) = 0 \rightarrow c_2 = 0$$

$$y = \cos t + \frac{t}{2}\sin t.$$

<u>PROBLEM 6</u>

$$(3-\lambda)(-3-\lambda) + 10 = 0$$

$$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

$$y = \underline{v}\,e^{it} \qquad \underline{x}' = A\underline{x}$$

$$y' = i\underline{v}\,e^{it} \qquad i\underline{v} = A\underline{v}$$

$$\begin{pmatrix} 3-i & 2 \\ 0 & 0 \end{pmatrix}\underline{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned}(3-i)\,V_1 &= -2V_2 \\ V_2 &= V_2\end{aligned}$$

$$V_1 = -\frac{2}{3-i}V_2$$

$$V_2 = V_2$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3-i \end{pmatrix}$$

PROBLEM 3: (20 Points)

Using Laplace transforms, find the solution to the initial value problem

$$y'' + y = \begin{cases} \cos t & 0 \le t < \pi/2 \\ 0 & \pi/2 \le t < \infty \end{cases} \qquad y(0) = 3\,, \quad y'(0) = -1\,.$$

PROBLEM 3

USING LAPLACE TRANSFORMS SOLVE

$$y'' + y = \begin{cases} \cos t & 0 \le t < \pi/2 \\ 0 & \pi/2 \le t < \infty \end{cases} \qquad y(0) = 3 \quad y'(0) = -1$$

NOW $\begin{cases} \cos t & 0 \le t < \pi/2 \\ 0 & \pi/2 \le t < \infty \end{cases} = \cos t - u_{\pi/2}(t) \cos t$. BUT $\cos t = -\sin(t - \pi/2)$

SO THAT

$$y'' + y = \cos t + u_{\pi/2}(t) \sin(t - \pi/2) \qquad (5 \text{ POINTS})$$

TAKING LAPLACE TRANSFORMS GIVES

$$(s^2+1)\, Y(s) = 3s - 1 + \frac{s}{(s^2+1)} + \frac{e^{-\pi/2\, s}}{(s^2+1)}$$

$$\text{SO} \quad Y(s) = 3\left(\frac{s}{s^2+1}\right) - \frac{1}{s^2+1} + H_1(s) + H_2(s)\, e^{-\pi s/2} \qquad (5 \text{ POINTS})$$

WHERE

$$H_1(s) = s/(s^2+1)^2 \qquad H_2(s) = 1/(s^2+1)^2 .$$

TO INVERT $H_1(s)$ AND $H_2(s)$ WE USE THE CONVOLUTION THEOREM

$$H_1(s) = \frac{s}{s^2+1}\,\frac{1}{s^2+1} \implies h_1(t) = \mathcal{L}^{-1}[H_1(s)] = \int_0^t \sin(t-\tau)\cos\tau\, d\tau$$

$$H_2(s) = \frac{1}{s^2+1}\,\frac{1}{s^2+1} \implies h_2(t) = \mathcal{L}^{-1}[H_2(s)] = \int_0^t \sin(t-\tau)\sin\tau\, d\tau$$

(8 POINTS)

$$\text{SO} \quad (1) \quad y(t) = \mathcal{L}^{-1}[Y(s)] = 3\cos t - \sin t + h_1(t) + u_{\pi/2}(t)\, h_2(t - \pi/2)$$

EVALUATE $h_1(t)$:

$$h_1(t) = \sin t \int_0^t \cos^2\tau\, d\tau - \cos t \int_0^t \cos\tau \sin\tau\, d\tau$$

$$= \frac{\sin t}{2}\int_0^t (1 + \cos 2\tau)\, d\tau - \frac{\cos t}{2}\int_0^t \sin 2\tau\, d\tau$$

$$h_1(t) = \frac{\sin t}{2}\left(t + \frac{1}{2}\sin 2t\right) - \frac{1}{4}\cos t\left(1 - \cos 2t\right)$$

$$h_1(t) = \frac{t}{2}\sin t - \frac{1}{4}\cos t + \frac{1}{4}\underbrace{\left(\sin 2t \sin t + \cos 2t \cos t\right)}_{\cos t}$$

$$(2) \quad h_1(t) = \frac{t}{2}\sin t$$

EVALUATE $h_2(t)$

$$h_2(t) = \sin t \int_0^t \cos\tau \sin\tau \, d\tau - \cos t \int_0^t \sin^2\tau \, d\tau$$

$$= \frac{\sin t}{4}(1-\cos 2t) - \frac{\cos t}{2}\left(\int_0^t (1-\cos 2\tau)\, d\tau\right)$$

$$= \frac{\sin t}{4}(1-\cos 2t) - \frac{\cos t}{2}\left[t - \frac{1}{2}\sin 2t\right] = -\frac{t\cos t}{2} + \frac{\sin t}{4} + \frac{1}{4}(\sin 2t \cos t - \sin t \cos 2t)$$

$$(3) \qquad h_2(t) = -\frac{t\cos t}{2} + \frac{\sin t}{2}$$

(2 POINTS)

THUS USING (2) AND (3) IN (1) GIVES

$$y(t) = 3\cos t - \sin t + \frac{t}{2}\sin t + \frac{1}{2}u_{\pi/2}(t)\left[\sin(t-\pi/2) - (t-\pi/2)\cos(t-\pi/2)\right]$$

$\cos(t-\pi/2) = \sin t \qquad \sin(t-\pi/2) = -\cos t$

$$y(t) = 3\cos t - \sin t + \frac{t}{2}\sin t + \frac{1}{2}u_{\pi/2}(t)\left[-(t-\pi/2)\sin t - \cos t\right]$$

$$y(t) = \begin{cases} 3\cos t - \sin t + \frac{t}{2}\sin t, & 0 \le t < \pi/2 \\ (\pi/4 - 1)\sin t + 5/2 \cos t, & t \ge \pi/2 \end{cases}$$

REMARK WE COULD ALSO INVERT $s/(s^2+1)^2$ BY USING

$$(*) \qquad \mathcal{L}[t f(t)] = -F'(s) \qquad \text{WHERE} \qquad F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

TO DERIVE (*) DIFFERENTIATE W.R.T. s TO GET

$$F'(s) = -\int_0^\infty t f(t) e^{-st}\, dt = -\mathcal{L}(t f(t))$$

IF $\quad F'(s) = s/(s^2+1)^2 \qquad F(s) = -\frac{1}{2}\frac{1}{s^2+1} \Rightarrow f(t) = -\frac{1}{2}\sin t$

$$\text{THUS } (*) \Rightarrow \mathcal{L}\left(\frac{t}{2}\sin t\right) = s/(s^2+1)^2$$

PROBLEM 2:

2a) Solve the following problem for $y = y(t)$ using Laplace transforms

$$y'' + 3y' + 2y = \begin{cases} 0\,, & 0 < t < 1 \\ 1/\tau\,, & 1 < t < 1+\tau \\ 0\,, & t > 1+\tau \end{cases} \qquad y(0) = 0\,, \quad y'(0) = 0\,.$$

Here τ is a positive constant.

2b) Draw a plot of y versus t when τ is small.

Michael Ward

$$\begin{cases} 0 \\ 1/\tau \\ 1/\tau \end{cases} - \begin{cases} 0 \\ 0 \\ 1/\tau \end{cases} \quad \frac{1}{\tau} U_1(t) - \frac{1}{\tau} U_{1+\tau}(t)$$

PROBLEM 2

2a) $$s^2 Y(s) + 3sY(s) + 2Y(s) = \frac{1}{\tau s} e^{-s} - \frac{1}{\tau s} e^{-(1+\tau)s}$$

$$(s+1)(s+2)Y(s) = \frac{1}{\tau s}\left(e^{-s} - e^{-(1+\tau)s}\right)$$

$$Y(s) = \frac{1}{s(s+1)(s+2)}\left[\frac{1}{\tau} e^{-s} - \frac{1}{\tau} e^{-(1+\tau)s}\right]$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{1}{2s} - \frac{1}{(s+1)} + \frac{1}{2(s+2)}$$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

$$s=0 \Rightarrow A = 1/2 \qquad s=-2 \Rightarrow C = \frac{1}{2}$$

$$s=-1 \Rightarrow B=-1$$

$$F(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

$$y(t) = \frac{1}{\tau}\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}\right) U_1(t)$$
$$- \frac{1}{\tau}\left[\frac{1}{2} - e^{-(t-1-\tau)} + \frac{1}{2} e^{-2(t-1-\tau)}\right] U_{1+\tau}(t)$$

$$y(t) = \begin{cases} 0 & 0 < t < 1 \\ \frac{1}{\tau}\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}\right) & 1 < t < 1+\tau \\ \frac{1}{\tau}\left(e^{-(t-1-\tau)} - e^{-(t-1)}\right) - \frac{1}{2\tau}\left(e^{-2(t-1-\tau)} - e^{-2(t-1)}\right) & t > 1+\tau \end{cases}$$

$$\frac{f(t+\tau) - f(t)}{\tau} \qquad f(t) = e^{-(t-1)}$$

WHEN τ IS SMALL

$$y'' + 3y' + 2y = \delta(t-1)$$

$$(s^2+3s+2)Y(s) = e^{-s}$$

$$Y(s) = \frac{1}{(s+1)(s+2)} e^{-s}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{(s+2)}$$

$A(s+2) + B(s+1) = 1$

$s=-1 \Rightarrow A=1$

$s=-2 \Rightarrow B=-1$

$$y(t) = u_1(t)\left[e^{-(t-1)} - e^{-2(t-1)}\right]$$

$$y(t) = \frac{1}{\tau}\left[f(t+\tau) - f(t)\right] - \frac{1}{2\tau}\left[g(t+\tau) - g(t)\right] \quad t > 1+\tau$$

$$f(t) = e^{-(t-1)} \qquad f'(t) = -e^{-(t-1)}$$

$$g(t) = e^{-2(t-1)} \qquad g'(t) = -2e^{-2(t-1)}$$

$$y(t) = \left(-e^{-(t-1)} + e^{-2(t-1)}\right) u_1(t)$$

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