MATH 215 HW 9 SOLUTIONS

SECTION 6.3 # 34

SUPPOSE $f(t+T) = f(t)$ FOR ALL t.

SHOW THAT $\mathcal{L}(f(t)) = \dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$.

PROOF

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \dots$$

$$= \sum_{j=0}^{\infty} \int_{jT}^{(j+1)T} f(t)e^{-st}\,dt$$

NOW LET $t = \sigma + jT$. THEN

$$\mathcal{L}(f(t)) = \sum_{j=0}^{\infty} \int_0^T f(\sigma+jT)\, e^{-s(\sigma+jT)}\,d\sigma \quad (*)$$

BUT $f(\sigma+jT) = f(\sigma+(j-1)T) = \dots = f(\sigma)$ SINCE f IS T-periodic.

SO $(*)$ BECOMES

$$\mathcal{L}(f(t)) = \sum_{j=0}^{\infty} e^{-sjT}\left(\int_0^T f(\sigma)e^{-s\sigma}\,d\sigma\right)$$

$$= \int_0^T f(\sigma)e^{-s\sigma}\,d\sigma \left(\sum_{j=0}^{\infty} e^{-sjT}\right)$$

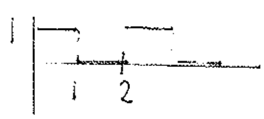
↑ geometric series

$$(+)\quad \left\{ \mathcal{L}(f(t)) = \frac{\int_0^T f(\sigma)e^{-s\sigma}\,d\sigma}{1-e^{-sT}} \right.$$

SECTION 6.3 # 35

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$$

WITH $f(t+2) = f(t)$



NOW TO CALCULATE $\mathcal{L}(f(t))$ WE USE $(+)$

$$\mathcal{L}(f(t)) = \int_0^2 f(\sigma)e^{-s\sigma}\,d\sigma \Big/ (1-e^{-2s})$$

$$= \int_0^1 e^{-s\sigma}\,d\sigma \Big/ (1-e^{-2s}) = \frac{-\frac{1}{s} e^{-s\sigma}\Big|_0^1}{(1-e^{-2s})}$$

THEN $\mathcal{L}(f(t)) = \frac{1}{s}(1-e^{-s}) \big/ (1-e^{-2s})$

WE CAN THEN WRITE

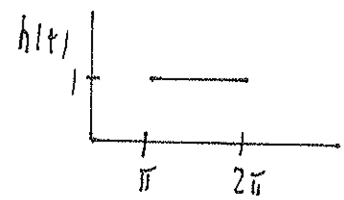
$$\mathcal{L}(F(t)) = \frac{1}{s}\frac{(1-e^{-s})}{(1-e^{-2s})} = \frac{1}{s}\frac{(1-e^{-s})}{(1-e^{-s})(1+e^{-s})}$$

THUS YIELDS $\mathcal{L}(F(t)) = \dfrac{1}{s(1+e^{-s})}$

SECTION 6.4 # 2

$$y'' + 2y' + 2y = h(t) \qquad h(t) = \begin{cases} 1 & \pi \le t < 2\pi \\ 0 & 0 \le t < \pi, \quad t \ge 2\pi \end{cases}$$

$y(0) = 0, \quad y'(0) = 1$



CLEARLY $h(t) = U_\pi(t) - U_{2\pi}(t)$.

THUS $y'' + 2y' + 2y = U_\pi(t) - U_{2\pi}(t)$

NOW $\left(s^2 Y - s y(0) - y'(0)\right) + 2\left(sY - y(0)\right) + 2Y = \dfrac{e^{-\pi s}}{s} - \dfrac{e^{-2\pi s}}{s}$

NOW $(s^2 + 2s + 2)Y - 1 = \dfrac{e^{-\pi s}}{s} - \dfrac{e^{-2\pi s}}{s}$

SO $Y = \dfrac{1}{s^2+2s+2} + \hat{F}(s)e^{-\pi s} - \hat{F}(s)e^{-2\pi s}$

$$\hat{F}(s) = \frac{1}{s(s^2+2s+2)}$$

WE WRITE $Y = \dfrac{1}{(s+1)^2+1} + \hat{F}(s)e^{-\pi s} - \hat{F}(s)e^{-2\pi s}$

SO THAT $y(t) = e^{-t}\sin t + U_\pi(t) f(t-\pi) - U_{2\pi}(t) f(t-2\pi) \quad (*)$

WHERE $f(t) = \mathcal{L}^{-1}[\hat{F}(s)] = \mathcal{L}^{-1}\left[\dfrac{1}{s(s^2+2s+2)}\right]$

NOW $$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

$$1 = A(s^2+2s+2) + s(Bs+C)$$

$$s^2: \quad A + B = 0 \rightarrow B = -1/2$$
$$s^1: \quad 2A + C = 0 \rightarrow C = -1$$
$$s^0: \quad 2A = 1 \rightarrow A = 1/2$$

$$\hat{f}(s) = \frac{1}{s(s^2+2s+2)} = \frac{1}{2s} - \left(\frac{s/2+1}{(s+1)^2+1}\right)$$
$$= \frac{1}{2s} - \frac{1}{2}\frac{(s+2)}{(s+1)^2+1}$$
$$\hat{f}(s) = \frac{1}{2s} - \frac{1}{2}\left[\frac{(s+1)}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right]$$

THUS $$f(t) = \mathcal{L}^{-1}[\hat{f}(s)] = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t.$$

FROM (*) THE EXACT SOLUTION IS

$$y(t) = e^{-t}\sin t + U_{\pi}(t)f(t-\pi) - U_{2\pi}(t)f(t-2\pi).$$

WE CAN WRITE THIS AS

$$f(t-\pi) = \frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\cos(t-\pi) - \frac{1}{2}e^{-(t-\pi)}\sin(t-\pi)$$

SO $$f(t-\pi) = \frac{1}{2} + \frac{1}{2}e^{-(t-\pi)}\cos t + \frac{1}{2}e^{-(t-\pi)}\sin t$$
$$f(t-2\pi) = \frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}\cos t - \frac{1}{2}e^{-(t-2\pi)}\sin t$$

THUS, $$\left\{ \begin{aligned} y(t) = e^{-t}\sin t &+ U_{\pi}(t)\left[\frac{1}{2} + \frac{1}{2}e^{-(t-\pi)}\cos t + \frac{1}{2}e^{-(t-\pi)}\sin t\right] \\ &+ U_{2\pi}(t)\left[\frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}\cos t - \frac{1}{2}e^{-(t-2\pi)}\sin t\right]. \end{aligned} \right.$$

SECTION 6.4 # 6

$$y'' + 3y' + 2y = u_2(t)$$
$$y(0) = 0, \quad y'(0) = 1$$

TAKE LAPLACE TRANSFORM TO GET

$$\left(s^2 Y - s\,y(0) - y'(0)\right) + 3\left(sY - y(0)\right) + 2Y = \frac{e^{-2s}}{s}$$

THEN $\quad (s^2 + 3s + 2)\,Y = 1 + \dfrac{e^{-2s}}{s}$

SO $\quad Y = \dfrac{1}{(s^2+3s+2)} + \hat{F}(s)\,e^{-2s}, \quad \hat{F}(s) = \dfrac{1}{s(s^2+3s+2)}$

SO $\quad Y = \dfrac{1}{(s+2)(s+1)} + \hat{F}(s)\,e^{-2s} = \dfrac{1}{(s+1)} - \dfrac{1}{(s+2)} + \hat{F}(s)\,e^{-2s}$

(BY PARTIAL FRACTIONS)

NOW $\quad \hat{F}(s) = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2} = \dfrac{1}{s(s+1)(s+2)}$

SO $\quad A(s+2)(s+1) + Bs(s+2) + Cs(s+1) = 1$

SO LET $\quad s = 0 \rightarrow 2A = 1 \rightarrow A = 1/2$

$s = -1 \rightarrow B = -1$

$s = -2 \rightarrow C = 1/2$

$\Longrightarrow f(t) = \mathcal{L}^{-1}[\hat{F}(s)]$

$f(t) = 1/2 - e^{-t} + 1/2\, e^{-2t}$

SO $\quad y(t) = \mathcal{L}^{-1}[Y(s)]$

$$y(t) = e^{-t} - e^{-2t} + u_2(t)\,f(t-2)$$

$$y(t) = e^{-t} - e^{-2t} + u_2(t)\left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right].$$

SECTION 6.4 # 9a)

$$y'' + y = g(t) = \begin{cases} t/2 & 0 \le t < 6 \\ 3 & t \ge 6 \end{cases}$$

$$y(0) = 0, \quad y'(0) = 1$$

SOLUTION LET $Y(s) = \mathcal{L}(y(t))$.

THEN $$g(t) = t/2 + \begin{cases} 0, & 0 \le t < 6 \\ 3 - t/2, & t \ge 6 \end{cases}$$

$$= t/2 + \begin{cases} 0, & 0 \le t < 6 \\ -(t-6)/2, & t \ge 6 \end{cases}$$

SO $g(t) = t/2 + u_6(t) f(t-6)$ WHERE $f(t) = -t/2$.

NOW $$\mathcal{L}(g(t)) = \frac{1}{2s^2} + e^{-6s}\left(\frac{-1}{2s^2}\right)$$

WE CALCULATE $$s^2 Y - s y(0) - y'(0) + Y = \frac{1}{2s^2} - \frac{1}{2s^2} e^{-6s}$$

WITH $y(0) = 0$, $y'(0) = 1$ THEN

$$(s^2+1) Y = 1 + \frac{1}{2s^2} - \frac{1}{2s^2} e^{-6s}$$

$$Y = \frac{1}{s^2+1} + \hat{F}(s) - \hat{F}(s) e^{-6s}, \quad \hat{F}(s) = \frac{1}{2s^2(s^2+1)}.$$

THEN $y(t) = \mathcal{L}^{-1}[Y(s)] = \sin t + F(t) - u_6(t) F(t-6)$

WHERE $F(t) = \mathcal{L}^{-1}[\hat{F}(s)]$. $\quad \hat{F}(s) = \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$ BY PARTIAL FRACTIONS.

SO $F(t) = \frac{1}{2}(t - \sin t)$. THIS GIVES,

$$y(t) = \sin t + \frac{1}{2}(t - \sin t) - u_6(t)\left[\frac{1}{2}(t - 6 - \sin(t-6))\right].$$

THIS GIVES

$$y(t) = t/2 + \sin t/2 - U_6(t)\left[t/2 - 3 - \tfrac{1}{2}\sin(t-6)\right].$$

SECTION 6.4 # 10

$$y'' + y' + \frac{5}{4} y = g(t) = \begin{cases} \sin t & , \quad 0 \le t < \pi \\ 0 & , \quad t \ge \pi \end{cases}$$

$y(0) = 0, \quad y'(0) = 0$

NOW $g(t) = \sin t + \begin{cases} 0 & , \quad 0 \le t < \pi \\ -\sin t & , \quad t \ge \pi \end{cases}$

$$g(t) = \sin t + \begin{cases} 0 & \quad 0 \le t < \pi \\ \sin(t-\pi) & \quad t \ge \pi \end{cases}$$

SINCE $\sin(t-\pi) = -\sin t$.

THIS GIVES, $\quad g(t) = \sin t + U_\pi(t)\, f(t-\pi)$

WITH $f(t) = \sin t$

NOW $\quad \mathcal{L}(g(t)) = \dfrac{1}{s^2+1} + \dfrac{e^{-\pi s}}{(s^2+1)}.$

NOW $Y(s) = \mathcal{L}(y(t))$ GIVES

$$s^2 Y + sY + \frac{5}{4} Y = \mathcal{L}(g(t)) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{(s^2+1)}$$

SO $$Y = \frac{1}{(s^2+1)(s^2+s+5/4)} + e^{-\pi s}\frac{1}{(s^2+1)(s^2+s+5/4)}$$

THIS GIVES $\quad y(t) = \mathcal{L}^{-1}[Y(s)] = f(t) + U_\pi(t)\, f(t-\pi) \quad (*)$

WHERE $\quad f(t) = \mathcal{L}^{-1}[F(s)] \qquad F(s) = \dfrac{1}{(s^2+1)(s^2+s+5/4)}$

NOW $\hat{f}(s) = \dfrac{1}{(s^2+1)(s^2+s+5/4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+s+5/4}$

so $(As+B)(s^2+s+5/4) + (Cs+D)(s^2+1) = 1$

so
$$
\begin{aligned}
s^3&: \quad A + C = 0 && (1)\\
s^2&: \quad A + B + D = 0 && (2)\\
s^1&: \quad 5/4\,A + B + C = 0 && (3)\\
s^0&: \quad 5/4\,B + D = 1 && (4)
\end{aligned}
$$

so
$$
\left.\begin{aligned}
B + D &= C && (1)-(2)\\
5B + 4D &= 4 && (4)
\end{aligned}\right\} \Rightarrow \quad D = 5C - 4
$$

$$-5/4\,C + B + C = 0 \quad (1)-(3) \rightarrow \quad B - C/4 = 0 \rightarrow B = C/4.$$

so $C/4 + D = C \longrightarrow D = 3C/4 = 5C - 4.$

so $5C = 4 + \dfrac{3C}{4} \longrightarrow \dfrac{17C}{4} = 4 \rightarrow C = 16/17$

so $C = 16/17,\quad B = 4/17,\quad D = 12/17,\quad A = -16/17$

THUS $\hat{f}(s) = -\dfrac{16}{17}\dfrac{s}{s^2+1} + \dfrac{4}{17}\dfrac{1}{(s^2+1)} + \dfrac{\left(\frac{16}{17}s + \frac{12}{17}\right)}{(s+1/2)^2 + 1}$

$$\hat{f}(s) = -\frac{16}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{(s^2+1)} + \frac{\frac{16}{17}(s+1/2)}{(s+1/2)^2+1} + \frac{4/17}{(s+1/2)^2+1}$$

so $f(t) = \mathcal{L}^{-1}[\hat{f}(s)] = -\dfrac{16}{17}\cos t + \dfrac{4}{17}\sin t + \dfrac{16}{17}e^{-t/2}\cos t + \dfrac{4}{17}e^{-t/2}\sin t. \quad (+)$

SO FROM (*) ON PREVIOUS PAGE

$$y(t) = f(t) + u_\pi(t)\, f(t-\pi)$$

WHERE $f(t)$ IS GIVEN IN (+).

SECTION 6.5 # 2a)

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$$

$$y(0) = 0, \quad y'(0) = 0.$$

THEN $Y(s) = \mathcal{L}(y(t))$ GIVES,

$$s^2 Y + 4Y = e^{-\pi s} - e^{-2\pi s}$$

THIS GIVES $$Y = \frac{1}{s^2+4} e^{-\pi s} - \frac{1}{s^2+4} e^{-2\pi s}$$

SO $$Y = \frac{1}{2}\left(\frac{2}{s^2+4}\right) e^{-\pi s} - \frac{1}{2}\left(\frac{2}{s^2+4}\right) e^{-2\pi s}$$

THIS GIVES $$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2} u_\pi(t) \sin[2(t-\pi)] - \frac{1}{2} u_{2\pi}(t) \sin[2(t-2\pi)]$$

BUT $\sin[2(t-\pi)] = \sin(2t)$, $\sin[2(t-2\pi)] = \sin 2t$, SO

$$y(t) = \frac{1}{2} u_\pi(t) \sin(2t) - \frac{1}{2} u_{2\pi}(t) \sin(2t)$$

SECTION 6.5 # 4a)

$$y'' - y = -20\,\delta(t-3)$$

$$y(0) = 1, \quad y'(0) = 0$$

SO $$s^2\bar{Y} - s\,y(0) - y'(0) - \bar{Y} = -20e^{-3s}$$

THIS GIVES, $$(s^2-1)\bar{Y} = s - 20e^{-3s}$$

SO $$\bar{Y} = \frac{s}{s^2-1} - \frac{20}{s^2-1}e^{-3s}$$

THEN, $$y(t) = \mathcal{L}^{-1}[\bar{Y}(s)] = \cosh t - 20\,U_3(t)\sinh[t-3]$$

SECTION 6.5 # 8a)

$$y'' + 4y = 2\,\delta(t-\pi/4) \qquad y(0) = y'(0) = 0.$$

NOW $$(s^2+4)\bar{Y} = 2e^{-\pi s/4}$$

SO $$\bar{Y} = \frac{2}{s^2+4}\,e^{-\pi s/4}$$

SO $$y(t) = U_{\pi/4}(t)\sin[2(t-\pi/4)]$$

EXTRA PROBLEM

$$y'' + 4y' - 5y = -5 + e^{-2t}$$

$$y(0) = \alpha, \quad y'(0) = 0.$$

TAKE LAPLACE TRANSFORMS. LET $\bar{Y}(s) = \mathcal{L}(y(t))$. THEN,

$$s^2 \bar{Y} - s y(0) - y'(0) + 4[s\bar{Y} - y(0)] - 5\bar{Y} = -\frac{5}{s} + \frac{1}{s+2}$$

SO $$(s^2 + 4s - 5)\bar{Y} - \alpha s - 4\alpha = -\frac{5}{s} + \frac{1}{s+2}$$

SO $$(s^2 + 4s - 5)\bar{Y} = \alpha(s+4) + \frac{s - 5(s+2)}{s(s+2)}$$

THUS, $$(s^2 + 4s - 5)\bar{Y} = \frac{\alpha(s+4)s(s+2) - 4s - 10}{s(s+2)}$$

THIS GIVES, $$\bar{Y}(s) = \frac{\alpha(s+4)(s)(s+2) - 4s - 10}{s(s+2)(s+5)(s-1)} = \frac{P(s)}{Q(s)} \quad (*)$$

$$\bar{Y}(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5} + \frac{D}{s-1}$$

INVERTING WOULD GIVE

$$y(t) = \mathcal{L}^{-1}[\bar{Y}(s)] = A + Be^{-2t} + Ce^{-5t} + De^{t} \quad (+)$$

SO TO ENSURE THAT y IS BOUNDED AS $t \to \infty$ WE NEED THAT $D = 0$. THIS IS TRUE WHEN $P(1) = 0$ IN $(*)$.

WE CALCULATE,

$$\alpha(5)(3) - 4 - 10 = 0$$

THUS $\alpha = 14/15$.

NOTICE THAT THE PARTICULAR SOLUTION MUST HAVE $y_{p_1} = 1$ (since it solves $y'' + 4y' - 5y = -5$). THUS $A = 1$, IN $(+)$.