MATH 215 HW # 1

DO THE FOLLOWING PROBLEMS FROM THE BOOK:

- SECTION 1.2 PAGES 17-18

  PROBLEMS 15, 17

- SECTION 2.1 PAGES 39-41

  PROBLEMS 2, 5, 11, 14, 19, 21, 24, 32.

MATH 215 HW 1

SECTION 1.2 #15

$$\frac{du}{dt} = -k(u - T).$$

(i) $u' + ku = kT$. INTEGRATING FACTOR $\phi = e^{\int^t k\,ds} = e^{kt}$.

THUS $$\frac{d}{dt}\left(e^{kt}u\right) = kTe^{kt}$$

$$\longrightarrow \quad e^{kt}u = Te^{kt} + C$$

SO $$u = T + Ce^{-kt}$$

NOW $u(0) = u_0 \;\rightarrow\; u_0 = T + C \;\rightarrow\; C = u_0 - T$

SO $$u = T + (u_0 - T)e^{-kt}$$ IS SOLUTION.

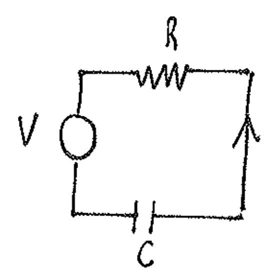
(ii) NOW FIND THE TIME τ FOR WHICH $u - T = \frac{1}{2}(u_0 - T)$.

THUS $$\frac{1}{2}(u_0 - T) = (u_0 - T)e^{-k\tau} \;\rightarrow\; \frac{1}{2} = e^{-k\tau} \;\rightarrow\; -\ln 2 = -k\tau.$$

THIS GIVES $$\tau = \frac{1}{k}\ln 2.$$

SECTION 1.2 #17

$$Q' + \frac{Q}{RC} = \frac{V}{R}$$



$R > 0$
$C > 0$

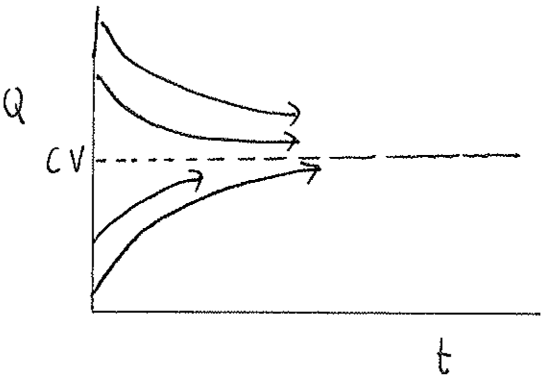
(i) INTEGRATING FACTOR IS $\phi = e^{\int^t 1/RC\,ds} = e^{t/RC}$

THUS $$\frac{d}{dt}\left(e^{t/RC}Q\right) = \frac{V}{R}e^{t/RC}$$

$$\longrightarrow \quad e^{t/RC}Q = CVe^{t/RC} + A$$ A: CONSTANT.

SO $Q = CV + Ae^{-t/RC}$ BUT $Q(0) = Q_0 \rightarrow Q_0 = CV + A$.

THUS $$Q = CV + \left(Q_0 - CV\right)e^{-t/RC}$$



(ii) AS $t \to \infty$ THEN $Q \to CV$

(iii) NOW SUPPOSE THAT BATTERY IS REMOVED FOR $t \geq t_1$.

THE PROBLEM IS

$$Q' + \frac{Q}{RC} = \begin{cases} V/R & 0 \leq t \leq t_1 \\ 0 & t > t_1 \end{cases}$$

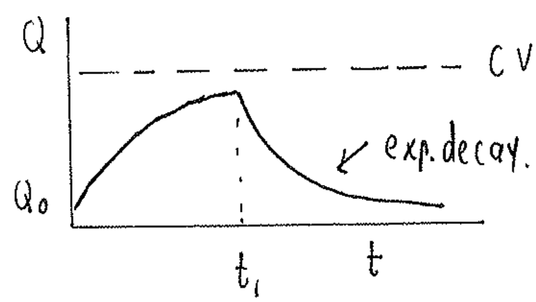
$Q(0) = Q_0$ AND Q CONTINUOUS ACROSS $t = t_1$

THE SOLUTION IS

$$Q(t) = \begin{cases} CV + (Q_0 - CV)e^{-t/RC}, & 0 \leq t \leq t_1 \\ Q(t_1)\, e^{-t/RC} & t \geq t_1 \end{cases}$$

WHERE $Q(t_1) = CV + (Q_0 - CV)e^{-t_1/RC}$

THUS



WHEN $Q_0 < CV$.

SECTION 2.1 # 2

$$y' - 2y = t^2 e^{2t}$$

THE INTEGRATING FACTOR IS $\phi(t) = e^{-\int^t 2\,dt} = e^{-2t}$.

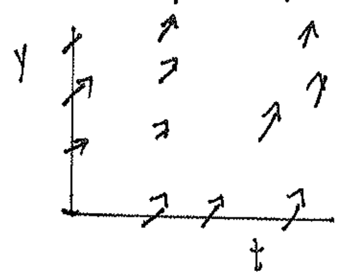
THUS $e^{-2t}(y' - 2y) = t^2 e^{2t} e^{-2t}$

$\longrightarrow \frac{d}{dt}\left(e^{-2t} y\right) = t^2.$

THUS $e^{-2t} y = t^3/3 + C \longrightarrow y = t^3/3\, e^{2t} + C e^{2t}$ IS GENERAL SOLUTION.

NOW AS $t \to \infty$ THEN $y \to \infty$.

DIRECTION FIELD FOR $y' = 2y + t^2 e^{2t} = f(t, y)$



EXPECT $y \to \infty$ AS $t \to \infty$ FROM DIRECTION FIELD, WHEN $y(0) > 0$.

SECTION 2.1 # 5

$$y' - 2y = 3e^t$$

INTEGRATING FACTOR IS $\phi(t) = e^{-\int^t 2\,dt} = e^{-2t}$.

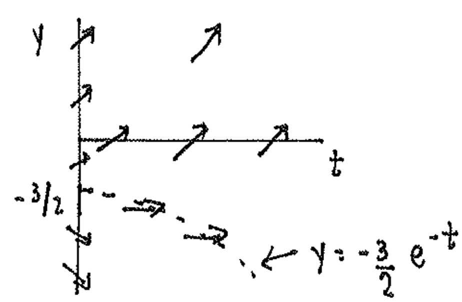
THUS $e^{-2t}(y' - 2y) = 3e^{-2t} e^t = 3e^{-t}$

$\frac{d}{dt}\left(e^{-2t} y\right) = 3e^{-t}$

THUS $e^{-2t} y = -3e^{-t} + C \longrightarrow y = -3e^t + Ce^{2t}$. is general solution.

NOW $y \to +\infty$ AS $t \to \infty$ IF $C > 0$. $y(0) = C - 3$

NOW $\frac{dy}{dt} = 2y + 3e^t$



SECTION 2.1 # 11

$$y' + y = 5 \sin 2t$$

integrating factor $\quad \phi(t) = e^{\int^t 1\, dt} = e^t$

THUS
$$\frac{d}{dt}\left(e^t y\right) = 5 e^t \sin(2t)$$

SO
$$e^t y = 5\int^t e^\lambda \sin(2\lambda)\, d\lambda + C. \qquad (*)$$

NOW
$$I = \int^t e^\lambda \sin(2\lambda)\, d\lambda = -\frac{1}{2} e^\lambda \cos(2\lambda)\Big|^t + \frac{1}{2}\int^t e^\lambda \cos(2\lambda)\, d\lambda .$$

$u = e^\lambda \quad du = e^\lambda d\lambda \qquad\qquad u = e^\lambda \quad du = e^\lambda d\lambda$

$dv = \sin(2\lambda)\, d\lambda \quad v = -\frac{1}{2}\cos(2\lambda) \qquad\qquad dv = \cos(2\lambda)\, d\lambda \quad v = \frac{1}{2}\sin(2\lambda)$

NOW INTEGRATE AGAIN BY PARTS

SO
$$I = -\frac{1}{2} e^t \cos(2t) + \frac{1}{2}\left[\frac{e^\lambda}{2}\sin(2\lambda)\Big|^t - \frac{1}{2}\int^t e^\lambda \sin 2\lambda\, d\lambda\right]$$

THUS
$$I = -\frac{1}{2} e^t \cos(2t) + \frac{1}{4} e^t \sin(2t) - \frac{1}{4} I .$$

THUS
$$5I/4 = 1/4\, e^t \sin(2t) - \frac{1}{2} e^t \cos(2t).$$

HENCE
$$I = \frac{1}{5} e^t \sin(2t) - \frac{2}{5} e^t \cos(2t)$$

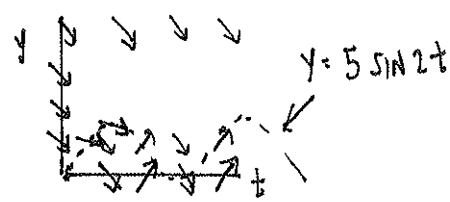
THUS FROM $(*)$,
$$e^t y = 5\left[\frac{1}{5} e^t \sin(2t) - \frac{2}{5} e^t \cos(2t)\right] + C.$$

THUS
$$y = \sin(2t) - 2\cos(2t) + Ce^{-t}. \quad \text{general solution.}$$

NOW $y(0) = C - 2$. NOW y IS BOUNDED AS $t \to \infty$ FOR ANY C.
IF t IS LARGE y IS APPROXIMATELY PERIODIC.
<u>NOTE</u>: y IS PERIODIC IF $C = 0$.

DIRECTION FIELD $\quad y' = 5\sin(2t) - y = f(t,y)$

TRAJECTORIES CANNOT ESCAPE TO INFINITY.



SECTION 2.1 # 14

$$y' + 2y = te^{-2t}, \quad y(1) = 0.$$

INTEGRATING FACTOR IS $\phi(t) = e^{\int^t 2\,dt} = e^{2t}$.

THUS $\frac{d}{dt}(e^{2t}y) = te^{-2t}e^{2t} = t$.

THUS $e^{2t}y = t^2/2 + C$.

IF $y(1) = 0 \rightarrow 0 = 1/2 + C \rightarrow C = -1/2$.

THUS $y = \frac{(t^2-1)}{2}e^{-2t}$.

SECTION 2.1 # 19

$$t^3 y' + 4t^2 y = e^{-t} \quad \text{WITH} \quad y(-1) = 0.$$

NOW $y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$. INTEGRATING FACTOR IS $\phi(t) = e^{\int^t 4/\lambda\, d\lambda} = e^{4\ln t}$

THUS $\phi = t^4$. HENCE, MULTIPLY BY t^4.

$$t^4 y' + 4t^3 y = te^{-t}$$

$$\rightarrow \frac{d}{dt}(t^4 y) = te^{-t}.$$

THUS, $t^4 y = \int^t \lambda e^{-\lambda}\, d\lambda + C$. (*)

INTEGRATE BY PARTS. $u = \lambda \quad du = d\lambda$
$dv = e^{-\lambda}d\lambda \quad v = -e^{-\lambda}$

$$\int^t \lambda e^{-\lambda} d\lambda = -\lambda e^{-\lambda}\Big|^t + \int^t e^{-\lambda} d\lambda$$

SO $\int^t \lambda e^{-\lambda} d\lambda = -te^{-t} - e^{-t}$

THUS FROM (*)

$$t^4 y = -te^{-t} - e^{-t} + C. \quad \text{(GENERAL SOLUTION)}$$

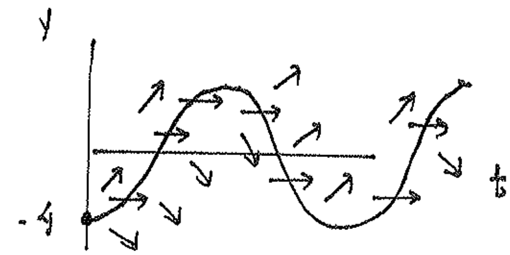
NOW PUT $y(-1) = 0 \rightarrow 0 = (-te^{-t} - e^{-t})\big|_{t=-1} + C \rightarrow C = 0$.

THUS $y = -\frac{1}{t^4}(t+1)e^{-t}$ EXACT SOLUTION SATISFYING $y(-1) = 0$.

SECTION 2.1 # 21

$$y' - \frac{1}{2} y = 2\cos t \qquad y(0) = a.$$

(i) $\quad y' = \frac{1}{2} y + 2\cos t = f(t, y).$



NOW $\quad y' = 0 \quad$ WHEN $\quad y = -4\cos t$

$\quad y' > 0 \quad$ WHEN $\quad y > -4\cos t$

$\quad y' < 0 \quad$ WHEN $\quad y < -4\cos t$

- WE MIGHT EXPECT THAT IF $y(0) > -4$ THAT y INCREASES AS $t \uparrow$
- EXPECT THAT IF $y(0) < -4$ THEN y DECREASES AS $t \uparrow$.

(ii) NOW SOLVE EXACTLY.

INTEGRATING FACTOR IS $\quad \phi(t) = e^{-\frac{1}{2}\int^t 1\, ds} = e^{-t/2}.$

THUS $\quad \frac{d}{dt}\left(e^{-t/2} y\right) = 2e^{-t/2}\cos t.$

NOW $\quad e^{-t/2} y = \int^t 2e^{-s/2}\cos s\, ds + C.$

SO $\quad e^{-t/2} y = 2\int^t e^{-s/2}\cos s\, ds + C. \quad (*)$

integrate by parts: $\quad u = e^{-s/2} \quad du = -\frac{1}{2} e^{-s/2}\, ds$

$\quad dv = \cos s\, ds \quad v = \sin s$

$$I \equiv \int^t e^{-s/2}\cos s\, ds = e^{-t/2}\sin t + \frac{1}{2}\int e^{-s/2}\sin s\, ds$$

$\quad u = e^{-s/2} \quad du = -\frac{1}{2} e^{-s/2}\, ds$

$\quad dv = \sin s\, ds \quad v = -\cos s$

$$I = e^{-t/2}\sin t + \frac{1}{2}\left[-e^{-t/2}\cos t - \frac{1}{2} I\right]$$

THUS, $\quad 5I/4 = e^{-t/2}\sin t - \frac{1}{2} e^{-t/2}\cos t \longrightarrow I = \frac{4}{5} e^{-t/2}\sin t - \frac{2}{5} e^{-t/2}\cos t$

THEN, $\quad e^{-t/2} y = 2\left[\frac{4}{5} e^{-t/2}\sin t - \frac{2}{5} e^{-t/2}\cos t\right] + C.$

THUS $\quad y = \frac{8}{5}\sin t - \frac{4}{5}\cos t + Ce^{t/2}.$

CHECK $\quad y' = \frac{8}{5}\cos t + \frac{4}{5}\sin t + \frac{c}{2} e^{t/2}.$

THUS $\quad y' - \frac{1}{2}y = \frac{8}{5}\cos t + \frac{4}{5}\sin t + \frac{c}{2}e^{t/2} - \frac{1}{2}\left[\frac{8}{5}\sin t + \frac{4}{5}\cos t + c e^{t/2}\right]$

$$= \left(\frac{8}{5} + \frac{4}{10}\right)\cos t + \left(4/5 - 4/5\right)\sin t = 2\cos t \ \checkmark\checkmark.$$

THEN EXACT SOLUTION IS

$$y = \frac{8}{5}\sin t - \frac{4}{5}\cos t + c e^{t/2}.$$

NOW $\quad y(0) = -\frac{4}{5} + C. \quad \rightarrow \quad C = y(0) + 4/5$

- IF $C > 0 \Rightarrow y \rightarrow +\infty$ AS $t \rightarrow +\infty \Rightarrow$ NEED $y(0) > -4/5$
- IF $C < 0 \Rightarrow y \rightarrow -\infty$ AS $t \rightarrow \infty \Rightarrow$ NEED $y(0) < -4/5$.

SECTION 2.1 # 24

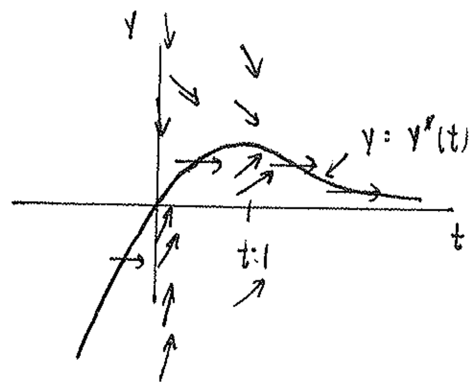
$$t y' + (t+1)y = 2te^{-t} \quad \text{WITH} \quad y(1) = a.$$

NOW $\quad y' + \left(1 + \frac{1}{t}\right)y = 2e^{-t}.$

$$y' = 2e^{-t} - \left(1 + \frac{1}{t}\right)y = f(t, y).$$

NOTE $\quad f(t,y) = 0$ WHEN $\quad y = \dfrac{2e^{-t}}{(1 + 1/t)} = \dfrac{2te^{-t}}{t+1} = y^*(t)$

THUS



IF $\quad y > y^* \longrightarrow y' < 0$

$\quad y < y^* \longrightarrow y' > 0$

NOTICE $\quad y' \rightarrow -\infty$ AS $t \rightarrow 0^+$ WHEN $y > 0$.

$\quad y' \rightarrow +\infty$ AS $t \rightarrow 0^+$ WHEN $y < 0$.

INTEGRATING FACTOR $\phi(t) = e^{\int^t (1+1/\lambda)\,d\lambda} = e^{t+\ln t} = te^t.$

THUS $\frac{d}{dt}(te^t y) = 2(te^t)e^{-t} = 2t.$

THEN, $te^t y = t^2/2 + C.$

THIS GIVES $y = \frac{t}{2}e^{-t} + \frac{Ce^{-t}}{t}.$

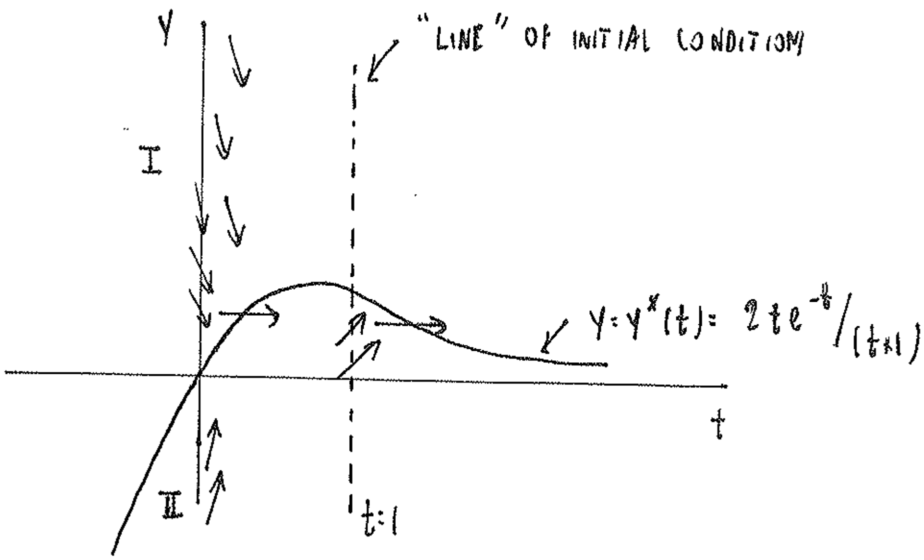
NOW $y(1) = a \rightarrow \quad a = \frac{1}{2}e^{-1} + Ce^{-1} \rightarrow C = e\left(a - \tfrac{1}{2}e^{-1}\right) = ae - 1/2.$

THUS $y(t) = \frac{t}{2}e^{-t} + \frac{1}{t}e^{-t}\left(ae - \frac{1}{2}\right)$

NOW $y \rightarrow +\infty$ AS $t \rightarrow 0^+$ WHEN $ae - 1/2 > 0 \rightarrow a > \frac{1}{2e}$ REGION I

$y \rightarrow -\infty$ AS $t \rightarrow 0^+$ WHEN $a < 1/2e$ REGION II

THIS IS SUGGESTED "ROUGHLY" BY DIRECTION FIELD



SECTION 2.1 # 32

$$2y' + ty = 2$$

SHOW THAT Y APPROACHES A LIMIT AS $t \to \infty$ AND FIND THIS LIMIT.

INTEGRATING FACTOR FOR $y' + \frac{t}{2} y = 1.$

$$\phi(t) = e^{\int^t \lambda/2 \, d\lambda} = e^{t^2/4}.$$

THUS $\frac{d}{dt}\left(e^{t^2/4} y\right) = e^{t^2/4}.$

NOW PUT $y(0) = y_0$ THEN

$$e^{t^2/4} y = \int_0^t e^{\lambda^2/4} \, d\lambda + C.$$

THUS EXACT SOLUTION IS

$$y = e^{-t^2/4} \int_0^t e^{\lambda^2/4} \, d\lambda + C e^{-t^2/4}.$$

NOW WHAT HAPPENS AS $t \to \infty$.

L'Hopital's RULE: $\lim_{t \to \infty} \frac{\int_0^t e^{\lambda^2/4} \, d\lambda}{e^{t^2/4}} = \lim_{t \to \infty} \frac{e^{t^2/4}}{t/2 \, e^{t^2/4}} = 0.$

THUS $y \to 0$ AS $t \to \infty$ AND FOR $t >> 1$, $y \approx 2/t$.