MATH 215 HW # 4

PLEASE TURN IN THE SOLUTIONS TO THE FOLLOWING PROBLEMS:

SECTION 3.2 # 26, 31

SECTION 3.1 # 11, 15, 22

SECTION 3.3 # 9, 17, 19, 25, 35

SECTION 3.4 # 5, 12, 14

MATH 215 HW 4 SOLUTIONS

SECTION 3.2 # 26

$L(y) \equiv x^2 y'' - x(x+2)y' + (x+2)y = 0 \quad x > 0; \quad y_1 = x, \quad y_2 = xe^x.$

(i) FOR y_1 WE CALCULATE $y_1' = 1, \quad y_1'' = 0.$

THUS $L(y_1) = 0 - x(x+2) + (x+2)x = 0 \rightarrow y_1$ IS A SOLUTION.

(ii) FOR y_2 WE CALCULATE $y_2' = e^x + xe^x, \quad y_2'' = xe^x + 2e^x.$

THUS
$$
\begin{aligned}
L(y_2) &= x^2(xe^x + 2e^x) - x(x+2)(x+1)e^x + (x+2)e^x x \\
&= x^3e^x + 2x^2e^x - (x^3 + x^2 + 2x^2 + 2x)e^x + (x^2 + 2x)e^x \\
&= e^x(x^3 + 2x^2 - x^3 - 3x^2 - 2x + x^2 + 2x) = 0.
\end{aligned}
$$
$\rightarrow y_2$ IS A SOLUTION.

NOW $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = x[e^x + xe^x] - [xe^x] = x^2 e^x \neq 0$ FOR $x > 0$.

THUS y_1, y_2 FUNDAMENTAL SET OF SOLUTIONS ON $x > 0$.

SECTION 3.2 # 31

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0.$$

WE WRITE AS $y'' + \frac{1}{x}y' + (1 - \nu^2/x^2)y = 0$ SO $p(x) = 1/x$

NOW $W' + p(x)W = 0 \rightarrow W' + \frac{1}{x}W = 0$

NOW $(xW)' = 0$ SO $xW = C$

OR $W = C/x$ WHICH DOES NOT VANISH AT ANY x WHEN $C \neq 0$.

THUS FOR ANY TWO SOLUTIONS y_1, y_2 WE MUST HAVE

THAT $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = C/x$ WHERE C IS A CONSTANT.

SECTION 3.1 # 11

$6y'' - 5y' + y = 0 \qquad y(0) = 4,\ y'(0) = 0$

let $y = e^{\lambda t} \rightarrow 6\lambda^2 - 5\lambda + 1 = (3\lambda - 1)(2\lambda - 1) = 0 \rightarrow \lambda = 1/3$ OR $\lambda = 1/2$

THUS $y = c_1 e^{t/3} + c_2 e^{t/2} \qquad y' = c_1/3\, e^{t/3} + c_2/2\, e^{t/2}$

NOW $y(0) = 4 \rightarrow 4 = c_1 + c_2 \qquad 4 = c_1 + c_2$

$y'(0) = 0 \rightarrow 0 = c_1/3 + c_2/2 \rightarrow 0 = 2c_1 + 3c_2$

THUS $8 = 2c_1 + 2c_2$
$0 = 2c_1 + 3c_2$ $\rightarrow$ subtract $8 = -c_2 \rightarrow c_2 = -8$

AND $c_1 = 12$

THUS $y = 12e^{t/3} - 8e^{t/2}$

$y \rightarrow -\infty$ AS $t \rightarrow \infty$.

SECTION 3.1 # 15

$y'' + 8y' - 9y = 0 \qquad y(1) = 1,\ y'(1) = 0$

let $y = e^{\lambda t} \rightarrow \lambda^2 + 8\lambda - 9 = (\lambda + 9)(\lambda - 1) = 0 \rightarrow \lambda = 1$ OR $\lambda = -9$.

THUS $y = c_1 e^{t-1} + c_2 e^{-9(t-1)} \Rightarrow y(1) = 1$ GIVES $c_1 + c_2 = 1$

$y' = c_1 e^{t-1} - 9c_2 e^{-9(t-1)} \Rightarrow y'(1) = 0 \longrightarrow c_1 - 9c_2 = 0$

THUS $c_2 = 1/10 \quad c_1 = 9/10$

HENCE $y = \dfrac{9}{10} e^{t-1} + \dfrac{1}{10} e^{-9(t-1)}$

SO $y \rightarrow +\infty$ AS $t \rightarrow \infty$

SECTION 3.1 # 22

$4y'' - y = 0 \qquad y(0) = 2,\ y'(0) = B.$ FIND B SO THAT $y \rightarrow 0$ AS $t \rightarrow \infty$.

PUT $y = e^{\lambda t} \rightarrow 4\lambda^2 - 1 = 0 \rightarrow \lambda = 1/2$ OR $\lambda = -1/2$.

THUS $y = c_1 e^{t/2} + c_2 e^{-t/2} \rightarrow y(0) = 2 \Rightarrow c_1 + c_2 = 2$

AND $y' = \dfrac{c_1}{2} e^{t/2} - \dfrac{c_2}{2} e^{-t/2} \rightarrow y'(0) = B \Rightarrow \dfrac{c_1}{2} - \dfrac{c_2}{2} = B$

WANT $c_1 = 0$ FOR $y \rightarrow 0$ AS $t \rightarrow \infty$.

THUS $B = -1$ IS NEEDED.

SECTION 3.3 # 9

$$y'' + 2y' - 8y = 0.$$

PUT $y = e^{\lambda t} \to \lambda^2 + 2\lambda - 8 = 0 \to \lambda = \dfrac{-2 \pm \sqrt{4+32}}{2} = -1 \pm 3 \to \lambda = -4$ OR $\lambda = 2.$

THUS $y = c_1 e^{-4t} + c_2 e^{2t}.$

SECTION 3.3 # 17

$$y'' + 4y = 0 \qquad y(0) = 0, \; y'(0) = 1.$$

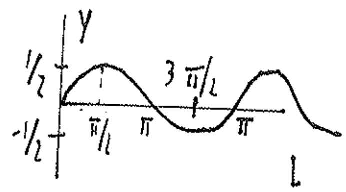
PUT $y = e^{\lambda t} \to \lambda^2 + 4 = 0$ SO $\lambda = \pm 2i.$

THUS $y_1 = \cos(2t)$, $y_2 = \sin(2t)$ ARE SOLUTIONS.

$y = c_1 \cos(2t) + c_2 \sin(2t)$; $\quad y(0) = 0 \longrightarrow c_1 = 0$

$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$; $\quad y'(0) = 1 \longrightarrow 2c_2 = 1 \to c_2 = 1/2.$

THUS $y = \dfrac{1}{2}\sin(2t)$ OSCILLATION WITH PERIOD π.



SECTION 3.3 # 19

$$y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2.$$

NOW PUT $y = e^{\lambda t} \to \lambda^2 - 2\lambda + 5 = 0 \to \lambda = \dfrac{+2 \pm \sqrt{4-20}}{2} = +1 \pm 2i$

THUS $y_1 = e^{+t}\cos(2t + \phi)$, $y_2 = e^{+t}\sin(2t+\phi)$ ARE SOLUTIONS.

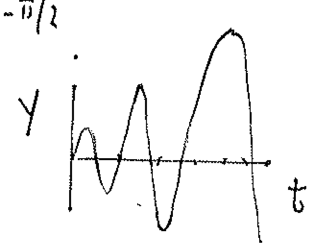
CHOOSE $\phi = 0$. THUS

$y = c_1 e^{+t}\cos(2t) + c_2 e^{+t}\sin(2t)$; $\quad y(\pi/2) = 0 \longrightarrow c_1 = 0$

$y' = +c_1 e^{+t}\cos(2t) - 2c_1 e^{+t}\sin(2t)$
$\quad + c_2 e^{+t}\sin(2t) + 2c_2 e^{+t}\cos(2t)$

NOW $y'(\pi/2) = 2 \longrightarrow 2c_2 e^{+\pi/2}\cos(\pi) = 2 \to c_2 = -e^{-\pi/2}.$

THUS $y = -e^{+(t-\pi/2)}\sin(2t)$ GROWING OSCILLATION



SECTION 3.3 # 25

$$y'' + 2y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = \alpha \geq 0.$$

(i) let $y = e^{\lambda t} \rightarrow \lambda^2 + 2\lambda + 6 = 0 \rightarrow \lambda = \dfrac{-2 \pm \sqrt{4-24}}{2} = -1 \pm \sqrt{5}\, i$;

THUS
$$y = c_1 e^{-t} \cos(\sqrt{5}\,t) + c_2 e^{-t} \sin(\sqrt{5}\,t)$$
$$y' = -c_1 e^{-t} \cos(\sqrt{5}\,t) - \sqrt{5}\, c_1 e^{-t} \sin(\sqrt{5}\,t) - c_2 e^{-t} \sin(\sqrt{5}\,t) + \sqrt{5}\, c_2 e^{-t} \cos(\sqrt{5}\,t)$$

NOW $y(0) = 2 \rightarrow c_1 = 2$

$y'(0) = \alpha \rightarrow -2 + \sqrt{5}\, c_2 = \alpha \rightarrow c_2 = \dfrac{1}{\sqrt{5}}(\alpha + 2)$.

THUS
$$y = 2e^{-t} \cos(\sqrt{5}\,t) + \frac{1}{\sqrt{5}}(\alpha+2) e^{-t} \sin(\sqrt{5}\,t)$$

(ii) FIND α SO THAT $y = 0$ AT $t = 1$

$$0 = 2e^{-1} \cos(\sqrt{5}) + \frac{1}{\sqrt{5}}(\alpha+2) e^{-1} \sin(\sqrt{5})$$

SO
$$2e^{-1} + \frac{1}{\sqrt{5}}(\alpha+2) \tan(\sqrt{5}) = 0$$

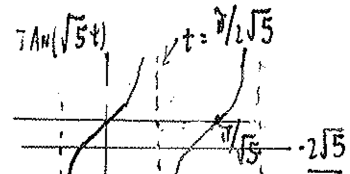
SO
$$\alpha = -2 + \frac{2\sqrt{5}e^{-1}}{\tan(\sqrt{5})} \cong 1.50878..$$

(iii) FIND SMALLEST t SO THAT $y = 0$.

$$y = e^{-t}\left(2\cos(\sqrt{5}\,t) + \frac{1}{\sqrt{5}}(\alpha+2)\sin(\sqrt{5}\,t)\right)$$

WANT
$$\frac{\alpha+2}{\sqrt{5}} = -2\cot(\sqrt{5}\,t) \rightarrow \cot(\sqrt{5}\,t) = -\left(\frac{\alpha+2}{2\sqrt{5}}\right)$$

THUS
$$\tan(\sqrt{5}\,t) = -\frac{2\sqrt{5}}{\alpha+2}$$



GRAPHICAL CONSTRUCTION OF ROOT

(iv) AS $\alpha \rightarrow \infty$, $\sqrt{5}\,t \rightarrow \pi \rightarrow t = \pi/\sqrt{5}$, FROM PICTURE ABOVE

SECTION 3.3 # 35

$$t^2 y'' + t y' + y = 0.$$

FOR EULER'S EQUATION PUT $y = t^\lambda$. THEN $y' = \lambda t^{\lambda-1}$, $y'' = \lambda(\lambda-1)t^{\lambda-2}$.

THUS $\quad t^2 (\lambda-1)\lambda t^{\lambda-2} + t\, t^{\lambda-1}\lambda + t^\lambda = t^\lambda(\lambda(\lambda-1)+\lambda+1) = 0.$

THUS $\quad \lambda^2 + 1 = 0 \quad$ SO $\lambda = \pm i$.

HENCE $\quad y = t^i = e^{i \ln t} = \cos(\ln t) + i \sin(\ln t)$ IS A COMPLEX SOLUTION.

THE <u>GENERAL</u> SOLUTION IS $\quad y = C_1 \cos(\ln t) + C_2 \sin(\ln t)$

SECTION 3.4 # 5

$$y'' - 2y' + 10y = 0.$$

PUT $y = e^{\lambda t} \rightarrow \lambda^2 - 2\lambda + 10 = 0 \qquad \lambda = \dfrac{2 \pm \sqrt{4-40}}{2} = +1 \pm 3i$

THUS $\quad y_1 = e^{+t}\cos(3t)$, $y_2 = e^{+t}\sin(3t)$ ARE SOLUTIONS.

SO $\quad y = C_1 e^{+t}\cos(3t) + C_2 e^{+t}\sin(3t)$ IS <u>GENERAL SOLUTION</u>

SECTION 3.4 # 12

$$y'' - 6y' + 9y = 0, \qquad y(0) = 0,\ y'(0) = 2$$

PUT $y = e^{\lambda t} \rightarrow \lambda^2 - 6\lambda + 9 = 0 \rightarrow (\lambda-3)^2 = 0 \rightarrow \lambda = 3$ REPEATED ROOT

THUS $\quad y = C_1 e^{3t} + C_2 t e^{3t}$ IS GENERAL SOLUTION

$y(0) = 0 \longrightarrow C_1 = 0 \qquad\qquad y' = 3C_1 e^{3t} + C_2 e^{3t} + 3C_2 t e^{3t}$

$y'(0) = 2 \longrightarrow \quad 2 = 3C_1 + C_2 \rightarrow C_2 = 2.$

THUS $\quad y = 2te^{3t}$ IS SOLUTION. $y \to +\infty$ AS $t \to \infty$.

SECTION 3.4 # 14

$$y'' + 4y' + 4y = 0 \qquad y(-1) = 2, \ y'(-1) = 1.$$

PUT $y = e^{\lambda t}$ SO $\lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0 \rightarrow \lambda = -2.$

THUS $y = c_1 e^{-2t} + c_2 t e^{-2t}$

HOWEVER IF WE PUT

$$y = \tilde{c}_1 e^{-2(t+1)} + \hat{c}_2 (t+1) e^{-2(t+1)}$$ THE ALGEBRA

IS EASIER.

NOW $y(-1) = 2 \rightarrow \tilde{c}_1 = 2$

$y'(-1) = 1 \rightarrow 1 = \tilde{c}_2 - 2\tilde{c}_1 \rightarrow \hat{c}_2 = 5.$

HENCE $y = 2e^{-2(t+1)} + 5(t+1)e^{-2(t+1)}$ IS SOLUTION.