

PROBLEM 1

$$y' - \frac{p}{x} y = x$$

integrating factor $\phi(x) = \exp\left[-\int \frac{p}{x} dx\right]$.

now $\int \frac{p}{x} dx = p \log x \rightarrow \phi(x) = x^{-p}$.

HENCE MULTIPLY BY x^{-p} .

$$(x^{-p} y)' = x^{1-p}$$

$$x^{-p} y = \begin{cases} \frac{1}{2-p} x^{2-p} + C & p \neq 2 \\ \log x + C & p = 2 \end{cases}$$

THUS

$$y = \frac{1}{2-p} x^2 + C x^p \quad \text{if } p \neq 2$$

$$y = x^2 (\log x + C) \quad \text{if } p = 2.$$

} general solution.

NOTICE: $\lim_{x \rightarrow 0} x^2 \log x = 0$. THUS y is well-defined at $x=0$ if $p=2$.

the general solution is not well-defined at $x=0$ if $p < 0$, since $x^p \rightarrow \infty$ as $x \rightarrow 0$ if $p < 0$.

PROBLEM 2

$$y' = \frac{x(y-1)^2}{x-1} \quad y(2) = 2$$

equation is separable: $\frac{x}{x-1} dx = (y-1)^{-2} dy$

now $\frac{x}{x-1} = \frac{(x-1)+1}{x-1} = 1 + \frac{1}{x-1}$.

THUS $\int \left(1 + \frac{1}{s-1}\right) ds = -(y-1)^{-1}$ upon integrating.

$$\rightarrow C + x + \log|x-1| = -(y-1)^{-1}$$

now $y(2) = 2 \rightarrow C + 2 = -1 \rightarrow C = -3$

$$y-1 = \frac{-1}{-3+x+\log|x-1|} \rightarrow y = 1 + \frac{1}{-3+x+\log|x-1|}$$

now if $y(2) = 1$ we have $C + 2 = -\frac{1}{0}$

we would like to let $C = \infty$. NOTICE THAT THE

FUNCTION $y(x) \equiv 1$ SATISFIES THE INITIAL CONDITION

AND THE ODE.

PROBLEM 3 LET $V > 0$ IF BODY IS MOVING AWAY FROM EARTH'S SURFACE AND $V < 0$ IF IT MOVES TOWARD THE EARTH.

THUS

$$m \frac{dv}{dt} = -\frac{mgR^2}{(x+R)}$$

$$V=0 \text{ WHEN } X=3R$$

X : distance from surface of earth.

now $\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$

(take distance as measured from center)

$$v \frac{dv}{dx} = -\frac{mgR^2}{(x+R)^2} \rightarrow \frac{1}{2} v^2 = \frac{mgR^2}{(x+R)} + C$$

BUT $V=0$ WHEN $X=3R \rightarrow \frac{mgR^2}{4R} = -C \rightarrow C = -\frac{mgR}{4}$

HENCE $\frac{1}{2} v^2 = mgR^2 \left[\frac{1}{x+R} - \frac{1}{4R} \right]$

need to take - square root since $V < 0$ if it goes down.

$$\rightarrow i) v = -(2mgR^2)^{1/2} \left[\frac{1}{x+R} - \frac{1}{4R} \right]^{1/2}$$

$x+R$: distance from center of earth.

now when $x=0$ at earth's surface we get

$$v = -(2gR^2)^{1/2} \left[\frac{1}{R} - \frac{1}{4R} \right]^{1/2}$$

$$ii) v = -(2gR)^{1/2} \left(\frac{3}{4} \right)^{1/2} \rightarrow v = -\sqrt{3}(gR)^{1/2}$$

\rightarrow velocity at impact.

now return to (i)

$$v = -(2gR^2)^{1/2} \left[\frac{1}{x+R} - \frac{1}{4R} \right]^{1/2}$$

but $v = dx/dt$:

$$\frac{dx}{dt} = -(2gR^2)^{1/2} \left[\frac{1}{x+R} - \frac{1}{4R} \right]^{1/2}$$

NOTE: $X(0) = 3R$. FIND T SUCH THAT $X(T) = 0$.

$$\left[\frac{1}{x+R} - \frac{1}{4R} \right]^{1/2} dx = -(2gR^2)^{1/2} dt$$

$$\rightarrow \int_{3R}^x \left[\frac{1}{s+R} - \frac{1}{4R} \right]^{-1/2} ds = -(2gR^2)^{1/2} \int_0^t ds$$

WANT $x=0$ WHEN $t=T$. (time of impact).

$$\rightarrow T = (2gR^2)^{-1/2} \int_0^{3R} \left[\frac{1}{s+R} - \frac{1}{4R} \right]^{-1/2} ds$$

NOTE: we could have also interpreted the initial distance as being measured from the earth's center in which case $X(0) = 4R$.

MATH 215 QUIZ #2 SOLUTIONS

PROBLEM 1

$$x^2 y' = 4x^2 + 7xy + 2y^2 \quad \text{WITH} \quad y(1) = 0$$

THIS GIVES

$$y' = 4 + 7y/x + 2y^2/x^2$$

$$\text{let } y = xv. \rightarrow y' = xv' + v.$$

$$xv' + v = 4 + 7v + 2v^2 \rightarrow xv' = 2v^2 + 6v + 4 = 2(v^2 + 3v + 2)$$

$$\frac{dv}{(v+2)(v+1)} = \frac{2dx}{x} \quad -\frac{1}{v+2} + \frac{1}{v+1} = \frac{1}{(v+1)(v+2)}$$

$$\left(\frac{1}{v+1} - \frac{1}{v+2} \right) dv = \frac{2dx}{x}$$

$$\text{Integrate} \quad \ln(v+1) - \ln(v+2) = 2 \ln(xC)$$

THIS

$$\ln \left| \frac{y+x}{y+2x} \right| = \ln(x^2 C^2)$$

$$\rightarrow \left| \frac{y+x}{y+2x} \right| = x^2 C^2.$$

$$\text{WHEN } x=1 \rightarrow y=0 \quad \frac{1}{2} = C^2 \rightarrow C^2 = 1/2$$

$$\left| \frac{y+x}{y+2x} \right| = \frac{x^2}{2}$$

NOW WE CAN UNDO $| |$ sign since when $x > 0$ $y > 0$.

$$\text{let } x > 0 \rightarrow \frac{y+x}{y+2x} = \frac{x^2}{2}$$

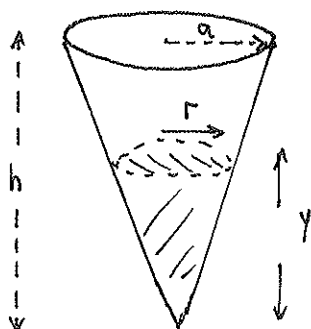
$$(y+x) = \frac{x^2}{2} (y+2x)$$

$$y[1 - x^2/2] = x^3 - x$$

$$y = \frac{x^3 - x}{1 - x^2/2}$$

DEFINED FOR $x < \sqrt{2}$.

PROBLEM 3



$$dV/dt = K - \alpha A \quad \alpha > 0$$

$a/h = r/y$ since we have similar triangles

$$V = \frac{1}{3} \pi r^2 y \quad A = \pi r^2$$

$$\text{since } r = \frac{a}{h} y \quad \text{THEN} \quad V = \frac{1}{3} \pi \frac{a^2}{h^2} y^3$$

$$dV/dt = \frac{\pi a^2}{h^2} y^2 dy/dt$$

$$A = \pi \frac{a^2}{h^2} y^2$$

$$i) \quad \frac{\pi a^2}{h^2} y^2 dy/dt = K - \alpha \frac{\pi a^2}{h^2} y^2$$

$$y(0) = y_0$$

$$ii) \quad \text{equilibrium is when } dy/dt = 0 \rightarrow y_{EQ} = \left(\frac{K h^2}{\alpha \pi a^2} \right)^{1/2} > 0$$

it is stable, as $\lim_{t \rightarrow \infty} y(t) = y_{EQ}$ FOR ANY INITIAL CONDITION.

$$iii) \quad \text{FOR the water to not overflow we need } y_{EQ} < h.$$

$$\text{This means } \left(\frac{K h^2}{\alpha \pi a^2} \right)^{1/2} < h$$

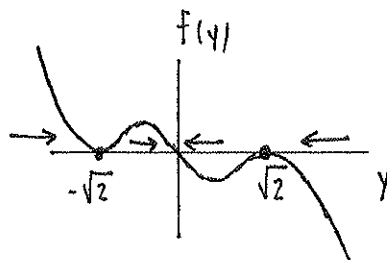
$$\text{OR } K < \alpha \pi a^2$$

$$K_c = \alpha \pi a^2$$

PROBLEM 4

$$\frac{dy}{dt} = -y(y^4 - 4y^2 + 4)$$

$$\frac{dy}{dt} = f(y) = -y(y^2 - 2)^2$$



EQ. POINTS AT $y=0$, $y = \pm \sqrt{2}$.

FROM THE FIGURE WE HAVE

- IF $y(0) = 2$ then $y(t) \rightarrow \sqrt{2}$ as $t \rightarrow \infty$
- IF $y(0) = -1/2$ then $y(t) \rightarrow 0$ as $t \rightarrow \infty$
- IF $y(0) = -4$ then $y(t) \rightarrow -\sqrt{2}$ as $t \rightarrow \infty$

MATH 215 QUIZ # 3PROBLEM 1

$$y'' + 2y' + 2y = 0 \quad y(\pi/4) = 2 \quad y'(\pi/4) = -2$$

let $y = e^{rx}$: $r^2 + 2r + 2 = 0 \rightarrow (r+1)^2 + 1 = 0 \quad (r+1) = \pm i$

$$r = -1 \pm i \quad e^{-(1 \pm i)x} \text{ is a solution}$$

$$\rightarrow y_1 = e^{-x} \cos x \quad y_2 = e^{-x} \sin x$$

$$y = C_1 y_1 + C_2 y_2 \text{ gen. solution}$$

$$y = e^{-x} (C_1 \cos x + C_2 \sin x)$$

much more convenient to write

$$y = e^{-(x-\pi/4)} (b_1 \cos(x-\pi/4) + b_2 \sin(x-\pi/4))$$

$$y(\pi/4) = 2 \rightarrow b_1 = 2$$

$$y'(\pi/4) = -b_1 + b_2 = -2 \rightarrow b_2 = 0$$

$$y = 2e^{-(x-\pi/4)} \cos(x-\pi/4)$$

PROBLEM 2

let $y = x^p$, $y' = px^{p-1}$, $y'' = p(p-1)x^{p-2}$

$$x^2 y'' + 3x y' + y = 0$$

$$\rightarrow p(p-1) + 3p + 1 = 0$$

$$p^2 + 2p + 1 = 0 \quad (p+1)^2 = 0 \quad p = -1 \quad (5 \text{ pts for this})$$

so $y_1 = 1/x$ is a solution.

now let $y = x^{-1} v$

$$y' = -x^{-2} v + x^{-1} v' \quad y'' = 2x^{-3} v - 2x^{-2} v' + x^{-1} v''$$

substitute into the ODE to get

$$x^2 (x^{-1} v'' - 2x^{-2} v' + 2x^{-3} v) + 3x (x^{-1} v' - x^{-2} v) + x^{-1} v = 0$$

$$\rightarrow x v'' - 2v' + 3v' = 0$$

$$\rightarrow x v'' + v' = 0 \rightarrow (x v')' = 0 \rightarrow x v' = C_1$$

THU, $v' = C_1/x$

$\rightarrow v = C_1 \log x + C_2$

THU, $y = \frac{1}{x} v \rightarrow y = \frac{C_1}{x} \log x + C_2/x$ is general solution.

PROBLEM 3 $y'' + p y' + y = 1 + e^{-x}$ $y(0) = y'(0) = 0$

homog. problem $y'' + p y' + y = 0$

let $y = e^{rx} \rightarrow r^2 + pr + 1 = 0$
 $r_{\pm} = \frac{-p \pm (p^2 - 4)^{1/2}}{2}$ (3)

particular solution

$y_{p1}'' + p y_{p1}' + y_{p1} = 1 \rightarrow y_{p1} = 1$ (2)

now

$y_{p2}'' + p y_{p2}' + y_{p2} = e^{-x}$

try

$y_{p2} = A e^{-x}$

$\rightarrow A - pA + A = 1$

$\rightarrow A = \frac{1}{2-p}$ $p \neq 2$.

$\lim_{x \rightarrow \infty} y(x) = 1$ for $p > 0$

(3)

if $p = 2$ not valid.

$y_{p2} = \frac{1}{2-p} e^{-x}$ $p \neq 2$.

(2)

notice if $p = 2$ then $r_{\pm} = -1$ which is a repeated root of multiplicity 2. hence y_{p2} must have a different form.

CASE 1 $0 < p < 2$. $r_{\pm} = \frac{-p \pm (4-p^2)^{1/2}}{2}$

$\rightarrow y_1 = e^{-px/2} \cos\left[\frac{(4-p^2)^{1/2}}{2} x\right]$ $y_2 = e^{-px/2} \sin\left[\frac{(4-p^2)^{1/2}}{2} x\right]$

THU, $y = e^{-px/2} \left[C_1 \cos\left(\frac{(4-p^2)^{1/2}}{2} x\right) + C_2 \sin\left(\frac{(4-p^2)^{1/2}}{2} x\right) \right] + 1 + \frac{e^{-x}}{2-p}$
 (gen. solution)

now $y(0) = 0$

$$\rightarrow C_1 + 1 + \frac{1}{2-p} = 0 \rightarrow C_1 = \frac{1}{p-2} - 1 = \frac{3-p}{p-2}$$

$y'(0) = 0$

$$\rightarrow -\frac{p}{2} C_1 + \frac{(4-p^2)^{1/2}}{2} C_2 + \frac{1}{p-2} = 0$$

$$C_1 = -1 + 1/p-2$$

$$\frac{(4-p^2)^{1/2}}{2} C_2 = \frac{p}{2} \left(\frac{3-p}{p-2} \right) - \frac{1}{p-2} = \frac{1}{p-2} \left[\frac{1}{2} p(3-p) - 1 \right]$$

$$\rightarrow C_2 = \frac{1}{(p-2)(4-p^2)^{1/2}} [p(3-p) - 2] = \frac{(1-p)}{(4-p^2)^{1/2}}$$

~~not~~

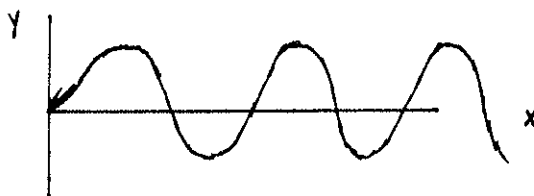
if $p=0$, then

$$C_2 = \frac{(1-p)}{(4-p^2)^{1/2}}$$

$$y = C_1 \cos x + C_2 \sin x + 1 + e^{-x/2}$$

$$C_1 = -3/2 \quad C_2 = 1/2 \quad (\text{substitute } p=0 \text{ into boxed formula})$$

now
we get a
graph sort of
like



(5)

NOTICE $\lim_{x \rightarrow \infty} y(x) = 1$ for any p with $2 > p > 0$.

(solution to homog. problem tend to zero as $x \rightarrow \infty$

for any $p > 0$). Also $y_{p_1} = 1$ and $y_{p_2} = \frac{1}{2-p} e^{-x}$, if $p \neq 2$.

There is no limiting behavior of $y(x)$ as $x \rightarrow \infty$ when $p=0$

CASE 2

$p > 2$.

$$\Gamma_{\pm} = \frac{-p \pm (p^2 - 4)^{1/2}}{2} \rightarrow \text{REAL}$$

$$y = C_1 e^{\Gamma_+ x} + C_2 e^{\Gamma_- x} + 1 + \frac{e^{-x}}{2-p}$$

$$y(0) = 0 \rightarrow C_1 + C_2 + 1 + \frac{1}{2-p} = 0$$

$$y'(0) = 0 \rightarrow \Gamma_+ C_1 + \Gamma_- C_2 - \frac{1}{2-p} = 0$$

$$\left. \begin{aligned} \Gamma_+ C_1 + \Gamma_+ C_2 + \Gamma_+ + \Gamma_+/2 \cdot p &= 0 \\ \Gamma_+ C_1 + \Gamma_- C_2 - 1/2 \cdot p &= 0 \end{aligned} \right\} \text{ subtract } \rightarrow (\Gamma_+ - \Gamma_-) C_2 = -\Gamma_+ - \Gamma_+/2 \cdot p + 1/p \cdot 2$$

$$C_2 = \frac{-1}{\Gamma_+ - \Gamma_-} \left[\Gamma_+ + \Gamma_+/2 \cdot p + 1/2 \cdot p \right] \quad (5)$$

$$\text{Now } \left. \begin{aligned} \Gamma_- C_1 + \Gamma_- C_2 + \Gamma_- + \Gamma_-/2 \cdot p &= 0 \\ \Gamma_+ C_1 + \Gamma_- C_2 - 1/2 \cdot p &= 0 \end{aligned} \right\} \text{ subtract } \rightarrow (\Gamma_- - \Gamma_+) C_1 = -\Gamma_- - \Gamma_-/2 \cdot p - 1/2 \cdot p$$

clearly since $\Gamma_+ > 0$ and $\Gamma_- > 0$ then $y \rightarrow 1$ as $x \rightarrow \infty$.

CASE 3 $p = 2$.

if $p = 2$ then $\Gamma_+ = \Gamma_- = -1$

so $y_1 = e^{-x}$ is a solution.

the second solution is $y_2 = x e^{-x}$.

(5)

$$\text{Now to find } y_{p2}: \quad y_{p2}'' + 2y_{p2}' + y_{p2} = e^{-x}$$

$$\text{let } y_{p2} = v e^{-x}$$

$$y_{p2}' = v' e^{-x} - v e^{-x} \quad y_{p2}'' = v'' e^{-x} - 2v' e^{-x} + v e^{-x}$$

$$\text{substitute to get } \rightarrow v'' - 2v' + v + 2v' - 2v + v = 1$$

$$\rightarrow v'' = 1 \quad \text{choose } v = x^2/2$$

$$\rightarrow y_{p2} = \frac{x^2}{2} e^{-x}$$

$$\text{Hence } y = C_1 e^{-x} + C_2 x e^{-x} + \frac{x^2}{2} e^{-x} + 1$$

$$y(0) = 0 \rightarrow C_1 + 1 = 0 \quad C_1 = -1$$

$$y'(0) = 0 \rightarrow -C_1 + C_2 = 0 \quad C_2 = C_1 = -1$$

$$y = -e^{-x} - x e^{-x} + \frac{x^2}{2} e^{-x} + 1$$

$$\text{clearly } \lim_{x \rightarrow \infty} y(x) = 1$$