M215 HW # 2 SOLUTIONS

SECTION 2.2

PROBLEM 9

$$y' = (1-2x)\,y^2 \qquad y(0) = -1/6$$

SO $$\frac{dy}{y^2} = (1-2x)\,dx \quad\rightarrow\quad -\frac{1}{y} = (x - x^2 + C)$$

NOW $$y(0) = -1/6 \quad\rightarrow\quad 6 = C \quad\rightarrow\quad y = \frac{1}{x^2 - x - 6}.$$

THUS $$y = \frac{1}{(x-3)(x+2)} \quad \text{DEFINED FOR} \quad -2 < x < 3.$$

PROBLEM 14

$$y' = \frac{x}{\sqrt{1+x^2}}\,y^3, \qquad y(0) = 1.$$

$$\frac{1}{y^3}\,dy = \frac{x}{(1+x^2)^{1/2}}\,dx \quad\rightarrow\quad -\frac{1}{2}\,y^{-2} = (1+x^2)^{1/2} + C.$$

NOW $$y(0) = 1 \quad\rightarrow\quad -\frac{1}{2} = 1 + C \quad \text{SO} \quad C = -3/2.$$

$$\rightarrow\quad -\frac{1}{2}\,y^{-2} = (1+x^2)^{1/2} - \frac{3}{2} \quad\rightarrow\quad \frac{1}{y^2} = 3 - 2(1+x^2)^{1/2}.$$

THUS $$y = \left[\,3 - 2(1+x^2)^{1/2}\,\right]^{-1/2}.$$

THIS IS DEFINED FOR $$3 - 2(1+x^2)^{1/2} \geq 0 \quad\rightarrow\quad 4(1+x^2) \leq 9.$$

THUS $$1 + x^2 \leq 9/4 \quad\rightarrow\quad |x| \leq \sqrt{5}/2$$ IS INTERVAL OF DEFINITION.

PROBLEM 15

$$y' = \frac{2x}{1+2y}, \qquad y(2) = 0.$$

NOW $$(1+2y)\,dy = 2x\,dx \quad\rightarrow\quad y + y^2 = x^2 + C.$$

NOW $$y(2) = 0 \quad\rightarrow\quad 0 = 4 + C \qquad C = -4.$$

THUS $$y^2 + y = x^2 - 4$$

$$y^2 + y + (4 - x^2) = 0 \quad\rightarrow\quad y = \frac{-1 \pm \sqrt{1 - 4(4 - x^2)}}{2}$$

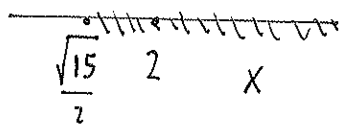
THIS GIVES $$y = \frac{-1 \pm \sqrt{4x^2 - 15}}{2}$$

WE NEED $y(2) = 0$ SO THAT $+$ SIGN IS NEEDED.

THUS $$y = \frac{-1 + \sqrt{4x^2 - 15}}{2}$$

EXISTS FOR $4x^2 - 15 \geq 0 \rightarrow |x| \geq \sqrt{15}/2$.

HOWEVER WE HAVE $x = 2$ IS INITIAL POINT THUS THIS INEQUALITY MEANS THAT $x > \sqrt{15}/2$.



PROBLEM 20 $(*)$ $y^2 (1 - x^2)^{1/2}\, dy = \sin^{-1}(x)\, dx \qquad y(0) = 1.$

$$y^2\, dy = \frac{\sin^{-1}(x)}{(1 - x^2)^{1/2}}\, dx \longrightarrow \int^y y^2\, dy = \int^x \frac{\sin^{-1}(x)}{\sqrt{1 - x^2}}\, dx.$$

LET $u = \sin^{-1}(x)$ $\quad du = \frac{1}{\sqrt{1 - x^2}}\, dx \longrightarrow \frac{1}{3} y^3 = \int^u u\, du = \frac{u^2}{2} + C.$

SO $\frac{1}{3} y^3 = \frac{1}{2}\left[\sin^{-1}(x)\right]^2 + C.$ NOW $y(0) = 1$ SO THAT

$$\frac{1}{3} = C \rightarrow C = 1/3.$$

HENCE $$y^3 = \frac{3}{2}\left(\sin^{-1} x\right)^2 + 1.$$

THUS $$y = \left[\frac{3}{2}\left(\sin^{-1} x\right)^2 + 1\right]^{1/3}$$

NOW THIS IS DEFINED FOR $-1 < x < 1$ SINCE IN $(*)$

WE NEED THIS RANGE

PROBLEM 22 $\quad y' = \dfrac{3X^2}{3y^2-4}$ (*) , $y(1) = 0$

NOW $\quad (3y^2-4)\,dy = 3x^2\,dX.$

THUS $\quad y^3 - 4y = X^3 + C \qquad y(1) = 0 \longrightarrow C = -1.$

THUS $\quad y^3 - 4y = X^3 - 1$ DEFINES $y(X)$ IMPLICITLY.

IT IS DEFINED ON AN INTERVAL ABOUT $X = 1$ PROVIDED y' IS FINITE.

NOW TAKE DERIVATIVE FORMULA (*). NOTICE $y = \pm\, 2/\sqrt{3}$ IS WHERE $y' = 0$. THEN FROM

$$y^3 - 4y = X^3 - 1 \quad \text{WE GET}$$

+ ROOT $\quad y = 2/\sqrt{3} \longrightarrow \left(2/\sqrt{3}\right)^3 - 4\left(2/\sqrt{3}\right) = X^3 - 1$

$$4/3\left(2/\sqrt{3}\right) - 4\left(2/\sqrt{3}\right) = X^3 - 1$$

$$\longrightarrow X^3 - 1 = +\left(2/\sqrt{3}\right)\left(-8/3\right) = -16/3\sqrt{3}.$$

- ROOT $\quad y = -2/\sqrt{3} \longrightarrow \left(-2/\sqrt{3}\right)^3 - 4\left(-2/\sqrt{3}\right) = X^3 - 1$

$$4/3\left(-2/\sqrt{3}\right) - 4\left(-2/\sqrt{3}\right) = X^3 - 1$$

$$\text{SO} \quad X^3 - 1 = 16/3\sqrt{3}$$

COMBINE : THUS WE NEED $\quad |X^3 - 1| \leq 16/3\sqrt{3} \Longrightarrow -1.28 < X < 1.60$
ESTIMATES

PROBLEM 32 $\quad \dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy} = \dfrac{x}{2y} + \dfrac{3y}{2x}$

(i) NOTICE $\quad dy/dx = \dfrac{1}{2(y/x)} + \left(\dfrac{y}{x}\right)\dfrac{3}{2} = F(y/x).$

THIS EQUATION IS HOMOGENEOUS FORM.

(ii) LET $v = y/x$ SO $y = xv \longrightarrow y' = xv' + v.$

SO $\quad xv' + v = \dfrac{1}{2v} + \dfrac{3v}{2} \quad \rightarrow \quad xv' = \dfrac{1}{2v} + \dfrac{v}{2}.$

THEN $\quad xv' = \dfrac{1}{2v}(1+v^2) \qquad x\dfrac{dv}{dx} = \dfrac{1}{2v}(1-v^2)$

THEN $\quad \dfrac{2v}{1+v^2}\,dv = \dfrac{dx}{x}.$

SO $\quad +\ln|1+v^2| = \ln|x| + \ln C.$

THEN $\quad \ln|1+v^2| = +\ln(|x|C)$

HENCE $\quad |1+v^2| = C|x|$

NOW PUT $v = y/x \longrightarrow |1 + y^2/x^2| = C|x|$

WE OBTAIN $\quad |y^2+x^2| = C|x|^3$

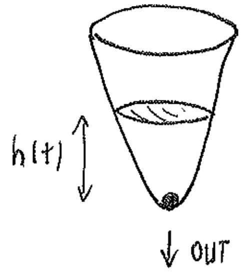
THUS $\quad y^2 + x^2 = Cx^3$ FOR $x > 0$ AND $C > 0.$

SECTION 2.3 # 6

(i) POTENTIAL ENERGY LOST = KINETIC ENERGY GAINED

$$mgh = \frac{1}{2} m v^2 \rightarrow v = \sqrt{2gh}. \qquad v: \text{speed.}$$

(ii)



$$\frac{d\bar{V}}{dt} = -[\text{OUTFLOW RATE}]. \qquad \bar{V}: \text{volume}$$

IN A TIME dt AMOUNT OF MATERIAL LEFT THE BOWL IS $= v\,a\,dt$ $\quad a$: AREA OF SMALL HOLE.

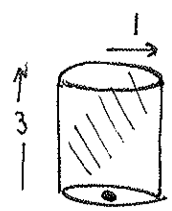
THUS $$\frac{d\bar{V}}{dt} = -\alpha a v, \quad 0 < \alpha < 1 \text{ IS CONTRACTION COEFFICIENT.}$$

NOW $$\bar{V} = \int_0^h A(s)\,ds$$ WHERE $A(s)$: CROSS-SECTIONAL AREA.

THUS $$d\bar{V}/dt = A(h)\,dh/dt$$

THEN, $$A(h)\,dh/dt = -\alpha a \sqrt{2gh}$$ ODE FOR $h(t)$.

(iii)



$A = \pi(1)$ IS AREA OF CROSS SECTION.

$\alpha = 0.6$ FOR WATER $\quad g = 9.8\ m/sec^2 \quad a = (1/10)^2 \pi$

NOW $$\pi\, dh/dt = -\alpha a \sqrt{2g}\, h^{1/2}. \qquad h(0) = 3$$

$$h^{-1/2}\,dh = -\frac{\alpha a \sqrt{2g}}{\pi}\,dt$$

THUS $$2h^{1/2} = -\frac{\alpha a \sqrt{2g}}{\pi} t + C.$$

WHEN $t = 0 \rightarrow 2\sqrt{3} = C \rightarrow 2h^{1/2} = -\dfrac{\alpha a \sqrt{2g}}{\pi} t + 2\sqrt{3}.$

THUS $h^{1/2} = \sqrt{3} - \dfrac{\alpha a \sqrt{2g}}{2\pi} t$.

BUT $a = \pi (1/10)^2$ SO $h = \left(\sqrt{3} - \dfrac{\alpha (1/10)^2 \sqrt{2g}}{2} t\right)^2$.

NOW $h = 0$ WHEN

$$t = \frac{2\sqrt{3}}{\alpha (1/10)^2 \sqrt{2g}} = \frac{\sqrt{6}\,(100)}{\alpha \sqrt{g}}.$$

WE GET $t = \dfrac{\sqrt{6}\,(100)}{(0.6)\sqrt{9.8}} \approx 130.4$ SECONDS

SECTION 2.3 # 18 a

$$du/dt = -k(u - T(t)) \qquad T(t) = T_0 + T_1 \cos(wt).$$

(a) CALCULATE SOLUTION IN GENERAL FORM.

NOW $du/dt = -k\left(u - T_0 - T_1 \cos(wt)\right)$

SO $u' + ku = k(T_0 + T_1 \cos(wt))$.

integrating factor is e^{kt}.

$$(ue^{kt})' = kT_0 e^{kt} + kT_1 e^{kt} \cos(wt).$$

THUS $ue^{kt} = T_0 e^{kt} + kT_1 \int^t e^{kt} \cos(wt)\, dt$ (*)

WE MUST CALCULATE $I = \int^t e^{kt} \cos(wt)\, dt$.

WE CAN INTEGRATE BY PARTS TWICE OR DO IT SLICK WAY.

NOW $\quad I = \mathrm{RE}\left(\int^t e^{Kt} e^{iwt}\, dt \right) = \mathrm{RE}\left(\int^t e^{(K+iw)t}\, dt \right)$

$$I = \mathrm{RE}\left(\frac{1}{K+iw} e^{(K+iw)t} \right) = \mathrm{RE}\left(\frac{(K-iw)}{K^2+w^2} \left(e^{(K+iw)t} \right) \right)$$

SO $\quad I = \dfrac{e^{Kt}}{K^2+w^2} \mathrm{RE}\left((K-iw)\left[\cos(wt) + i\sin wt \right] \right)$

$$I = \frac{e^{Kt}}{K^2+w^2} \left(K\cos(wt) + w\sin(wt) \right)$$

FROM $(*)$ THEN

$$U e^{Kt} = T_0 e^{Kt} + \frac{K T_1 e^{Kt}}{K^2+w^2} \left(K\cos(wt) + w\sin(wt) \right) + C$$

THUS $\quad U = C e^{-Kt} + T_0 + \dfrac{K T_1}{K^2+w^2} \left(K\cos(wt) + w\sin(wt) \right)$

$\uparrow$ $\quad\longleftarrow$ steady-state part $\longrightarrow$

transient

part

decays as $t \to \infty$

SECTION 2.3 # 16

$$(*) \begin{cases} dT/dt = -K(T - T_{ENV}) \\ T(0) = T_0. \end{cases}$$

WE ARE GIVEN $T(0) = 200$ AND $T(1) = 190$ $T_{ENV} = 70$.

FIND t SUCH THAT $T = 150$.

THE SOLUTION TO $(*)$ IS

$$T = T_{ENV} + (T_0 - T_{ENV})e^{-Kt}.$$

SO $$T = 70 + 130e^{-Kt}.$$

NOW $T(1) = 190 \longrightarrow 190 = 70 + 130e^{-K} \longrightarrow 120/130 = e^{-K}$.

THUS $-K = \ln(20/130) \longrightarrow K = \ln(130/120) = \ln(13/12)$

NOW WHEN IS $T = 150$?

SET $150 = 70 + 130e^{-Kt}$

$$\frac{80}{130} = e^{-Kt} \longrightarrow \ln(80/130) = -Kt.$$

$$t = \frac{\ln(130/80)}{K} = \frac{\ln(130/80)}{\ln(130/120)} = \frac{.4855}{.0800}$$

THUS

$$t = \frac{\ln(130/80)}{\ln(130/120)} = \frac{\ln(13/8)}{\ln(13/12)} \cong \frac{.4855}{.0800} \cong 6.06 \text{ MINUTES}$$

SECTION 2.4 # 3

$y' + \tan t \, y = \sin t \qquad y(\pi) = 0.$

NOW $p(t) = \tan t$ IS CONTINUOUS ON $\pi/2 < t < 3\pi/2$.

BY THEOREM, y EXISTS ON INTERVAL $\pi/2 < t < 3\pi/2$

SECTION 2.4 # 16

$$y' = \frac{t^2}{y(1+t^3)} = f(t,y), \quad y(0) = y_0.$$

NOTICE THAT f, f_y ARE CONTINUOUS EXCEPT AT $t = -1$ AND $y = 0$.

WE CALCULATE $\qquad y \, dy = \dfrac{t^2}{1+t^3} \, dt.$

$$\frac{1}{2} y^2 = \frac{1}{3} \ln(1+t^3) + C.$$

BUT $y(0) = y_0 \longrightarrow \qquad \dfrac{1}{2} y^2 = \dfrac{1}{3} \ln(1+t^3) + \dfrac{1}{2} y_0^2.$

THUS $\qquad y = \left[\dfrac{2}{3} \ln(1+t^3) + y_0^2 \right]^{1/2}.$

THIS SOLUTION IS DEFINED ON $\qquad \dfrac{2}{3} \ln(1+t^3) + y_0^2 > 0.$

THUS $\qquad \ln(1+t^3) > -3y_0^2/2$

$$1 + t^3 > e^{-3y_0^2/2}$$

OR $\qquad t^3 > e^{-3y_0^2/2} - 1$

THIS MEANS THAT TAKING NEGATIVE ROOT

$$t > -\left[1 - e^{-3y_0^2/2} \right]^{1/3}.$$

SECTION 2.4 # 31

$$dy/dt = (\Gamma \cos t + T)\, y - y^3. \quad ; \quad y(t_0) = y_0$$

THIS IS BERNOULLI'S EQUATION. LET $V = y^{-2}$.

OR $y = V^{-1/2}$.

NOW $y' = -\frac{1}{2} V^{-3/2} V' = \mu(t) V^{-1/2} - V^{-3/2}$. $\mu_0(t) = \Gamma \cos t + T$.

THUS $-\frac{1}{2} V' = \mu V - 1$.

NOW $V' = -2\mu V + 2$.

$V' + 2\mu V = 2$ INTEGRATING FACTOR $\phi(t) = \exp\left(\int_0^t 2\mu(s)\, ds\right)$.

NOW $(\phi V)' = 2\phi$.

THUS $\phi V = 2\int_{t_0}^{t} \phi(s)\, ds + C$.

HENCE $$V = \frac{2\int_{t_0}^{t} \phi(s)\, ds + C}{\phi(t)} = \frac{1}{y^2}.$$

HENCE $$y(t) = \left[\frac{\phi(t)}{2\int_{t_0}^{t} \phi(s)\, ds + C}\right]^{1/2}$$

NOW $\phi(t) = \exp\left(\int^t (2T + 2\Gamma \cos t)\, dt\right) = e^{2tT + 2\Gamma \sin t}$.

THUS $$y(t) = \left[\frac{\phi(t)}{2\int_{t_0}^{t} \phi(s)\, ds + C}\right]^{1/2}, \quad \phi(t) = e^{2tT + 2\Gamma \sin t}$$

IS A ONE-PARAMETER FAMILY OF SOLUTIONS.