M215 HW # 6 SOLUTIONS

PROBLEM 1

(i) $L(y) = x^2 y'' + x y' - y = x \ln x \qquad y_1(x) = x.$

LET $y = xv$. THEN $y' = xv' + v, \quad y'' = xv'' + 2v'$

SO $x^2(xv'' + 2v') + x(xv' + v) - xv = x \ln x$

SO $x^3 v'' + 3x^2 v' = x \ln x$

integrate by parts

$\rightarrow \quad (x^3 v')' = x \ln x \rightarrow x^3 v' = \int^x x \ln x = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C_1$

SO $v' = \frac{1}{2} \frac{\ln x}{x} - \frac{1}{4x} + \frac{C_1}{x^3} \rightarrow v = \frac{1}{2} \int^x \frac{\ln x}{x} dx - \frac{1}{4} \ln x + \frac{\hat{C}_1}{x^2} + C_2$

SO $v = \frac{1}{4} (\ln x)^2 - \frac{1}{4} \ln x + \frac{\hat{C}_1}{x^2} + C_2.$

SO $y = xv \rightarrow y = \frac{x}{4} (\ln x)^2 - \frac{1}{4} x \ln x + \frac{\hat{C}_1}{x} + C_2 x$

IS GENERAL SOLUTION.

THE HOMOGENEOUS SOLUTION IS $y_1 = x, \ y_2 = 1/x$

(ii) $L(y) = x^2 y'' + 7xy' + 5y = x \qquad y_1 = 1/x.$

WE PUT $y = x^{-1} v$ SO $y' = -x^{-2} v + x^{-1} v', \ y'' = x^{-1} v'' - 2x^{-2} v' + 2x^{-3} v$

SO

$x^2 (x^{-1} v'' - 2x^{-2} v' + 2x^{-3} v) + 7x (x^{-1} v' - x^{-2} v) + 5x^{-1} v = x$

$\rightarrow \quad x v'' - 2v' + 7v' = x$

$\rightarrow \quad v'' + \frac{5}{x} v' = 1 \qquad \phi = \exp\left(\int 5/x \, dx\right) = e^{5 \ln x} = x^5.$

SO $(x^5 v')' = x^5 \rightarrow x^5 v' = \frac{1}{6} x^6 + C_1$

SO $v' = \frac{1}{6} x + \frac{C_1}{x^5}$

$v = \frac{1}{12} x^2 + \frac{\tilde{C}_1}{x^4} + C_2$ WITH $y = x^{-1} v$

SO $y = \frac{1}{12} x + \frac{\hat{C}_1}{x^5} + \frac{C_2}{x}$ IS GENERAL SOLUTION

WITH $y_1 = 1/x$, $y_2 = 1/x^5$ SOLUTION TO HOMOGENEOUS PROBLEM.

PROBLEM 2 $y'' + y' + 2y = F_0 \cos(\omega t)$

WE PUT $\tilde{y}'' + \tilde{y}' + 2\tilde{y} = F_0 e^{i\omega t}$.

NOW $\tilde{y} = A_0 e^{i\omega t} \longrightarrow (-\omega^2 + i\omega + 2) A_0 = F_0$.

SO $A_0 = \frac{F_0}{(2-\omega^2) + i\omega} = \frac{F_0}{(2-\omega^2)+i\omega} \frac{(2-\omega^2) - i\omega}{(2-\omega^2) - i\omega} = \frac{F_0 [(2-\omega^2) - i\omega]}{(2-\omega^2)^2 + \omega^2}$

THUS $y_p = \mathrm{IM}\left[\frac{F_0((2-\omega^2) - i\omega)}{(2-\omega^2)^2 + \omega^2} (\cos \omega t + i \sin \omega t) \right]$

$= \frac{F_0}{(2-\omega^2)^2 + \omega^2} \left[(2-\omega^2)\sin(\omega t) - \omega \cos(\omega t) \right]$

WANT $= A \cos(\omega t - \phi)$

$= A \cos\phi \cos(\omega t) + A \sin\phi \sin \omega t$

SO $A \cos\phi = -\omega$

$A \sin\phi = (2-\omega^2)$

$\longrightarrow A = \sqrt{[2-\omega^2]^2 + \omega^2}$

$\tan\phi = -(2-\omega^2)/\omega$.

THUS $y_p = R(w)\cos(wt-\phi)$

WHERE $R = \dfrac{F_0}{\left([2-w^2]^2 + w^2\right)^{1/2}}$, $\tan\phi = (w^2-2)/w$.

DEFINE $g(w) = (2-w^2)^2 + w^2$.

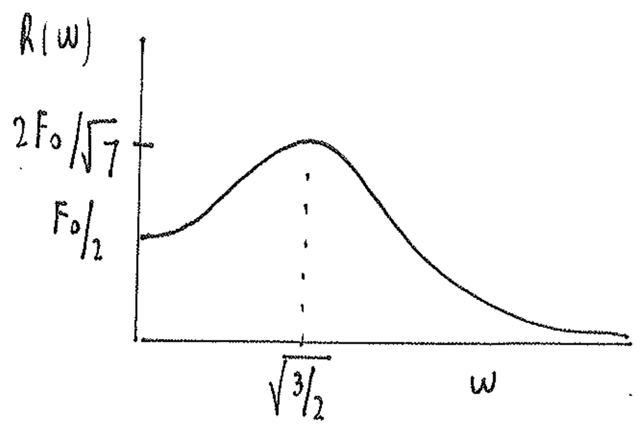
THEN $R = F_0\,[g(w)]^{-1/2}$

R HAS A MAXIMUM WHEN $R' = 0 \to -\dfrac{F_0}{2}[g(w)]^{-3/2}\, g'(w) = 0$.

NOW $g'(w) = 0$ WHEN $2(2-w^2)(-2w) + 2w = 0$

$\to 2(2-w^2) = 1 \to w = \sqrt{3/2}$.

NOW AT $w = \sqrt{3/2}$, $R(\sqrt{3/2}) = \dfrac{F_0}{[g(\sqrt{3/2})]^{1/2}} = \dfrac{2F_0}{\sqrt{7}}$.



PROBLEM 3

(i) $y'''' - 4y = 0$ PUT $y = e^{\lambda t} \rightarrow \lambda^4 - 4 = 0.$

THUS $\lambda^4 = 4 \rightarrow \lambda^2 = 2$ OR $\lambda^2 = -2$

$\downarrow$ $\qquad$ $\downarrow$

$\lambda = \pm\sqrt{2}$ $\qquad$ $\lambda = \pm\sqrt{2}\,i$

THUS $e^{\sqrt{2}t}$, $e^{-\sqrt{2}t}$, $e^{\sqrt{2}it}$, $e^{-\sqrt{2}it}$ ARE SOLUTIONS.

SO $y = C_1 Y_1 + C_2 Y_2 + C_3 Y_3 + C_4 Y_4$ general solution

$Y_1 = e^{\sqrt{2}t}$, $Y_2 = e^{-\sqrt{2}t}$, $Y_3 = \cos(\sqrt{2}t)$, $Y_4 = \sin(\sqrt{2}t)$.

(ii) $y'''' - 4y'' + 3y = t^2.$

HOMOGENEOUS PROBLEM PUT $y = e^{\lambda t} \rightarrow \lambda^4 - 4\lambda^2 + 3 = 0$

let $\mu = \lambda^2 \rightarrow \mu^2 - 4\mu + 3 = (\mu-3)(\mu-1) = 0 \rightarrow \mu = 1, 3.$

THUS $\lambda = 1$, $\lambda = -1$, $\lambda = \sqrt{3}$, $\lambda = -\sqrt{3}$ ARE ROOTS OF CHARACTERISTIC polynomial. $Y_1 = e^{t}$, $Y_2 = e^{-t}$, $Y_3 = e^{\sqrt{3}t}$, $Y_4 = e^{-\sqrt{3}t}$

ARE SOLUTIONS TO HOMOG. PROBLEM.

PARTICULAR SOLUTION PUT $y = A_0 + A_1 t + A_2 t^2.$

THEN $y'' = 2A_2$ SO $-4(2A_2) + 3A_0 + 3A_1 t + 3A_2 t^2 = t^2.$

THUS $3A_2 = 1$, $A_1 = 0$, $-8A_2 + 3A_0 = 0 \rightarrow A_2 = 1/3$, $A_1 = 0$, $A_0 = 8/9$

$Y_P = 8/9 + t^2/3$

$\rightarrow y = c_1 e^{t} + c_2 e^{-t} + c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t} + \left(8/9 + t^2/3\right)$ GEN. SOL'N.

PROBLEM 4 WE SOLVE

$$y'' + .1y' + \frac{1}{c}y = F_0 \sin t\omega$$

TO FIND THE PARTICULAR SOLUTION. WE WILL TAKE $F_0 = 1$ AND $w = 1$

THEN $F_0 = 4$ AND $w = 5$.

$$\hat{y}'' + .1\hat{y}' + \frac{1}{c}\hat{y} = F_0 e^{iwt}.$$

let $\hat{y} = A e^{iwt} \longrightarrow A\left[\left(-w^2 + \frac{1}{c}\right) + .1 i w\right] = F_0$

$$A = \frac{F_0\left[\left(-w^2 + 1/c\right) - .1 i w\right]}{\left(w^2 - 1/c\right)^2 + .01 w^2}$$

NOW $\quad Y_p = \mathrm{IM}\left[A e^{iwt}\right] = \dfrac{F_0\left[\left(-w^2 + 1/c\right)\right]}{\Delta}\sin(wt) + -\dfrac{.10w \cos(wt) F_0}{\Delta}$

$$\Delta \equiv \left(w^2 - 1/c\right)^2 + .01 w^2.$$

THUS $\quad Y_p = \dfrac{F_0}{\Delta^{1/2}} \sin(wt + \phi)$ $\qquad \sin\phi = \dfrac{\left(-w^2 + 1/c\right)}{\Delta^{1/2}}$

$$\cos\phi = -\frac{.10w}{\Delta^{1/2}}$$

THEREFORE WE HAVE THAT THE STEADY STATE SOLUTION IS

$$Y_p = Y_{p_1} + Y_{p_2}$$

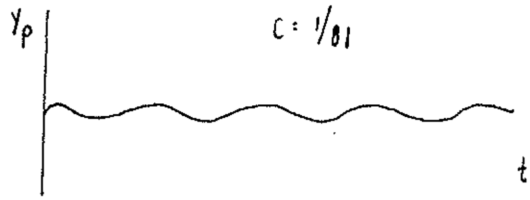
$$Y_p = \left[\frac{\sin(t + \phi_1)}{\left[\left(1 - 1/c\right)^2 + .01\right]^{1/2}} + \frac{4 \sin(5t + \phi_2)}{\left[\left(25 - 1/c\right)^2 + .25\right]^{1/2}}\right]$$

where $\sin\phi_1 = \dfrac{\left(-1 + 1/c\right)}{\Delta_1^{1/2}}$ $\quad \cos\phi_1 = -\dfrac{.10}{\Delta_1^{1/2}}$ $\quad \Delta_1 = \left(1 - 1/c\right)^2 + .($

$\sin\phi_2 = \dfrac{\left(-25 + 1/c\right)}{\Delta_2^{1/2}}$ $\quad \cos\phi_2 = -\dfrac{.50}{\Delta_2^{1/2}}$ $\quad \Delta_2 = \left(25 - \frac{1}{c}\right)^2 +$

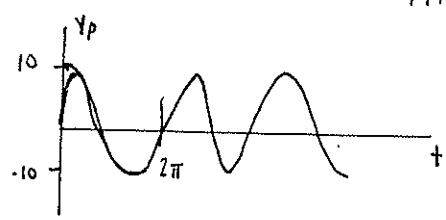
NOTICE IF $C = 1/81$, THEN BOTH Δ_1 AND Δ_2 HAVE ROUGHLY THE SAME ORDER OF MAGNITUDE

i.e. $$y_p \approx \frac{1}{81}\sin(t+\phi_1) + \frac{1}{14}\sin(t5+\phi_2)$$ (small amplitude response)



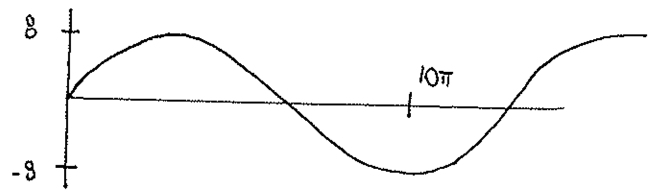
IF $C = 1$ THEN THE TERM PROPORTIONAL TO $\sin(t+\phi_1)$ dominates. Thus,

$$y_p \approx 10\sin(t+\phi_1)$$



OW IF $C = 1/25$, then the term proportional to $\sin(5t+\phi_2)$ dominates AND

$$y_p \approx 8\sin(5t+\phi_2)$$



therefore by tuning the parameter C appropriately we can pick up either a response that oscillates at frequency 1 or at frequency 5. These are the two frequencies in the input signal.

PROBLEM 5

$$y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2}.$$ page 189 SECTION 3.6 # 13

$y_1 = t^2, \quad y_2 = t^{-1}, \quad g = 3 - 1/t^2.$

$$y_p = -y_1(t)\int^t \frac{y_2(\lambda)\, g(\lambda)}{W(y_1, y_2)}\, d\lambda + y_2(t)\int^t \frac{y_1(\lambda)\, g(\lambda)}{W(y_1, y_2)}\, d\lambda.$$

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \lambda^2\left(-1/\lambda^2\right) - 2\lambda\left(1/\lambda\right) = -3.$$

THUS $$y_p = -t^2 \int^t \frac{\lambda^{-1}(3 - 1/\lambda^2)}{(-3)}\, d\lambda + \frac{1}{t}\int^t \frac{\lambda^2 (3 - 1/\lambda^2)}{(-3)}\, d\lambda$$

SO $$y_p = \frac{t^2}{3}\left(3 \ln t + \frac{1}{2} t^{-2}\right) - \frac{1}{3t}\left(t^3 - t\right)$$

SO $$y_p = t^2 \ln t + \frac{1}{2} - \frac{1}{3} t^2$$

solves homog. problem.

SO WE CAN TAKE $y_p = t^2 \ln t + 1/2$

PROBLEM 6 page 190 # 23

$$y'' + y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0.$$

NOW $y_1 = \sin t, \quad y_2 = \cos t$

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \sin t(-\sin t) - \cos t(\cos t) = -1.$$

SO $$y_p(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(\Lambda)\, g(\Lambda)}{W(y_1, y_2)}\, d\Lambda + y_2(t)\int_{t_0}^{t} \frac{y_1(\Lambda)\, g(\Lambda)}{W(y_1, y_2)}\, d\Lambda$$

SO $$y_p = -\sin t \int_{t_0}^{t} \frac{\cos(\Lambda)\, g(\Lambda)}{(-1)}\, d\Lambda + \cos t \int_{t_0}^{t} \frac{\sin \Lambda\, g(\Lambda)}{(-1)}\, d\Lambda$$

SO $$y_p = \int_{t_0}^{t} g(\Lambda)\left[\cos \Lambda \sin t - \cos t \sin \Lambda\right] d\Lambda$$

$$y_p = \int_{t_0}^{t} g(\Lambda) \sin(t - \Lambda)\, d\Lambda$$

SO general solution is

$$y = c_1 \cos t + c_2 \sin t + \int_{t_0}^{t} g(\Lambda) \sin(t - \Lambda)\, d\Lambda.$$

NOW $y(t_0) = c_1 \cos t_0 + c_2 \sin t_0 = 0 \qquad (1)$

$$y'(t) = -c_1 \sin t + c_2 \cos t + \int_{t_0}^{t} g(\Lambda) \cos(t - \Lambda)\, d\Lambda + g(t)\sin(0)$$

$y'(t_0) = -c_1 \sin t_0 + c_2 \cos t_0 = 0 \qquad (2)$

NOW (1) AND (2) $\longrightarrow$ $c_1 = c_2 = 0$ $\Longrightarrow$ $y = \int_{t_0}^{t} g(\Lambda) \sin(t - \Lambda)\, d\Lambda.$

PROBLEM 7 $\qquad y' + y = T_0 + T_1 \cos(\omega t)$

THE HOMOGENEOUS SOLUTION SATISFIES $y' = -y \rightarrow y = e^{-t}$.

NOW FOR PARTICULAR SOLUTION. LET y_{p_1}, y_{p_2} SATISFY

$$y_{p_1}' + y_{p_1} = T_0 \qquad\qquad y_{p_2}' + y_{p_2} = T_1 \cos(\omega t)$$

GUESS $y_{p_1} = A_0$

$\rightarrow A_0 = T_0$

SO $\underline{y_{p_1} = A_0}$

let $\tilde{y}_{p_2}' + \hat{y}_{p_2} = T_0 e^{i\omega t}$

PUT $\hat{y}_{p_2} = A e^{i\omega t}$

$$\rightarrow A i\omega + A = T_0 \rightarrow A = \frac{T_0}{1 + i\omega}$$

SO
$$y_{p_2} = \mathrm{RE}\left[\frac{T_1}{1+i\omega} e^{i\omega t}\right]$$

$$= \mathrm{RE}\left[\frac{T_1 (1 - i\omega)}{1 + \omega^2} (\cos \omega t + i \sin \omega t)\right]$$

$$y_{p_2} = \frac{T_1}{(1+\omega^2)}\left[\cos(\omega t) + \omega \sin \omega t\right].$$

THUS GENERAL SOLUTION IS

$$y = c_1 e^{-t} + A_0 + \frac{T_1}{(1+\omega^2)}\left[\cos(\omega t) + \omega \sin(\omega t)\right].$$