MATH 215 HW # 7 SOLUTIONS

PROBLEM 10 PAGE 320.

FIND INVERSE L.T. OF $\dfrac{2s-3}{s^2+2s+10} = F(s)$

$$\text{NOW}\quad \frac{2s-3}{(s^2+2s+1)+9} = \frac{2s-3}{(s+1)^2+9} = \frac{2(s+1)-2-3}{(s+1)^2+9} = \frac{2(s+1)}{(s+1)^2+9} - \frac{5}{(s+1)^2+9}$$

$$F(s) = \frac{2(s+1)}{(s+1)^2+9} - \frac{5}{3}\,\frac{3}{(s+1)^2+9}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t$$

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SOLVE $\quad y'' + 2y' + 5y = 0 \qquad y(0) = 2 \qquad y'(0) = -1$

TAKE LAPLACE TRANSFORMS:

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + 2\big(s\,\mathcal{L}(y) - y(0)\big) + 5\,\mathcal{L}(y) = 0$$

$$(s^2+2s+5)\,Y(s) = 2s - 1 + 4$$

$$Y(s) = \frac{2s+3}{s^2+2s+5} = \frac{2s+3}{(s^2+2s+1)+4} = \frac{2(s+1)+1}{(s+1)^2+4}$$

$$Y(s) = \frac{2(s+1)}{(s+1)^2+4} + \frac{1}{2}\,\frac{2}{(s+1)^2+4}$$

$$\Rightarrow \qquad y(t) = 2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin 2t$$

PROBLEM 20 PAGE 320

SOLVE $\quad y'' + w^2 y = \cos 2t \qquad w^2 \neq 4 \qquad y(0) = 1,\ y'(0) = 0 \quad$ BY L.T.

$$\big(s^2\mathcal{L}(y) - s\,y(0) - y'(0)\big) + w^2\big(\mathcal{L}(y)\ \cancel{\ldots}\big) = \frac{s}{s^2+4}$$

$$(s^2+w^2)\,\mathcal{L}(y) = s + \frac{s}{s^2+4}$$

$$Y(s) = \frac{s}{s^2+w^2} + \frac{s}{(s^2+w^2)(s^2+4)}$$

$$\text{NOW}\quad \frac{s}{(s^2+w^2)(s^2+4)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+w^2}$$

$$s = (As + B)(s^2 + w^2) + (Cs + D)(s^2 + 4)$$

$O(s^3)$: $A + C = 0$ $\qquad$ $O(s)$: $Aw^2 + 4C = 1$

$O(s^2)$: $B + D = 0$ $\qquad$ $O(1)$: $Bw^2 + 4D = 0$

THUS $B = D = 0$ AND $C = -A$, $A = \dfrac{1}{w^2 - 4}$ $\qquad C = \dfrac{1}{4 - w^2}$

SO THAT
$$Y(s) = \frac{s}{s^2 + w^2} - \frac{1}{w^2 - 4}\,\frac{s}{s^2 + w^2} + \frac{1}{4 - w^2}\,\frac{s}{s + 4}$$

$$y(t) = \cos wt - \frac{1}{w^2 - 4}\cos(wt) + \frac{1}{w^2 - 4}\cos(2t)$$

$$y(t) = \left(\frac{w^2 - 5}{w^2 - 4}\right)\cos(wt) + \frac{1}{w^2 - 4}\cos(2t)$$

PROBLEM 23 P. 321

$y'' + 2y' + y = 4e^{-t}$ $\qquad y(0) = 2$ $\qquad y'(0) = -1$ $\qquad$ SOLVE BY L.T.

$$\left(s^2 \mathcal{L}(y) - s\,y(0) - y'(0)\right) + 2\left(s\,\mathcal{L}(y) - y(0)\right) + \mathcal{L}(y) = \frac{4}{s+1}$$

$$(s^2 + 2s + 1)\,Y(s) = s\,y(0) + y'(0) + 2\,y(0) + \frac{4}{s+1}$$

$$(s+1)^2\,Y(s) = 2s + 3 + \frac{4}{s+1}$$

$$Y(s) = \frac{2s + 3}{(s+1)^2} + \frac{4}{(s+1)^3}$$

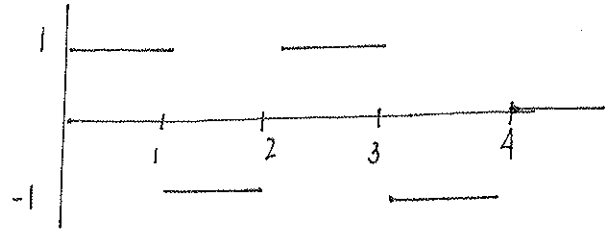
$$Y(s) = \frac{2(s+1) + 1}{(s+1)^2} + \frac{4}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

$$y(t) = 2e^{-t} + te^{-t} + 2t^2e^{-t}$$

PROBLEM 8 PAGE 329

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ -1 & 3 \le t < 4 \\ 0 & t \ge 4 \end{cases}$$

a)



b) THEN SINCE $u_c(t) = \begin{cases} 0, & t < c \\ 1, & t > c \end{cases}$

$$f(t) = 1 - 2u_1(t) + 2u_2(t) - 2u_3(t) + u_4(t)$$

PROBLEM 13 PAGE 329

WE RECALL $\mathcal{L}(u_c(t) f(t-c)) = e^{-cs} F(s)$ WITH $F(s) = \mathcal{L}(f(t))$.

SO $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \ge 2 \end{cases}$

WE WRITE $f(t) = u_2(t)\, g(t-2)$ WITH $g(t) = t^2$.

THUS $\mathcal{L}(f(t)) = \mathcal{L}(u_2(t) g(t-2)) = e^{-cs} \mathcal{L}(g(t)) = \dfrac{e^{-cs}\, 2!}{s^3}$

PROBLEM 15 PAGE 329

WE WRITE $f(t) = \begin{cases} 0 & t < \pi \\ t-\pi & \pi \le t < 2\pi \\ 0 & t > 2\pi \end{cases} = \begin{cases} 0 & t < \pi \\ t-\pi & \pi \le t < 2\pi \\ t-\pi & t > 2\pi \end{cases} - \begin{cases} 0, & t < \pi \\ 0, & \pi \le t < 2\pi \\ (t-\pi), & t > 2\pi \end{cases}$

THUS $f(t) = (t-\pi)\, u_\pi(t) - (t-\pi)\, u_{2\pi}(t) = g_1(t-\pi)\, u_\pi(t) - [(t-2\pi) + \pi]\, u_{2\pi}(t)$

$f(t) = g_1(t-\pi)\, u_\pi(t) - g_2(t-2\pi)\, u_{2\pi}(t)$ $g_1(t) = t, \quad g_2(t) = t + \pi$

THUS $\mathcal{L}(f(t)) = e^{-\pi s} G_1(s) - e^{-2\pi s} G_2(s)$ $\mathcal{L}(g_1) = 1/s^2 \quad \mathcal{L}(g_2) = \dfrac{1}{s^2} + \dfrac{\pi}{s}$

$\longrightarrow \quad \mathcal{L}(f(t)) = e^{-\pi s}/s^2 - e^{-2\pi s} \left(\dfrac{1}{s^2} + \dfrac{\pi}{s} \right)$

PROBLEM 21 PAGE 329

WE WRITE $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2} = \dfrac{2(s-1)e^{-2s}}{(s-1)^2+1}$

NOW $\mathcal{L}^{-1}\left(\dfrac{2(s-1)}{(s-1)^2+1}\right) = 2e^{t}\cos(t) = g(t)$

THUS SINCE $\mathcal{L}\left[g(t-c)\,u_c(t)\right] = e^{-cs}G(s)$

THEN $\mathcal{L}^{-1}[F(s)] = u_2(t)\,g(t-2) = 2e^{(t-2)}\cos(t-2)\,u_2(t).$

PROBLEM 22 PAGE 329

WE WRITE $F(s) = \dfrac{2e^{-2s}}{s^2-4} = G(s)e^{-2s}$ WITH $G(s) = \dfrac{2}{s^2-4}$.

NOW $g(t) = \mathcal{L}^{-1}[G(s)] = \sinh[2t].$

AND SINCE $\mathcal{L}[g(t-c)\,u_c(t)] = e^{-cs}G(s)$

THEN $\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[e^{-cs}G(s)] = u_2(t)\sinh[2(t-2)].$

PROBLEM 24 PAGE 329 $F(s) = \dfrac{e^{-s}}{s} + \dfrac{e^{-2s}}{s} - \dfrac{e^{-3s}}{s} - \dfrac{e^{-4s}}{s}$

$\mathcal{L}^{-1}[F(s)] = u_1(t) + u_2(t) - u_3(t) - u_4(t)$, SINCE $\mathcal{L}^{-1}(1/s) = 1.$

PROBLEM 25 PAGE 329

(i) PROVE THAT IF $S > ca$ THEN $\mathcal{L}(F(ct)) = \frac{1}{c} \bar{F}(S/c)$

WHERE $F(S) = \mathcal{L}(f(t))$ EXISTS FOR $S > a \geq 0$.

PROOF $\mathcal{L}(F(ct)) = \int_0^\infty F(ct) e^{-st} dt = \frac{1}{c}\int_0^\infty F(\lambda) e^{-(S/c)\lambda} d\lambda$

let $\lambda = ct$ be new variable $dt = 1/c \, d\lambda$

$= \frac{1}{C} F(S/c)$

VALID FOR $S/c > a$.

(ii) PROVE THAT IF $K > 0$ THEN

$\mathcal{L}^{-1}[F(KS)] = \frac{1}{K} f(t/K)$

NOW $\mathcal{L}\left[\frac{1}{K} f(t/K)\right] = \frac{1}{K}\int_0^\infty F(t/K) e^{-st} dt = \int_0^\infty f(\lambda) e^{-(SK)\lambda} d\lambda$

let $\lambda = t/K$

$dt = K d\lambda$

$= F(SK)$

We need $K > 0$ so THAT $\int_0^\infty dt \longrightarrow \int_0^\infty d\lambda$.

(iii) SHOW THAT IF a, b CONSTANTS WITH $a > 0$ THEN

$\mathcal{L}^{-1}[F(as+b)] = \frac{1}{a} f(t/a) e^{-bt/a}$

PROOF $\mathcal{L}\left(\frac{1}{a} f(t/a) e^{-bt/a}\right) = \int_0^\infty \frac{1}{a} F(t/a) e^{-bt/a} e^{-st} dt = I$

Let $\lambda = t/a$ so $dt = a\, d\lambda$. THEN $I = \int_0^\infty f(\lambda) e^{-b\lambda} e^{-sa\lambda} d\lambda$

so $I = \int_0^\infty f(\lambda) e^{-(as+b)\lambda} d\lambda = F(as+b)$. (Need $a > 0$ so THAT $\int_0^\infty dt \to \int_0^\infty d\lambda$)

PROBLEM 27 PAGE 330

$$F(s) = \frac{2s+1}{(4s^2+4s+5)} = \frac{2s+1}{4(s^2+s)+5} = \frac{2(s+1/2)}{4\left(s^2+s+\frac{1}{4}-\frac{1}{4}\right)+5}$$ complete square

$$F(s) = \frac{2(s+1/2)}{4(s+1/2)^2+4} = \frac{1}{2}\left[\frac{(s+1/2)}{(s+1/2)^2+1}\right]$$

BUT $\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$ SO $f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2}e^{-t/2}\cos(t)$.

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$$F(s) = \frac{1}{9s^2-12s+3} = \frac{1}{9\left(s^2-\frac{4}{3}s\right)+3}$$

COMPLETE THE SQUARE

$$F(s) = \frac{1}{9\left(s^2-\frac{4s}{3}+\frac{4}{9}-\frac{4}{9}\right)+3} = \frac{1}{9\left(s^2-\frac{4s}{3}+\frac{4}{9}\right)-1}$$

SO $$F(s) = \frac{1}{9(s-2/3)^2-1} = \frac{1}{9[(s-2/3)^2-1/9]} = \frac{1}{3}\left[\frac{1/3}{(s-2/3)^2-1/9}\right]$$

SO $$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{3}e^{2t/3}\sinh[t/3] = \frac{1}{6}e^{2t/3}\left(e^{t/3}-e^{-t/3}\right)$$

SO $$f(t) = \frac{1}{6}e^{t/3}\left(e^{2t/3}-1\right).$$

PROBLEM 29 PAGE 330

$$F(s) = \frac{e^2 e^{-4s}}{2(s-1/2)} = \frac{e^2}{2}\left(\frac{e^{-4s}}{(s-1/2)}\right) = \frac{e^2}{2}\left(e^{-4s}G(s)\right)$$

RECALL $\mathcal{L}^{-1}\left(\frac{1}{(s-1/2)}\right) = e^{t/2} = g(t)$ $\quad \mathcal{L}^{-1}[e^{-4s}G(s)] = u_4(t)\,g(t-4)$

THUS $$f(t) = \mathcal{L}^{-1}[F(s)] = e^{1/2(t-4)}\frac{e^2}{2}u_4(t)$$

SO $f(t) = \frac{e^{t/2}}{2}u_4(t)$ WHICH IS SAME AS $f(t) = \frac{e^{t/2}}{2}u_2(t/2)$.