(III) 1. $\epsilon y'' + y' = e^{-x}$; $y(0) = 1$; $y(1) = 1$

Let $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$

Substituting

$$\epsilon\left[y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + \dots\right] + \left[y_0' + \epsilon y_1' + \epsilon^2 y_2' + \dots\right] = e^{-x}$$

with $y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \dots = 1$

& $y_0(1) + \epsilon y_1(1) + \epsilon^2 y_2(1) + \dots = 1$

$\underline{O(1)}$:

$$\left.\begin{aligned} y_0' &= e^{-x} \\ \Rightarrow y_0 &= C_0 - e^{-x} \end{aligned}\right\} \text{Outer Solution}$$

To find C_0, we need to apply the boundary condition at $x = 0$ or $x = 1$. which of these is correct?

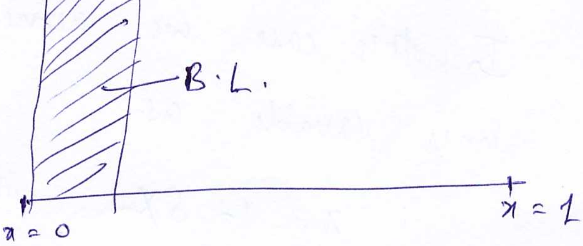
Looking at $y_0(x)$, it is clear that $y_0(x)$ & y_0' are both of order 1 for all values of x.

Let us check from the inner solution if the boundary layer should be at $x = 0$ or $x = 1$.

Let $\underline{x = 0}$ be $\underline{\text{the location of}}$ the $\underline{\text{boundary layer}}$.

Let $x = \delta X$ where $X \sim \text{Ord}(1)$ & $\delta \ll 1$.

$y(x) = Y(X)$



$$\therefore \frac{dy}{dx} = \frac{1}{\delta}\frac{dY}{dX}.$$

$$\Rightarrow \text{we get} \quad \frac{\epsilon}{\delta} Y'' + Y' = \delta e^{-\delta X}$$

Therefore $\delta = \epsilon$.

$\Rightarrow$ we get $Y'' + Y' = \epsilon e^{-\epsilon X}$

Let $Y = Y_0(X) + \epsilon Y_1(X) + \epsilon^2 Y_2(X) + \dots$

$\therefore$ $\underline{O(1)}$: $Y_0'' + Y_0' = 0$

$\Rightarrow Y_0' + Y_0 = C_1$

$\Rightarrow Y_0' = C_1 - Y_0$

$\therefore Y_0 = C_1 + C_2 e^{-X}$

Since we assumed that the boundary layer is at $x = 0$,

we have $Y_0(X = 0) = 1$

$\Rightarrow 1 = C_1 + C_2 \Rightarrow C_2 = 1 - C_1$

$$\therefore \boxed{Y_0(X) = e^{-X} + C_1(1 - e^{-X})}$$

This solution clearly decays away from $X = 0$. Therefore, this solution is acceptable since it can be matched to an $O(1)$ outer solution.

If $\underline{x = 1}$ $\underline{\text{has}}$ the $\underline{\text{boundary layer}}$:-

In this case, we define the inner variable as

$x = 1 - \delta X$, with $\delta << 1$
& $X \sim ord(1)$.

As $X \to \infty$; $x \to 0$, i.e; as $X \to \infty$, we go towards the outer solution — necessary for matching purposes.

$$\frac{dy}{dx} = \frac{dY}{dX}\cdot\frac{dX}{dx} = -\frac{1}{\delta}\frac{dY}{dX}$$

& $$\frac{d^2y}{dx^2} = \frac{1}{\delta^2}\frac{d^2Y}{dX^2}$$

Substituting, we have, $\frac{\epsilon}{\delta^2}Y'' - \frac{1}{\delta}Y' = e^{-(1-\delta X)}$

or $\frac{\epsilon}{\delta}Y'' - Y' = \delta e^{-(1-\delta X)}$

Again, we require $\delta = \epsilon$

$\Rightarrow Y'' - Y' = \epsilon e^{-(1-\epsilon X)}$

Using $Y = Y_0 + \epsilon Y_1 + \cdots$, we get at $O(1)$:

$Y_0'' - Y_0' = 0$

or $Y_0 = C_2 e^{X} - C_1$

This solution exponential grows as $X \to \infty$, and hence cannot be matched to the $y_0(x)$ solution obtained earlier. Hence the boundary layer must be at $x = 0$.

Boundary layer at $x = 0$:-

Outer Solution :-

$y_0(x) = C_0 - e^{-x}$

with $y_0(1) = 1$

$\Rightarrow 1 = C_0 - e^{-1} \Rightarrow C_0 = 1 + \frac{1}{e}$

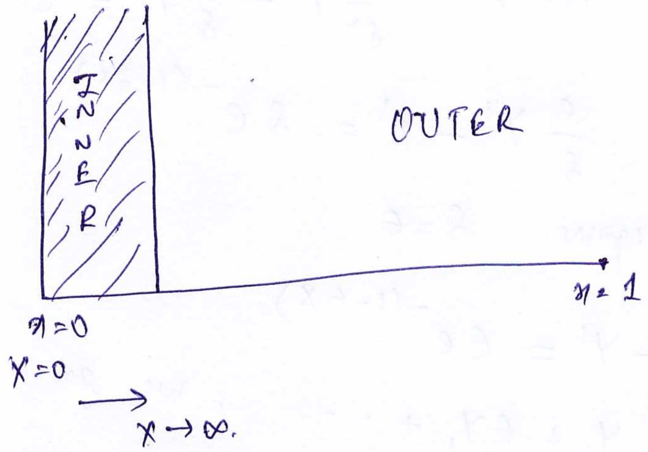
$\therefore \boxed{y_0(x) = 1 + e^{-1} - e^{-x}}$

Inner Solution: $Y_0(X) = e^{-X} + C_1(1 - e^{-X})$ with $X = \frac{x}{\epsilon}$

with $Y_0(0) = 1$

The only quantity to be determined is C_1. This is obtained by matching the inner & outer solutions.

<u>Matching</u>:-



$$\lim_{X \to \text{towards Outer}} Y(X) = \lim_{x \to \text{Inner (towards } x=0)} y(x)$$

$$\Rightarrow \lim_{X \to \infty} Y_0(X) = \lim_{x \to 0} y_0(x)$$

$$\Rightarrow C_1 = e^{-1}$$

$$\Rightarrow C_1 = \frac{1}{e}$$

With $C_1 = \frac{1}{e}$ & $X \to \infty$; $\lim_{X \to \infty} Y_0(X) = \frac{1}{e}$

Similarly, as $x \to \infty$; $\lim_{x \to 0} y_0(x) = \frac{1}{e}$

Therefore the common value of $y_0(x)$ & $Y_0(x)$ is $\frac{1}{e}$.

$$\Rightarrow y_{mid} = \frac{1}{e}$$

<u>Composite solution</u>:- $y_{composite} = y_{inner} + y_{outer} - y_{middle}$

$$= Y_0(X) + y_0(x) - \frac{1}{e}$$

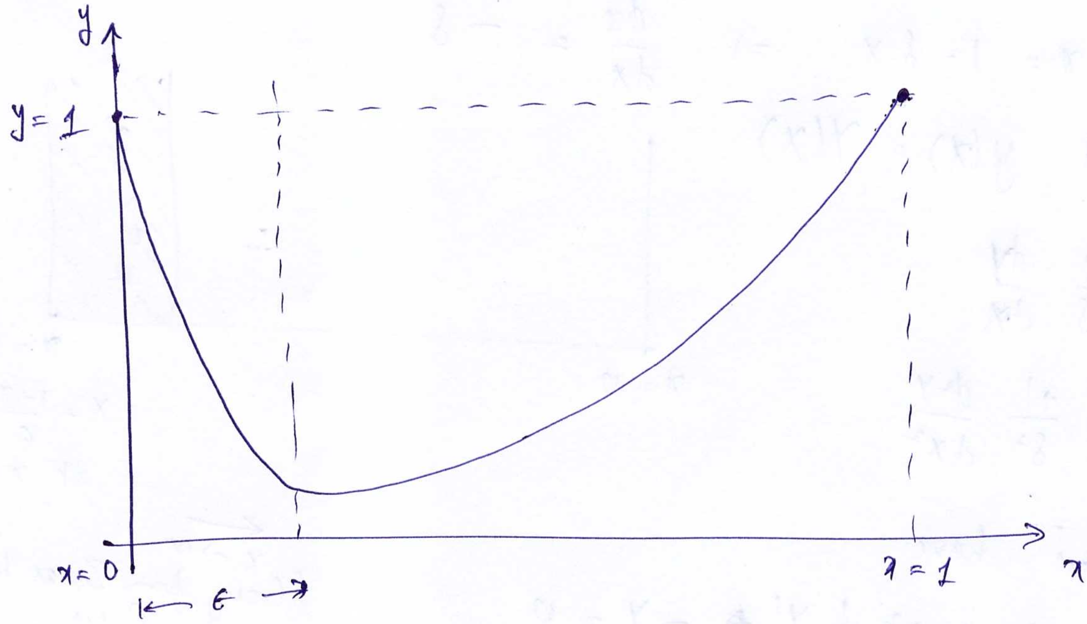
$$= [1 + e^{-1} - e^{-x}] + [e^{-X} + e^{-1}(1 - e^{-X})] - \frac{1}{e}$$

(iii)

$$= 1 + \cancel{e^{-1}} - e^{-x} - e^{-X} + e^{-1} - e^{-1}e^{-X} - \cancel{1/e}$$

$$= 1 - e^{-x} + e^{-x/\epsilon} - e^{-(1+\frac{x}{\epsilon})} - \frac{1}{e}$$

↳ This solution agrees with the exact solution as $\epsilon \to 0$.



(III) 2.
$$\epsilon y'' - x^2 y' - y = 0$$
$$y(0) = 1$$
$$y(1) = 1$$

Let $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$

Substituting

$$\epsilon\left[y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + \cdots\right] - x^2\left[y_0' + \epsilon y_1' + \epsilon^2 y_2' + \cdots\right]$$
$$- \left[y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right] = 0$$

$\underline{O(1)}$: $\quad -x^2 y_0' - y_0 = 0$

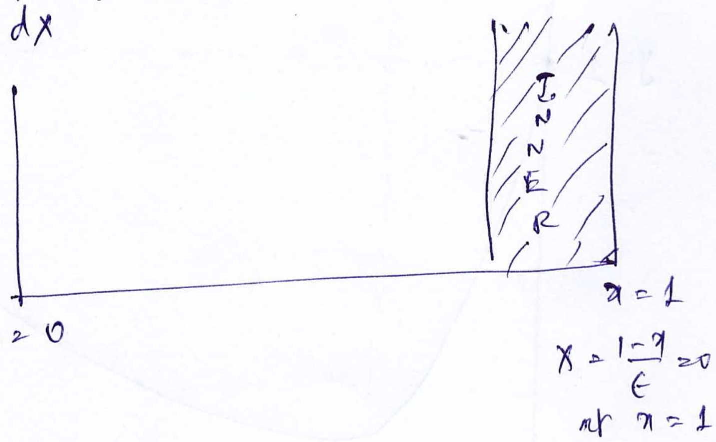
$$\Rightarrow \boxed{y_0 = C_0 e^{1/x}}$$

If $x = 0$, then $\frac{1}{x} \to \infty$ & hence $y_0 \to \infty$. Let us examine the two limits $x = 0$ & $x = 1$ for a boundary layer.

<u>Putting boundary layer at $x = 1$:-</u>

Let $x = 1 - \delta X \Rightarrow \frac{dx}{dX} = -\delta$

and $y(x) = Y(X)$



$$\frac{dy}{dx} = \frac{-1}{\delta} \frac{dY}{dX}$$

& $$\frac{d^2y}{dx^2} = \frac{+1}{\delta^2} \frac{d^2Y}{dX^2}$$

$\overleftrightarrow{X \to \infty}$ (Going from Inner to Outer)

Substituting, we have

$$\frac{\epsilon}{\delta^2} Y'' + (1 - \delta X)^2 \frac{1}{\delta} Y' - Y = 0$$

$$\Rightarrow \frac{\epsilon}{\delta} Y'' + (1 - \delta X)^2 Y' - \delta Y = 0$$

natural choice for δ is $\boxed{\delta = \epsilon}$ $\Rightarrow X = \frac{1-x}{\epsilon}$

$$\Rightarrow Y'' + (1 - \epsilon X)^2 Y' - \epsilon Y = 0$$

Let $Y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \cdots$

<u>$O(1)$:</u> $Y_0'' + Y_0' = 0$

with $Y_0(X = 0) = 1$

$$\Rightarrow Y_0(X) = C_1 + C_2 e^{-X} \quad \text{with } Y_0(0) = 1.$$

$$\Rightarrow 1 = C_1 + C_2 \Rightarrow C_2 = 1 - C_1$$

$$\therefore Y_0(X) = C_1 + (1 - C_1) e^{-X}$$: This solution decay as $X \to \infty$ and is <u>acceptable</u>

The constant C_1 is unknown and has to be determined by matching.

Clearly, having a boundary layer at $x = 1$ is acceptable. But since $y_0(x) \to \infty$ as $x \to 0$, we cannot have $y_0(x) = C_0 e^{1/x}$ as the outer solution.

We therefore require another boundary layer at $x = 0$, to satisfy the condition $y(0) = 1$. We use another boundary layer scale Z such that

$$x = Z \cdot \beta \quad \text{where} \quad Z \sim \mathrm{Ord}(1) \ \& \ \beta << 1$$

$$\& \ y(x) = Y_{left}(Z)$$

Substituting $x = Z\beta$, we have

$$\frac{\epsilon}{\beta^2} \frac{d^2 Y_{left}}{dZ^2} - Z^2 \beta^2 \cdot \frac{1}{\beta} \frac{dY_{left}}{dZ} - Y_{left} = 0$$

$$\Rightarrow \frac{\epsilon}{\beta^2} \frac{d^2 Y_{left}}{dZ^2} - \beta Z^2 \frac{dY_{left}}{dZ} - Y_{left} = 0$$

There are two possibilities :

$$\frac{\epsilon}{\beta^2} \sim \beta \quad \text{such that} \quad \beta \sim \epsilon^{1/3}$$

$$\text{or} \quad \frac{\epsilon}{\beta^2} \sim 1 \quad \text{such that} \quad \beta \sim \epsilon^{1/2}$$

In the first case, the third term dominates and since $Y_{left} \sim \mathrm{Ord}(1)$, this cannot be balanced.

The only consistent boundary layer for the inner solution on the left boundary is $\boxed{\beta = \epsilon^{1/2}}$.

With this choice, we have

$$\frac{d^2 Y_{left}}{dz^2} - Y_{left} = \epsilon^{1/2} z^2 \frac{dY_{left}}{dz}$$

Let $Y_{left} = Y_0 + \epsilon^{1/2} Y_1 + \cdots$

Leading order $O(1)$:-

$$\frac{d^2 Y_{0,left}}{dz^2} - Y_{0,left} = 0$$

$$\Rightarrow \quad Y_{0,left}(z) = C_3 e^{z} + C_4 e^{-z}$$

with $Y_{0,left}(0) = 1$, we have $C_3 + C_4 = 1$

As $z \to \infty$, $e^{z} \to \infty$. To suppress this large growth,

we require $C_3 = 0$.

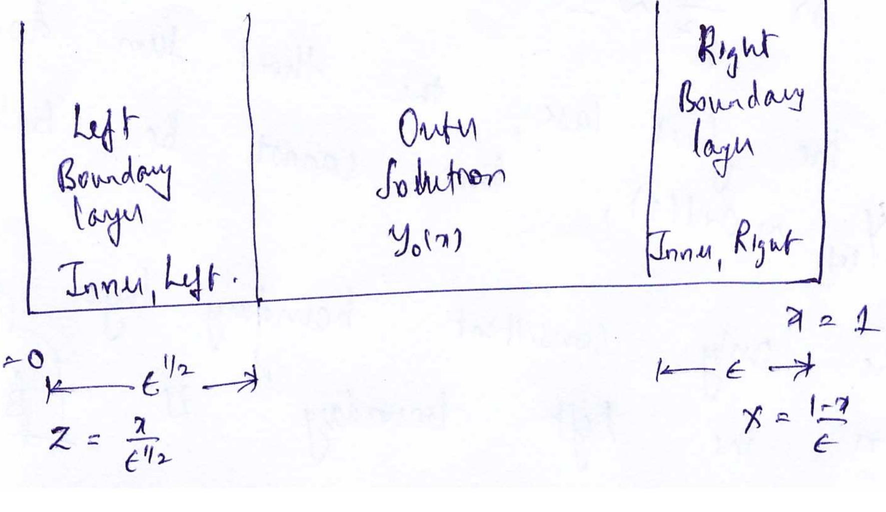
Matching also requires that $C_0 = 0$ since $y_0(x) \to \infty$

as $x \to 0$.

Therefore, we have $C_0 = 0$, and hence outer solution is.

$$y_0(x) = 0$$

General picture:



(v)

Matching the right boundary layer to the outer solution :-

$$\lim_{x \to 1} y_0(x) = \lim_{X \to \infty} Y_{right}.$$

where $y_0(x) = C_0 e^{1/x} = 0$ since $C_0 = 0$

& $Y_{right} = C_1 + (1 - C_1) e^{-X}$

$\Rightarrow \quad 0 = C_1 \quad \Rightarrow \quad \boxed{C_1 = 0}$

Now we have all the necessary constants except C_4. In this problem, there are two boundary layers, one on left & one on right, & an outer solution, $y_0(x)$, in the middle.

$$Y_{0,left} = C_4 e^{-Z}$$

$$Y_{0,right} = e^{-X}$$

$$y_0(x) = 0 \quad \text{(outer solution)}$$

~~Matching the left boundary layer to the outer solution.~~

$$\cancel{\lim_{x \to 0} Y_{outer} = \lim_{Z \to \infty} Y_{left}}$$

$$\cancel{\Rightarrow 0 =}$$

Since $Y_{0,left}(Z = 0) = 1$, we have $C_4 = 1$.

Uniform or Composite Solution :-

$$y_{unif} = y_0(x) + Y_{0,left}(z) + Y_{0,right}(X)$$
$$- y_{left,\,match} - y_{right,\,match}$$

$$= 0 + e^{-\left(\frac{1-x}{\epsilon}\right)} + e^{-x/\epsilon^{1/2}} - 0 - 0$$

$$\boxed{y_{unif.} = e^{\frac{x-1}{\epsilon}} + e^{-x/\epsilon^{1/2}}}$$

See the mathematica file "assign. Soln 3b.nb" for a comparison between exact solution & perturbation solution.

IV. MULTIPLE SCALES

1. $y'' + y + \epsilon(y')^3 = 0$

$y(0) = 1$

$y'(0) = 0$

If $\epsilon > 0$, we expect the solution to decay to zero as $t \to \infty$. To verify this, we can construct an integral relation for the energy, $E = \frac{1}{2}y^2 + \frac{1}{2}(y')^2$.

We get $y'y'' + yy' = -\epsilon(y')^4$

$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}(y')^2\right] + \frac{d}{dt}\left[\frac{1}{2}y^2\right] = -\epsilon(y')^4$

$\Rightarrow \frac{d}{dt}E = -\epsilon(y')^4$

< 0

Since $\epsilon > 0$ & $(y')^4 > 0$.

Since $\frac{dE}{dt} < 0$ when $\epsilon > 0$, the energy decreases to 0 as $t \to \infty$.

Similarly, as $\epsilon < 0$, we expect E to increases with time.

To obtain the long term behaviour of the solution, we now use multiple-scale analysis:

Let $y(t) = Y_0(t, \tau) + \epsilon Y_1(t, \tau) + \cdots$ when $\epsilon \to 0$

Here $\tau = \epsilon t$.

Substituting & equating various terms with the same power of ϵ_1 we get

$O(\epsilon^0)$: $$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0 \qquad \text{(a)}$$

$O(\epsilon^1)$: $$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t\,\partial \tau} - \left(\frac{\partial Y_0}{\partial t}\right)^3 \qquad \text{(b)}$$

General solution of (a) as shown in class is:

$$Y_0(t,\tau) = A(\tau)\,e^{it} + A^*(\tau)\,e^{-it}$$

Substituting Y_0 in (b), we get

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -e^{it}\left[2i\frac{dA}{d\tau} + 3i\,A^2A^*\right] - e^{-it}\left[-2i\frac{dA^*}{d\tau} - 3i(A^*)^2A\right] + i\,e^{3it}A^3 - i\,e^{-3it}(A^*)^3$$

Since e^{it} & e^{-it} are the fundamental solution of the homogeneous problem $\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = 0$ (since this is identical to (a)), the RHS terms involving e^{it} & e^{-it} lead to secular growth.

To eliminate secular terms, we set the terms in square bracket to zero.

$$2i \frac{dA}{d\tau} + 3i A^2 A^* = 0$$

$$-2i \frac{dA^*}{d\tau} - 3i (A^*)^2 A = 0$$

Both are the same equations, complex conjugates of each other.

To solve the above equations, we use

$A(\tau) = R(\tau) e^{i\theta(\tau)}$ where $R(\tau)$ & $\theta(\tau)$ are real.

Substituting, we have

$$\frac{dR}{d\tau} = -\frac{3}{2} R^3 \quad ; \quad \frac{d\theta}{d\tau} = 0$$

Therefore, $$R(\tau) = \frac{R(0)}{\sqrt{3\tau R(0)^2 + 1}}$$

and $\theta(\tau) = \theta(0)$.

Here $R(0)$ & $\theta(0)$ are the initial values of $R(\tau)$ & $\theta(\tau)$.

These two are determined by the initial conditions

$y(0) = 1$ & $y'(0) = 0$.

These initial conditions become:

$$Y_0(0,0) = 1 \quad \& \quad \frac{\partial Y_0(0,0)}{\partial t} = 0$$

Hence $R(0) = \frac{1}{2}$, $\theta(0) = 0$

Thus, to leading order in ϵ, $y(t) = Y_0(t, \tau)$

$$\Rightarrow \quad y(t) = \underbrace{R(\tau) e^{i\theta(\tau)}}_{A(\tau)} \cdot e^{it} + \underbrace{R(\tau) e^{-i\theta(\tau)}}_{A^*(\tau)} e^{-it}$$

$= Re^{it} + Re^{-it}$ with $\theta(\tau) = \theta(0) = 0$

$$= R(\tau) \cdot \left[e^{it} + e^{-it} \right]$$

$$= R(\tau) \cdot 2\cos t$$

$$= 2\cos t \cdot \frac{R(0)}{\sqrt{3\tau \cdot R(0)^2 + 1}}$$

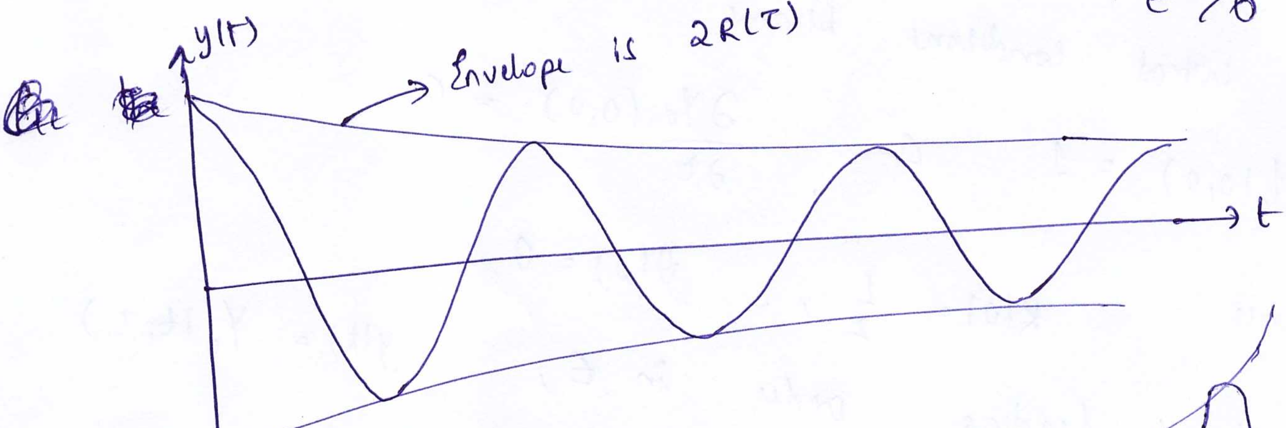
Using $R(0) = \frac{1}{2}$, we have

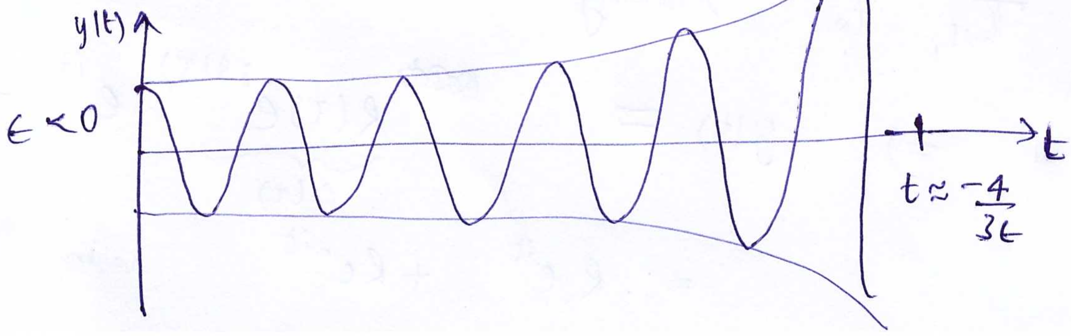
$$y(t) = \frac{\cos t}{\sqrt{1 + \frac{3\epsilon t}{4}}}$$ where we have used $\tau = \epsilon t$

If $\epsilon > 0$, then $y(t)$ decays as $t^{-1/2}$ for large t.

when $\epsilon < 0$, then $y(t) \to \infty$ as $t \to -\frac{4}{3\epsilon}$.

Also note that the solution does not exhibit any phase-shift, i.e; frequency shift, to leading order in ϵ.





(VIII)

IV

2. $$y'' + y = \epsilon\left[y' - \frac{1}{3}(y')^3\right]$$

$$y(0) = 0$$
$$y'(0) = \alpha$$

Performing a multiple-scale analysis, we have

$y(t) \sim Y_0(t,\tau) + \epsilon\, Y_1(t,\tau) + \dots$ where $\tau = \epsilon t$ & $\epsilon \to 0$.

Substituting $y(t)$ & ordering terms according to ϵ:

$O(\epsilon^0)$: $$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0 \qquad \text{—— (a)}$$

$O(\epsilon^1)$: $$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t\,\partial \tau} + \frac{\partial Y_0}{\partial t} - \frac{1}{3}\left(\frac{\partial Y_0}{\partial t}\right)^3 \qquad \text{—— (b)}$$

General solution of (A) :-

$$Y_0(t,\tau) = A(\tau)\, e^{it} + A^*(\tau)\, e^{-it}$$

Substituting $Y_0(t,\tau)$ into (b), and setting terms involving coefficients of e^{it} & e^{-it} in the RHS of (b) to zero, we have

$$2i\frac{dA}{d\tau} + iA - iA^2A^* = 0$$

$$2i\frac{dA^*}{d\tau} - iA^* + i(A^*)^2A = 0$$

Using $A(\tau) = R(\tau)\, e^{i\theta(\tau)}$ where $R(\tau)$ & $\theta(\tau)$

are real functions, we have

$$2\frac{dR}{d\tau} = R - R^3$$

and $\dfrac{d\theta}{d\tau} = 0$

$\theta(\tau) = \theta(0)$ is constant

& $R(\tau) = R(0)\left[e^{-\tau} + R^2(0)\{1 - e^{-\tau}\}\right]^{-1/2}$

↳ Need to use Partial fractions,

ie; $R - R^3 = R(1-R^2)$
$= R(1-R)(1+R)$

$$\frac{dR}{R(1-R)(1+R)} = \frac{dR}{R} - \frac{dR}{2(R+1)} - \frac{dR}{2(R-1)}$$

Using the initial conditions $y(0) = 0$ & $y'(0) = \alpha$, we

have $R(0) = \frac{\alpha}{2}$, $\theta(0) = \frac{-\pi}{2}$.

Putting everything together, at leading order, we have

$$y(t) \sim \frac{\alpha \sin t}{\sqrt{e^{-\tau} + \alpha^2(1-e^{-\tau})/4}}, \quad \tau = \epsilon t$$

As $t \to \infty$; $e^{-\tau} = e^{-\epsilon t} \to 0$

$\Rightarrow y(t) \sim 2\sin t$ as $t \to \infty$. : Limit cycle.