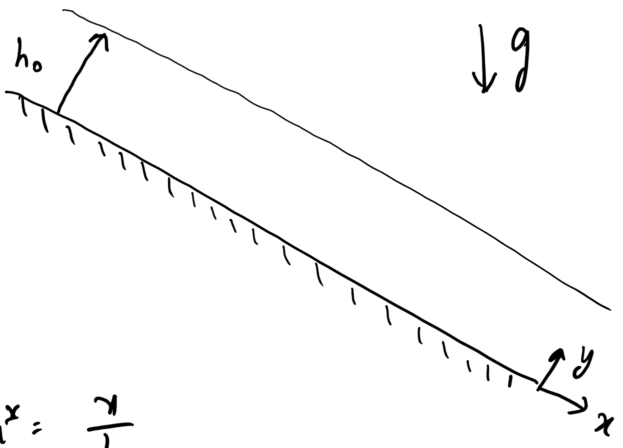
Hydrodynamics of free-surface flows:-



$x^* = \frac{x}{L}$

$y^* = \frac{y}{h_0}$

Let $\epsilon = \frac{h_0}{L} << 1$

Continuity:- $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

If $u^* = \frac{u}{U}$, $v^* \sim ?$

$$\frac{U}{L} \sim \frac{v}{h_0} \Rightarrow v \sim \frac{h_0}{L} \cdot U$$

x-Mom:- $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g_x$

$$\frac{U^2}{L}\cdot\frac{D^* u^*}{Dt^{*2}} = -\frac{1}{\rho}\cdot\frac{p_c}{L}\frac{\partial p^*}{\partial x^*} + \mu U\left(\frac{1}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{1}{h_0^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right) + g_x$$

Options for p_c:- $p_c = \rho U^2$

$p_c = \frac{1}{\epsilon}\frac{\rho\mu U}{h_0}$

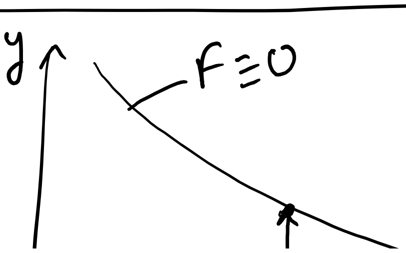
$\frac{1}{\rho}\frac{p_c}{L} \sim \frac{\mu U}{h_0^2}$

$p_c = \rho\mu U \frac{L}{h_0^2}$

$\hat{n}\cdot\underline{\underline{T}}\cdot\hat{n}$

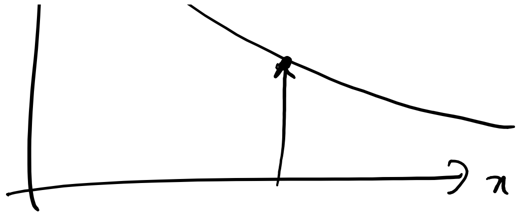
Eqn. of free surface:-

$F(x, y, t) = C$



$$F(x, y, t) = C$$

or $\boxed{y = h(x,t)}$



$$F(x,y,t) \equiv y - h(x,t) = 0$$

$$\hat{n} = \frac{\nabla F}{|\nabla F|} \quad ; \quad \vec{\nabla F} = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j}$$

$$\vec{\nabla F} = -h_x \hat{i} + \hat{j}$$

$$\hat{n} = \frac{-h_x \hat{i} + \hat{j}}{\sqrt{1+h_x^2}}$$

$$\hat{t} = \frac{\hat{i} + h_x \hat{j}}{\sqrt{1+h_x^2}}$$

$$\underline{\underline{S}} = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$\underline{\nabla u} = \left[\frac{\partial u}{\partial x} + \right.$$

$$\underline{\underline{T}} = -p\,\underline{\underline{I}} + 2\mu\,\underline{\underline{S}}$$

$$= -p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} \end{bmatrix}$$

$\underline{\hat{n}} \cdot \underline{\underline{T}} \cdot \underline{\hat{n}}$:-

$$\underline{\underline{T}} \cdot \underline{\hat{n}} = \left\{ -p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) \\ \frac{1}{2}(u_y + v_x) & v_y \end{bmatrix} \right\} \cdot \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

$$\begin{bmatrix} n_x & n_y \end{bmatrix}_{1\times 2} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_{2\times 2} \begin{bmatrix} n_x \\ n_y \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} n_x & n_y \end{bmatrix}_{1\times 2} \begin{bmatrix} n_x T_{11} + n_y T_{12} \\ n_x T_{21} + n_y T_{22} \end{bmatrix}_{2\times 1}$$

$$= \left\{ n_x \left(n_x T_{11} + n_y T_{12} \right) + n_y \left(n_x T_{21} + n_y T_{22} \right) \right\}_{1\times 1}$$

$$n_x = \frac{-h_x}{\sqrt{1+h_x^2}} \; ; \quad n_y = \frac{1}{\sqrt{1+h_x^2}}$$

$$T_{11} = -p + 2\mu u_x$$

$$T_{12} = \mu \left(u_y + v_x \right)$$

$$T_{21} = \mu \left(u_y + v_x \right)$$

$$T_{22} = -p + 2\mu v_y$$

$$\underline{\hat{n}} \cdot \underline{\underline{T}} \cdot \underline{\hat{n}} = -p + \frac{2\mu}{(1+h_x^2)} \left[-u_x \left(1-h_x^2\right) - h_x \left(u_y + v_x \right) \right]$$

$$\left(\underline{n} \cdot \underline{\underline{T}} \cdot \underline{\hat{n}} \right)_{\ominus} - \left(\underline{\hat{n}} \cdot \underline{\underline{T}} \cdot \underline{\hat{n}} \right)_{\oplus} = -\sigma \nabla \cdot \underline{\hat{n}}$$

$$\Rightarrow \left\{ -p_- + \frac{2\mu_-}{1+h_x^2} \left[-u_x \left(1-h_x^2\right) - h_x \left(u_y + v_x \right) \right]_{\ominus} \right\} - \left(-p_+ \right)$$

$$= -\sigma \nabla \cdot \underline{\hat{n}}$$

$$\Rightarrow -\left(p_- - p_+ \right) - \frac{2\mu_w}{1+h_x^2} \left(u_x \left(1-h_x^2\right) + h_x \left(u_y + v_x \right) \right) = -\sigma \nabla \cdot \underline{\hat{n}}$$

$\tilde{p}_\infty$

Replacing p_- with p, p_+ with p_∞, $\mu_w \to \mu$:-

$$(p - p_\infty) + \frac{2\mu}{1+h_x^2}\left[u_x(1-h_x^2) + h_x(u_y+v_x)\right] = \sigma\, \nabla\cdot\underline{\hat{n}}$$

$\underline{\underline{\nabla\cdot\hat{n}}}$:

$$\frac{\partial}{\partial x} n_x + \cancel{\frac{\partial}{\partial y} n_y}^{\to 0} = \frac{\partial n_x}{\partial x} = \frac{\partial}{\partial x}\left\{\frac{-h_x}{\sqrt{1+h_x^2}}\right]$$

$$= -\left\{\frac{\sqrt{1+h_x^2}\cdot h_{xx} - h_x\cdot\frac{1}{\cancel{2}\sqrt{1+h_x^2}}\cdot\cancel{2}h_x h_{xx}}{1+h_x^2}\right]$$

$$= -\frac{(1+h_x^2)h_{xx} - h_x^2\cdot h_{xx}}{(1+h_x^2)^{3/2}}$$

$$\nabla\cdot\underline{\hat{n}} = \frac{-h_{xx}}{(1+h_x^2)^{3/2}}$$

Normal stress balance :-

$$(p - p_\infty) + \frac{2\mu}{1+h_x^2}\left[u_x(1-h_x^2) + h_x(u_y+v_x)\right] = \frac{-\sigma\, h_{xx}}{(1+h_x^2)}$$

$\circledast$

Tangential stress balance :-

$$\underline{\hat{n}}\cdot\underline{\underline{T}}\cdot\underline{\hat{t}} = \frac{\mu}{\sqrt{1+h_x^2}}\left[-4h_x u_x + (u_y+v_x)(1-h_x^2)\right]$$

$$\underline{\hat{n}} \cdot \underline{\underline{I}} \cdot \underline{\hat{t}} = \hat{t} \cdot \nabla_s \sigma$$

$$\nabla_s = [\underline{I} - \underline{\hat{n}}\,\underline{\hat{n}}] \cdot \nabla \longrightarrow \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

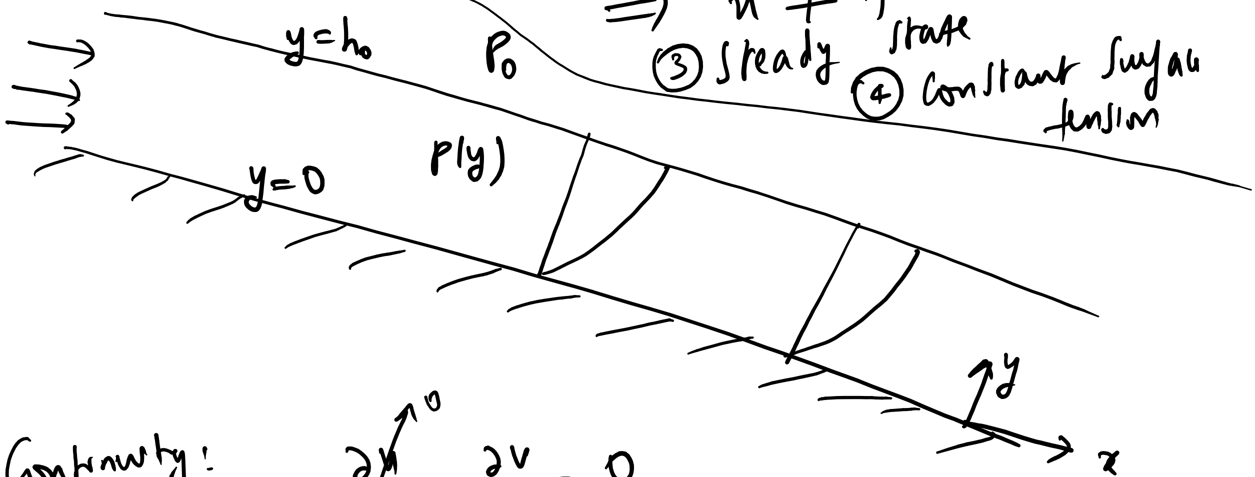
$$\underbrace{\begin{bmatrix} n_x \\ n_y \end{bmatrix}_{2\times 1} \begin{bmatrix} n_x & n_y \end{bmatrix}_{1\times 2}}$$

$$\begin{bmatrix} & \\ & \end{bmatrix}_{2\times 2}$$

Returning to the film flowing down an incline:-

Assumptions:-

1. Film thickness is constant $\Rightarrow h = h_0 \neq f(x)$
2. Fully developed flow:- $\Rightarrow u \neq f(x)$
3. Steady state
4. Constant surface tension



Continuity:

$$\cancelto{0}{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v = \text{constant}$$

Since $v = 0$ at $y = 0$, $v = 0$ everywhere

x-Mom. Eqn:-

$$\cancelto{0}{\frac{\partial u}{\partial t}} + u \cancelto{0}{\frac{\partial u}{\partial x}} + \cancelto{0}{v \frac{\partial u}{\partial y}} = -\frac{1}{\rho} \cancelto{0}{\frac{\partial p}{\partial x}} + \nu \left(\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} \right) + g_x$$

$$\cancel{\frac{\partial u}{\partial t}}^{0} + u\cancel{\frac{\partial u}{\partial x}} + \cancel{v}\frac{\partial u}{\partial y} = -\frac{1}{\rho}\cancel{\frac{\partial p}{\partial x}} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right) + g_x$$

We get $\quad \nu \dfrac{\partial^2 u}{\partial y^2} + g_x = 0$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\frac{1}{\nu} g_x$$

$$\frac{\partial u}{\partial y} = -\frac{1}{\nu} g_x \cdot y + C_1$$

$$\boxed{u(y) = -\frac{1}{\nu} g_x \cdot \frac{y^2}{2} + C_1 y + C_2}$$

y-Mom. Equation:-

$$\cancel{\frac{\partial v}{\partial t}}^{0} + u\cancel{\frac{\partial v}{\partial x}}^{0} + v\cancel{\frac{\partial v}{\partial y}}^{0} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}^{0} + \cancel{\frac{\partial^2 v}{\partial y^2}}^{0}\right) + g_y$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} + g_y = 0 \quad \Rightarrow \quad \frac{\partial p}{\partial y} = \rho g_y$$

$$\Rightarrow \quad \boxed{p = \rho g_y \cdot y + C_3}$$

At $y = h_0$, $\quad p = P_0$

$$\Rightarrow \quad P_0 = \rho g_y \cdot h_0 + C_3$$

$$\Rightarrow \quad C_3 = P_0 - \rho g_h \cdot h_0$$

$$\Rightarrow \quad \boxed{p(y) = P_0 + \rho g_y (h - y_0)}$$

Tangential stress balance:-

$$\hat{t} \cdot \cancel{\nabla_s \sigma}^{0}$$

This reduces to

This reduces to

$$\frac{\mu}{1} \times (u_y) = 0 \Rightarrow u_y = 0 \Big|_{\text{at } y = h_0}$$

Equation for $u(y)$:-

$$u(y) = -\frac{1}{\nu} g_x \cdot \frac{y^2}{2} + c_1 y + c_2$$

with $u = 0$ at $y = 0$

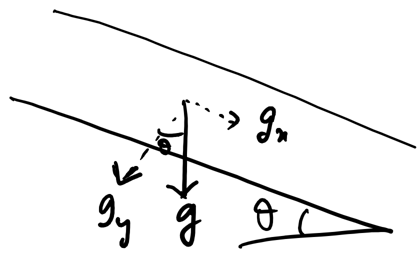
& $\frac{\partial u}{\partial y} = 0$ at $y = h_0$

$u = 0$ at $y = 0 \Rightarrow c_2 = 0$

$$\frac{\partial u}{\partial y} = -\frac{1}{\nu} g_x \cdot y + c_1$$

$$0 = -\frac{1}{\nu} g_x \cdot h_0 + c_1 \Rightarrow c_1 = \frac{g_x}{\nu} \cdot h_0$$

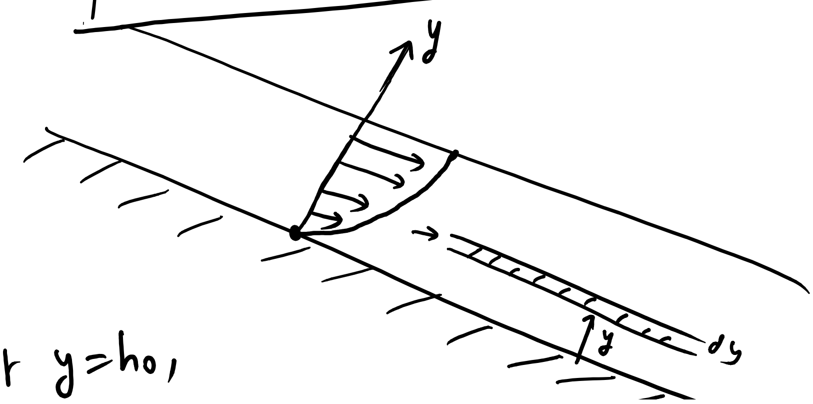
$$\therefore u(y) = -\frac{g_x}{\nu} \cdot \frac{y^2}{2} + \frac{g_x}{\nu} \cdot h_0 y$$



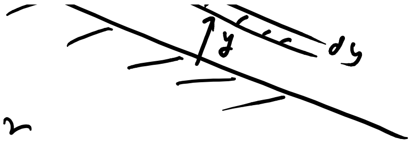
$$u(y) = -\frac{g_x}{\nu}\left(\frac{y^2}{2} - h_0 y\right)$$

$$g_x = +g \sin\theta$$

$$\therefore \boxed{u(y) = -\frac{g \sin\theta}{\nu}\left(\frac{y^2}{2} - h_0 y\right)} \qquad \boxed{g > 0}$$



At $y = h_0$,



At $y = h_0$,

$$u(y=h_0) = \frac{g\sin\theta}{\nu} \cdot \frac{h_0^2}{2}$$

Flow rate:

$$Q = \int_0^{h_0} u(y)\, dy$$

$$= \int_0^{h_0} -\frac{g\sin\theta}{\nu}\left(\frac{y^2}{2} - h_0 y\right) dy$$

$$= -\frac{g\sin\theta}{\nu} \cdot \left[\frac{y^3}{6} - h_0\frac{y^2}{2}\right]_0^{h_0}$$

$$= -\frac{g\sin\theta}{\nu}\left[\frac{h_0^3}{6} - \frac{h_0^3}{2} \times \frac{3}{3}\right]$$

$$= -\frac{g\sin\theta}{\nu} \cdot -\frac{2}{6} h_0^3$$

$$\boxed{Q = \frac{1}{3}\frac{g\sin\theta}{\nu} \cdot h_0^3}$$