26/March/2012

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 5e^{2t} \\ 3e^{-t} \end{bmatrix}$$

METHOD I :- $T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$; $T^{-1} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix}$

Let $\vec{x} = T\vec{y} \Rightarrow \vec{x}' = T\vec{y}'$

$\vec{x}' = A\vec{x} + \vec{g}(t) \Rightarrow \vec{y}' = T^{-1}AT\vec{y} + T^{-1}\vec{g}(t)$

$\Rightarrow T\vec{y}' = AT\vec{y} + \vec{g}(t)$

$\vec{y}' = D\vec{y} + T^{-1}\vec{g}(t)$

where $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$$T^{-1}\vec{g}(t) = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix}_{2\times 2} \begin{bmatrix} 5e^{2t} \\ 3e^{-t} \end{bmatrix}_{2\times 1} = \begin{bmatrix} \frac{5}{2}e^{2t} + \frac{3}{4}e^{-t} \\ \frac{5}{2}e^{2t} - \frac{3}{4}e^{-t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \frac{5}{2}e^{2t} + \frac{3}{4}e^{-t} \\ \frac{5}{2}e^{2t} - \frac{3}{4}e^{-t} \end{bmatrix}$$

$\Rightarrow y_1' = 3y_1 + \frac{5}{2}e^{2t} + \frac{3}{4}e^{-t}$ ~~$\Rightarrow y_1' - 3y_1$~~

$y_2' = -y_2 + \frac{5}{2}e^{2t} - \frac{3}{4}e^{-t}$

$y_1' - 3y_1 = \frac{5}{2}e^{2t} + \frac{3}{4}e^{-t} \Rightarrow \phi = e^{-3t} \Rightarrow (y_1 e^{-3t})' = \frac{5}{2}e^{-t} + \frac{3}{4}e^{-4t}$

$\Rightarrow y_1 e^{-3t} = -\frac{5}{2}e^{-t} - \frac{3}{16}e^{-4t} + c_1$

$\Rightarrow y_1 = -\frac{5}{2}e^{2t} - \frac{3}{16}e^{-t} + c_1 e^{3t}$

$$y_2' + y_2 = \frac{5}{2}e^{2t} - \frac{3}{4}e^{-t}$$

$$\Rightarrow \phi = e^{t} \qquad \Rightarrow (y_2 e^{t})' = \frac{5}{2}e^{3t} - \frac{3}{4}$$

$$\Rightarrow y_2 e^{t} = \frac{5}{2}\cdot\frac{e^{3t}}{3} - \frac{3}{4}t + c_2$$

$$\Rightarrow y_2 = \frac{5}{6}e^{2t} - \frac{3t}{4}e^{-t} + c_2 e^{-t}$$

$$-\frac{5}{2}+\frac{5}{6} = \frac{-6+2}{12}\cdot 5 = -\frac{4}{12}5 =$$

$$5+\frac{5}{3} =$$

$$\therefore \vec{X} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -\frac{5}{2}e^{2t} - \frac{3}{16}e^{-t} + c_1 e^{3t} \\ \frac{5}{6}e^{2t} - \frac{3t}{4}e^{-t} + c_2 e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{2}e^{2t} - \frac{3}{16}e^{-t} + c_1 e^{3t} + \frac{5}{6}e^{2t} - \frac{3t}{4}e^{-t} + c_2 e^{-t} \\ -5e^{2t} - \frac{3}{8}e^{-t} + 2c_1 e^{3t} - \frac{5}{3}e^{2t} + \frac{3t}{2}e^{-t} - 2c_2 e^{-t} \end{bmatrix}$$

$$= c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-t} + \begin{bmatrix} -\frac{5}{3} \\ -\frac{20}{3} \end{bmatrix}e^{2t} + \begin{bmatrix} -3/16 \\ -3/8 \end{bmatrix}e^{-t} + \begin{bmatrix} -\frac{3}{4} \\ \frac{3}{2} \end{bmatrix}te^{-t}$$

METHOD II:- $\vec{X}' = A\vec{X} + \vec{g}(t)$

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} 5e^{2t} \\ 3e^{-t} \end{bmatrix}$$

Homogeneous Problem:

$$\vec{X}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{3t} \quad ; \quad \vec{X}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-t}$$

General solution of Homogeneous Part:-

$$\vec{X} = c_1\vec{X}^{(1)} + c_2\vec{X}^{(2)} = \left[\vec{X}^{(1)} \quad \vec{X}^{(2)}\right]_{2\times2}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{2\times1} = \vec{\psi}(t)\,\vec{c}$$

$$\text{where } \vec{\psi}(t) = \left[\vec{X}^{(1)} \quad \vec{X}^{(2)}\right] = \begin{bmatrix} \vec{X}_1^{(1)} & \vec{X}_1^{(2)} \\ \vec{X}_2^{(1)} & \vec{X}_2^{(2)} \end{bmatrix}$$

↳ Fundamental Matrix

$$\vec{X}'^{(1)} = A\vec{X}^{(1)} \quad ; \quad \vec{X}'^{(2)} = A\vec{X}^{(2)}$$

$$\Rightarrow \left[\vec{X}'^{(1)} \quad \vec{X}'^{(2)}\right] = \left[A\vec{X}^{(1)} \quad A\vec{X}^{(2)}\right]$$

$$\Rightarrow \boxed{\vec{\psi}'(t) = A\,\vec{\psi}(t)}$$

Particular Solution:-

$$\vec{X} = \vec{\psi}(t)\,\vec{u}(t)$$

↳ vector.

$$\vec{X}' = \vec{\psi}(t)\,\vec{u}'(t) + \vec{\psi}'(t)\,\vec{u}(t) = A\,\vec{\psi}(t)\,\vec{u}(t) + \vec{g}(t)$$

$$\Rightarrow \vec{\psi}(t)\,\vec{u}'(t) + \vec{\psi}'(t)\,\vec{u}(t) - A\,\vec{\psi}(t)\,\vec{u}(t) = \vec{g}(t)$$

$$\Rightarrow \vec{\psi}(t)\,\vec{u}'(t) + \left[\underbrace{\vec{\psi}'(t) - A\,\vec{\psi}(t)}_{=0}\right]\vec{u}(t)$$

$$\Rightarrow \vec{\psi}(t)\,\vec{u}'(t) = \vec{g}(t)$$

$$\Rightarrow \begin{bmatrix} e^{3t} & e^{-t} \\ \boxed{2e^{3t}} & -2e^{-t} \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 5e^{2t} \\ 3e^{-t} \end{bmatrix}$$

↳ NOT A CONSTANT MATRIX $\Rightarrow$ $e^{\lambda t}$ type solution does not WORK.

~~Row reduction:~~ ~~$R_2 \to R_2 - 2R_1$~~

$\cancel{\Rightarrow \begin{bmatrix} e^{3t} & e^{-t} \end{bmatrix} \begin{bmatrix} u_1' \end{bmatrix}}$

ROW REDUCTION :.

$$\vec{\Psi}(t) \,|\, g(t) = \left[\begin{array}{cc|c} e^{3t} & e^{-t} & 5e^{2t} \\ 2e^{3t} & -2e^{-t} & 3e^{-t} \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\Rightarrow \vec{\Psi}(t) \,|\, g(t) = \left[\begin{array}{cc|c} e^{3t} & e^{-t} & 5e^{2t} \\ 0 & -4e^{-t} & 3e^{-t} - 10e^{2t} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} e^{3t} & e^{-t} \\ 0 & -4e^{-t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 5e^{2t} \\ 3e^{-t} - 10e^{2t} \end{bmatrix}$$

$5 - \frac{10}{4}$

$$e^{3t}u_1' + e^{-t}u_2' = 5e^{2t}$$

$$-4e^{-t}u_2' = 3e^{-t} - 10e^{2t}$$

$$\Rightarrow u_2' = \frac{-3}{4} + \frac{10}{4}e^{3t} \Rightarrow u_2 = \frac{-3t}{4} + \frac{10}{12}e^{3t} + C_2$$

$$u_1' e^{3t} + e^{-t}\left[\frac{-3}{4} + \frac{10}{4}e^{3t}\right] = 5e^{2t}$$

$$\Rightarrow u_1' e^{3t} - \frac{3}{4}e^{-t} + \frac{10}{4}e^{2t} = 5e^{2t} \Rightarrow u_1' e^{3t} = \frac{5}{2}e^{2t} + \frac{3}{4}e^{-t}$$

$$\Rightarrow u_1' = \frac{5}{2}e^{-t} + \frac{3}{4}e^{-4t}$$

$$\Rightarrow u_1 = \frac{-5}{2}e^{-t} - \frac{3}{16}e^{-4t} + C_1$$

$$\vec{X} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{5}{2}e^{-t} - \frac{3}{16}e^{-4t} + C_1 \\ \frac{-3t}{4} + \frac{5}{6}e^{3t} + C_2 \end{bmatrix}$$

$\frac{5}{6} - \frac{5}{2} = \frac{10-30}{12} = -\frac{20}{12}$

$$\Rightarrow \vec{x} = \begin{bmatrix} -\frac{5}{2}e^{2t} - \frac{3}{16}e^{-t} + c_1 e^{3t} - \frac{3t}{4}e^{-t} + \frac{5}{6}e^{2t} + c_2 e^{-t} \\ -5e^{2t} - \frac{3}{8}e^{-t} + 2c_1 e^{3t} + \frac{3t}{2}e^{-t} - \frac{5}{3}e^{2t} - 2c_2 e^{-t} \end{bmatrix}.$$

$$= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + \begin{bmatrix} -5/3 \\ -\frac{20}{3} \end{bmatrix} e^{2t} + \begin{bmatrix} -3/16 \\ -3/8 \end{bmatrix} e^{-t}$$

$$+ \begin{bmatrix} -3/4 \\ \frac{3}{2} \end{bmatrix} t e^{-t}$$

26/Mar/2012

NONLINEAR SYSTEMS: (Geometric approach to differential equations)

The Phase Plane: (LINEAR SYSTEMS)

$\vec{X}' = A\vec{X}$

$\hookrightarrow \quad (A - rI)\vec{\xi} = 0$ $\qquad r$: Eigenvalue, $\vec{\xi}$: Eigenvector

$|A - rI| = 0$

$\vec{X} = 0$: critical points or equilibrium solutions.

Assume: A is non singular $\Rightarrow |A| \neq 0$

$\Rightarrow \vec{X} = \vec{0}$ is the only critical point of the system.

Solution of $\vec{X}' = AX$: $\vec{X} = \vec{\phi}(t)$: Trajectory of a moving particle with velocity $\frac{d\vec{x}}{dt}$

(x_1, x_2) : Phase plane.

Set of Trajectories: Phase portrait.

Classification:

$$\underbrace{\begin{pmatrix} x' \\ y' \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{X}}$$

Eigenvalues: $|A - rI| = 0 \Rightarrow \begin{vmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{vmatrix} = 0$

$$(a_{11}-r)(a_{22}-r) - a_{12}a_{21} = 0$$

$$\Rightarrow \quad r^2 - \underbrace{(a_{11}+a_{22})}_{Tr(A)}\, r + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\cancel{\Delta(A)}\ |A|} = 0$$

$$\Rightarrow \quad r^2 - \underbrace{Tr(A)}_{p}\, r + \underbrace{|A|}_{q} = 0$$

$$\left.\begin{aligned} p &= a_{11}+a_{22} \\ q &= a_{11}a_{22} - a_{21}a_{12} \end{aligned}\right\} \quad r^2 - pr + q = 0$$

$$\boxed{r = \frac{p \pm \sqrt{p^2-4q}}{2}}$$

Let $\Delta = p^2 - 4q$

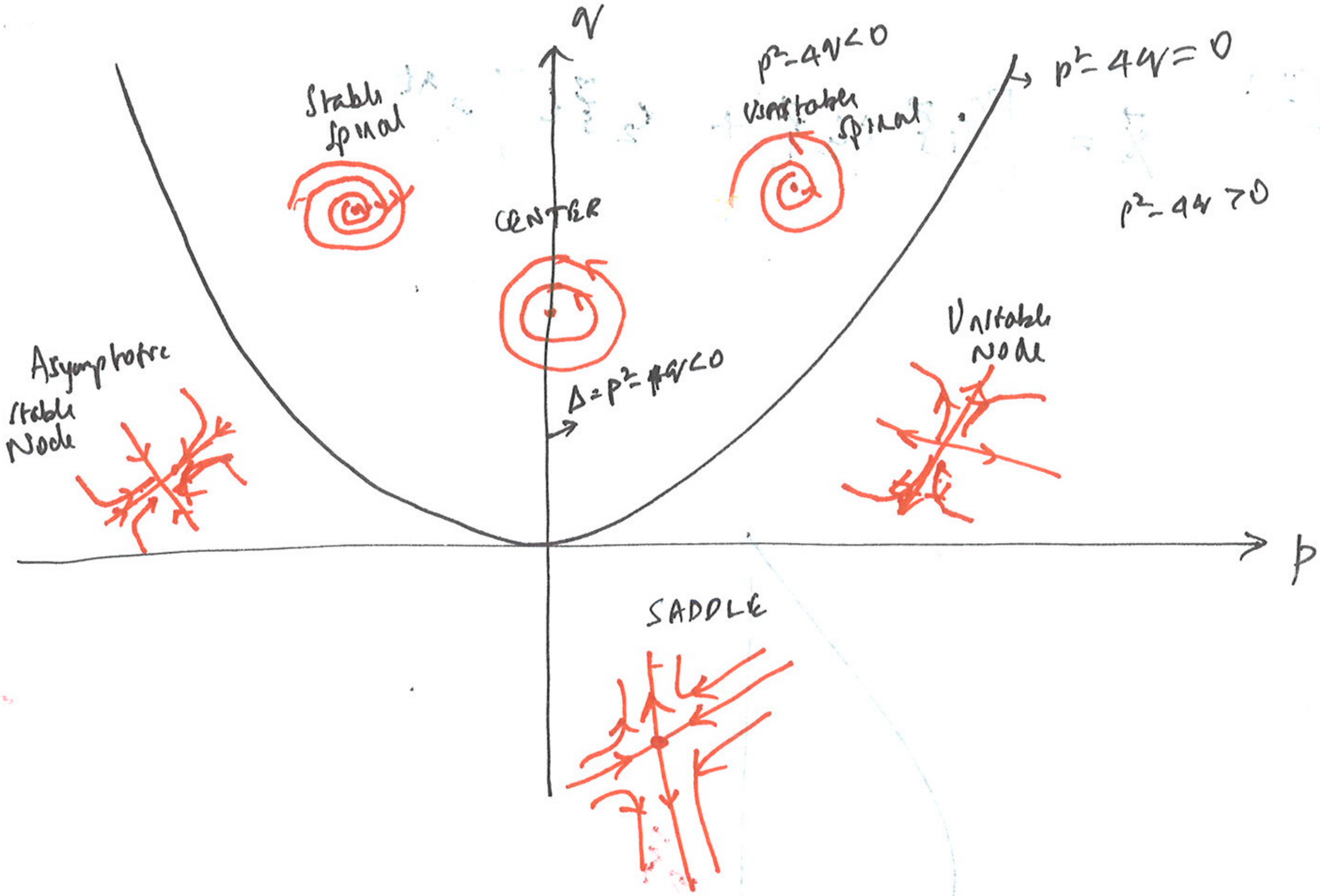
Cases:

(i) $\Delta = 0$ $\Rightarrow$ r_1 & r_2 are purely real & and equal.
Unstable node if $p > 0$
stable node if $p < 0$

(ii) $\Delta > 0$ $\Rightarrow$ Unstable node if $p > 0$
Stable " " $p < 0$

(iii) $\Delta < 0$ $\Rightarrow$ r_1 & r_2 are complex conjugates.
Unstable spiral if $p > 0$
Stable " " $p < 0$

(iv) $\Delta < 0$, $p = 0$ $\Rightarrow$ Purely imaginary. (complex conjugates)
$\Rightarrow$ Center.



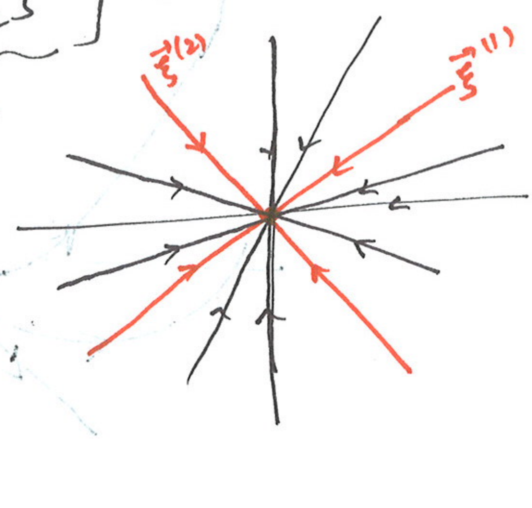
Additional Special Cases.-

Phase portrait for $r_1 = r_2 < 0$:-

Ex. Case I: $\vec{\xi}^{(1)}$ & $\vec{\xi}^{(2)}$ are two independent eigenvectors associated with $r = r_1 = r_2$

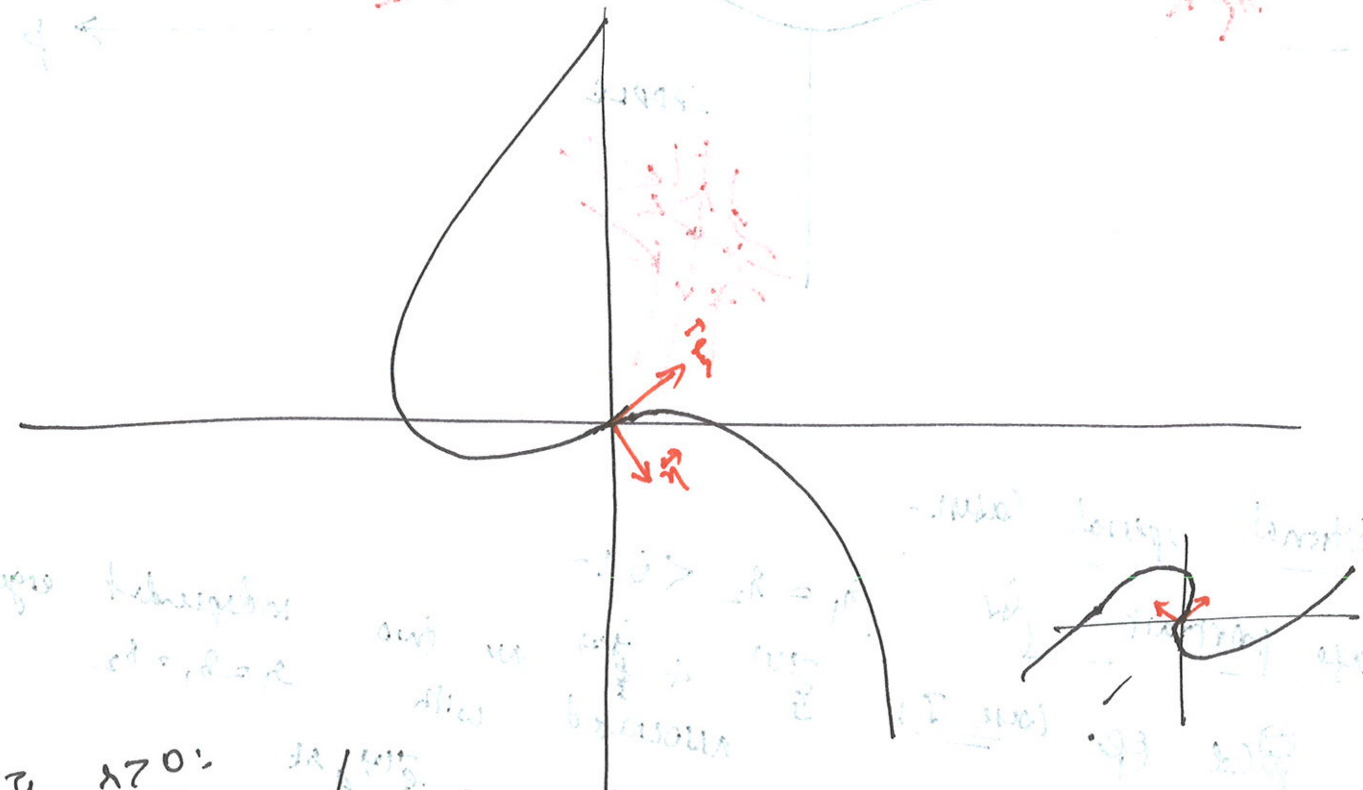
$$\Rightarrow \vec{x} = c_1 \vec{\xi}^{(1)} e^{rt} + c_2 \vec{\xi}^{(2)} e^{rt}$$

$$= \Big[\underbrace{c_1 \vec{\xi}^{(1)} + c_2 \vec{\xi}^{(2)}}_{\vec{\xi}^{(3)}}\Big] e^{rt}$$



Case II:

$$\vec{x} = \left[(c_1 \vec{\xi} + c_2 \vec{\eta}) + c_2 \vec{\xi} t \right] e^{\lambda t}$$



If $\lambda > 0$: