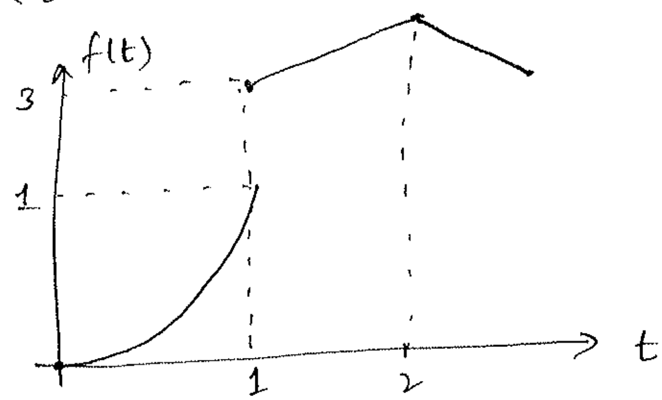
MATH 215/255

HW #8

H. DIXIT

Section 6.1 #1

(1) $f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 2+t & 1 \le t \le 2 \\ 6-t & 2 < t \le 3 \end{cases}$

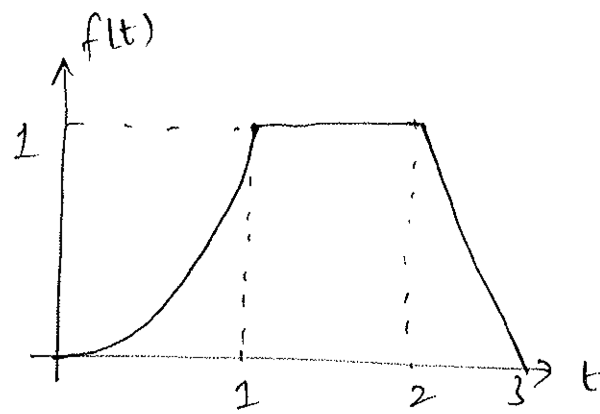


There is a discontinuity at $t = 1$.

The function is piecewise continuous.

Section 6.1 #3

(2) $f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 1 & 1 < t \le 2 \\ 3-t & 2 < t \le 3 \end{cases}$



$f(t)$ is continuous.

Section 6.1 # 5(b)

$f(t) = t^2$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} \cdot t^2 \, dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot t^2 \, dt$$

$$\cancel{= \lim_{R\to\infty} \left[e^{-st} \frac{t^3}{3} - \int \right.}$$

$$= \lim_{R\to\infty}\left[t^2 \frac{e^{-st}}{-s}\Big|_0^R - \int_0^R 2t\,\frac{e^{-st}}{-s}\,dt \right]$$

$$= \lim_{R\to\infty}\left[t^2 \frac{e^{-st}}{-s}\Big|_0^R + \frac{2}{s}\left\{ t\,\frac{e^{-st}}{-s}\Big|_0^R - \int_0^R \frac{e^{-st}}{-s}\,dt \right.\right]$$

$$= \lim_{R\to\infty}\left\{\left(\frac{R^2 e^{-sR} - 0}{-s}\right) + \frac{2}{s}\left\{\frac{Re^{-sR} - 0}{-s}\right\} + \frac{2}{s^2}\,\frac{e^{-st}}{-s}\Big|_0^R\right\}$$

$$= 0 + 0 - \frac{2}{s^3}\left(e^{-sR} - 1\right)$$

$$= \frac{2}{s^3} \quad \text{if } s > 0.$$

Note that we require $s > 0$ such that $\lim_{R\to\infty} e^{-sR} \to 0$

$$\therefore \mathcal{L}\{t^2\} = \frac{2}{s^3}$$

We can get the same result by looking at the table.

from #3 in the table, if $n = 2$,

$$\mathcal{L}\{t^2\} = \frac{n!}{s^{n+1}} = \frac{2!}{s^3} = \frac{2}{s^3}, \quad s > 0.$$

Section 6.1 # 15

$$f(t) = te^{at}$$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\cdot te^{at}\,dt$$

$$= \lim_{R\to\infty} \int_0^R e^{-st} \cdot t e^{at}\, dt = \lim_{R\to\infty} \int_0^R t e^{-(s-a)t}\, dt$$

Using integration by parts, we have

$$F(s) = \lim_{R\to\infty} \left\{ t \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^R - \int_0^R 1 \cdot \frac{e^{-(s-a)t}}{-(s-a)}\, dt \right\}$$

$$= \lim_{R\to\infty} \left\{ \frac{Re^{-(s-a)R} - 0}{-(s-a)} + \frac{1}{s-a} \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^R \right\}$$

$$= \lim_{R\to\infty} \left\{ \frac{Re^{-(s-a)R}}{-(s-a)} - \frac{1}{(s-a)^2} \left(e^{-(s-a)R} - 1 \right) \right\}$$

If $s > a$, $\lim_{R\to\infty} e^{-(s-a)R} \to 0$

$$\Rightarrow F(s) = \frac{1}{(s-a)^2}, \quad s > a$$

Section 6.2 #1

① $F(s) = \dfrac{3}{s^2+4}$

Rewriting the function as follows:

$$F(s) = \frac{3}{2} \cdot \frac{2}{s^2+4}$$

Using #5 in the table, $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$

If $a = 2$, we have $f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{3}{2}\sin(2t)$

Section 6.2 # 6

$$F(s) = \frac{2s-3}{s^2-4}$$

Rewriting $F(s)$, we have $\quad F(s) = \dfrac{2s}{s^2-4} - \dfrac{3}{s^2-4}$

$$= 2\,\frac{s}{s^2-4} - \frac{3}{2}\cdot\frac{2}{s^2-4}$$

from the table, using # 7 and # 8, we have

$$f(t) = 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} - \frac{3}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\}$$

$$= 2\cosh(2t) - \frac{3}{2}\sinh(2t), \qquad s > 2$$

Alternative approach:

$$F(s) = \frac{2s-3}{(s+2)(s-2)} = \frac{a}{s-2} + \frac{b}{s+2}$$

Using partial fractions, we get $\quad a = \frac{1}{4}, \; b = \frac{7}{4}$

$$\therefore\ F(s) = \frac{1}{4}\left[\frac{1}{s-2} + \frac{7}{s+2}\right]$$

$$\therefore\ f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{4}e^{2t} + \frac{7}{4}e^{-2t}$$

The formula of result above is same as obtained before.

To verify this, recall that

$$\cosh(2t) = \frac{e^{2t}+e^{-2t}}{2}, \text{ and } \sinh(2t) = \frac{e^{2t}-e^{-2t}}{2}.$$

Since $f(t) = 2\cosh(2t) - \frac{3}{2}\sinh(2t)$

$$= 2\,\frac{e^{2t}+e^{-2t}}{2} - \frac{3}{2}\cdot\frac{e^{2t}-e^{-2t}}{2}$$

$$= e^{2t}\left(1-\frac{3}{4}\right) + e^{-2t}\left(1+\frac{3}{4}\right)$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{-2t}$$

: Exactly same as that obtained in the alternative approach.

Section 6.2 # 7

$$F(s) = \frac{2s+1}{s^2-2s+2} = \frac{2s+1}{(s-1)^2+1}$$

$$= \frac{2(s-1+1)+1}{(s-1)^2+1} = \frac{2(s-1)+3}{(s-1)^2+1}$$

$$= 2\,\frac{s-1}{(s-1)^2+1} + 3\cdot\frac{1}{(s-1)^2+1}$$

Using formulae #9 & #10, we have with $a=1$, $b=1$

$$f(t) = 2e^t\cos t + 3e^t\sin t$$

Section 6.2 # 10 : $F(s) = \dfrac{2s-3}{s^2+2s+10}$:

Now $$\frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s^2+2s+1)+9} = \frac{2s-3}{(s+1)^2+3^2} = \frac{2(s+1)-2-3}{(s+1)^2+3^2}$$

$$= \frac{2(s+1)}{(s+1)^2+3^2} - \frac{5}{(s+1)^2+3^2}$$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2e^{-t}\cos(3t) - \frac{5}{3} \cdot e^{-t}\sin(3t)$

Section 6.2 #16   Solve the IVP:

$y'' + 2y' + 5y = 0 \quad ; \quad y(0) = 2, \quad y'(0) = -1$

Take the Laplace transform on both sides:

$[s^2 Y(s) - s\,y(0) - y'(0)] + 2[s\,Y(s) - y(0)] + 5\,Y(s) = 0$

where $Y(s) = \mathcal{L}\{y(t)\}$

Using the initial conditions, we have

$(s^2 + 2s + 5)\,Y(s) = 2s + 3$

$$\therefore \quad Y(s) = \frac{2s+3}{s^2+2s+5} = \frac{2s+3}{s^2+2s+1+4} = \frac{2(s+1)+1}{(s+1)^2+2^2}$$

$$= \frac{2(s+1)}{(s+1)^2+2^2} + \frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

$$\therefore \quad y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)$$

Section 6.2 #19

$y^{(4)} - 4y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$

Taking the Laplace transform, we get

$[s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0)] - 4\,Y(s) = 0$

$$\Rightarrow \quad s^4 Y(s) - s^3 + 2s \quad - 4Y(s) = 0$$

$$\Rightarrow \quad (s^4 - 4) Y(s) = s^3 - 2s$$

$$\Rightarrow \quad Y(s) = \frac{s(s^2-2)}{s^4-4} = \frac{s(s^2-2)}{(s^2-2)(s^2+2)}$$

$$= \frac{s}{s^2+2} = \frac{s}{s^2+(\sqrt{2})^2}$$

$$\therefore \quad y(t) = \cos(\sqrt{2}t)$$

<u>Section 6.2 #20</u>

$$y'' + \omega^2 y = \cos(2t), \quad \omega^2 \neq 4, \quad y(0) = 1; \quad y'(0) = 0$$

Taking the Laplace transform, we have

$$\left[s^2 Y(s) - s\, y(0) - y'(0) \right] + \omega^2 Y(s) = \mathcal{L}\{\cos 2t\}$$

$$\Rightarrow \quad (s^2+\omega^2) Y(s) - s = \frac{s}{s^2+4}$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2+\omega^2}\left[s + \frac{s}{s^2+4} \right]$$

$$= \frac{s}{s^2+\omega^2} + \frac{s}{(s^2+\omega^2)(s^2+4)}$$

Now $$\frac{s}{(s^2+\omega^2)(s^2+4)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+\omega^2}$$

$$\Rightarrow \quad s = (As+B)(s^2+\omega^2) + (Cs+D)(s^2+4)$$

$O(\underline{s^3})$: $\quad A + C = 0$

$O(s^2)$: $\quad B + D = 0$

$O(s)$: $\quad A\omega^2 + 4C = 1$

$O(1)$: $\quad B\omega^2 + 4D = 0$

Thus $\quad B = D = 0, \quad C = -A, \quad A = \frac{1}{\omega^2-4}, \quad C = \frac{1}{4-\omega^2}$

$$\therefore \cancel{Y(s) = \frac{s}{s^2+\omega^2} + \frac{1}{\omega^2-4}\cdot\frac{s}{s^2-4\omega^2} - \frac{1}{\omega^2-4}\,\frac{1}{s^2+4}}$$

$$Y(s) = \frac{s}{s^2+\omega^2} + \frac{1}{\omega^2-4}\,\frac{s}{s^2+4} - \frac{1}{\omega^2-4}\,\frac{s}{s^2+\omega^2}$$

$$\therefore y(t) = \cos(\omega t) + \frac{1}{\omega^2-4}\cos(2t) - \frac{1}{\omega^2-4}\cos(\omega t)$$

$$= \left(\frac{\omega^2-5}{\omega^2-4}\right)\cos(\omega t) + \frac{1}{\omega^2-4}\cos(2t)$$

Section 6.2 # 23

$$y'' + 2y' + y = 4e^{-t}; \qquad y(0) = 2, \quad y'(0) = -1$$

Take the Laplace transform:

$$[s^2 Y(s) - s y(0) - y'(0)] + 2[s Y(s) - y(0)] + Y(s) = \frac{4}{s+1}$$

$$\Rightarrow (s^2 + 2s + 1) Y(s) - 2s + 1 - 4 = \frac{4}{s+1}$$

$$\Rightarrow (s^2 + 2s + 1) Y(s) = 2s + 3 + \frac{4}{s+1}$$

$$\Rightarrow \quad Y(s) = \frac{2s+3}{(s+1)^2} + \frac{4}{(s+1)^3}$$

$$= \frac{2(s+1)+1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

$$= \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

$$\boxed{\begin{array}{l} \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \\ n=1,\ a=-1 \\ \mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2} \\ n=2,\ a=-1 \\ \mathcal{L}\{t^2e^{-t}\} = \frac{2}{(s+1)^3} \end{array}}$$

Using formulae #2, #11, we have

$$y(t) = 2e^{-t} + te^{-t} + \frac{4}{2}t^2e^{-t}$$

$$\therefore \quad y(t) = 2e^{-t} + te^{-t} + 2t^2e^{-t}$$