MATH 215/255 HW # 7

Section 3.7 # 2 $\quad u = -\cos t + \sqrt{3}\sin t \quad \text{---} \quad (*)$

$$u = R\cos(\omega_0 t - \delta)$$

Expand $\cos(\omega_0 t - \delta)$: $u(t) = R\cos\delta\cos(\omega_0 t) + R\sin\delta\sin(\omega_0 t) \quad \text{---} \quad (**)$

Comparing $(*)$ and $(**)$, we have

$$\left.\begin{aligned} R\cos\delta &= -1 \\ R\sin\delta &= \sqrt{3} \end{aligned}\right\} \quad R^2(\cos^2\delta + \sin^2\delta) = 1+3 = 4$$

$$\Rightarrow R = 2$$

and $\omega_0 = 1$

and $\delta = \tan^{-1}(-\sqrt{3})$

$$= \frac{-2\pi}{3}$$

$$\therefore u = 2\cos\left(t - \frac{2\pi}{3}\right)$$

Section 3.7 # 3 $\quad u = 4\cos 3t - 2\sin 3t$

Now $\left.\begin{aligned} R\cos\delta &= 4 \\ R\sin\delta &= -2 \end{aligned}\right\} \quad R^2 = 4^2 + (-2)^2 = 16 + 4 = 20 \Rightarrow R = \sqrt{20}$

and $\omega_0 = 3$ $\quad$ And $\tan\delta = \left(\frac{-1}{2}\right) \Rightarrow \delta = \tan^{-1}\left(\frac{1}{2}\right)$

$$\approx -0.46$$

$$\therefore u(t) = \sqrt{20}\cos(3t + 0.46)$$

Section 3.7 # 5

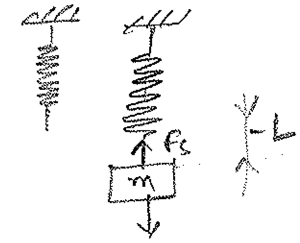
We use the following units for length, time and mass:

length $\rightarrow$ in feet

time $\rightarrow$ in seconds

mass $\longrightarrow$ lb.

Find spring constant, K :- Given that the weight from a 2 lb mass stretches the spring by 6 in. $= \frac{1}{2}$ ft.



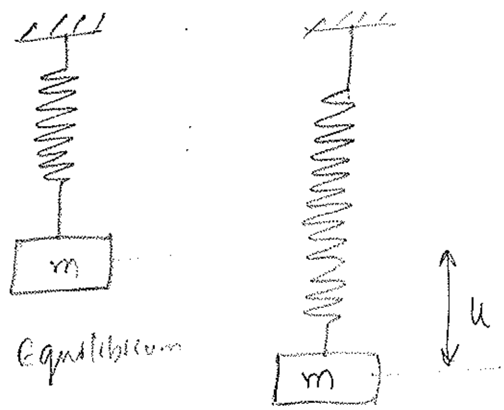
Using gravity $g = 32 \frac{ft}{sec^2}$, we have

$F_s = -KL$ where $L = \frac{-1}{2}$ ft

$\Rightarrow mg = -KL \Rightarrow 2\text{ lb} \times 32 \frac{ft}{sec^2} = -K \times \frac{-1}{2}$ ft

$\Rightarrow K = 128 \frac{lb}{sec^2}$

Additional extension from equilibrium :-



Mass pulled by 3 inches

$\Rightarrow u(0) = \frac{1}{4}$ ft.

Since the mass is not pushed, the initial velocity at release $= 0$

$\Rightarrow u'(0) = 0$

Now $F = ma$

$\Rightarrow mu'' = mg - K(L+u)$

$= -Ku$

$\Rightarrow 2 \cdot u'' + 128u = 0 \Rightarrow \boxed{u'' + 64u = 0}$

with $u(0) = \frac{1}{4}$

$u'(0) = 0$

$u'' + 64u = 0 \Rightarrow r^2 + 64 = 0$

$\Rightarrow r = \pm 8i$

$\therefore \quad u(t) = c_1 \cos(8t) + c_2 \sin(8t)$

Now $\quad u(0) = \frac{1}{4} \Rightarrow \frac{1}{4} = c_1$

$u'(0) = 0 \Rightarrow 0 = -8c_1 \sin(0) + 8c_2 \cos(0)$

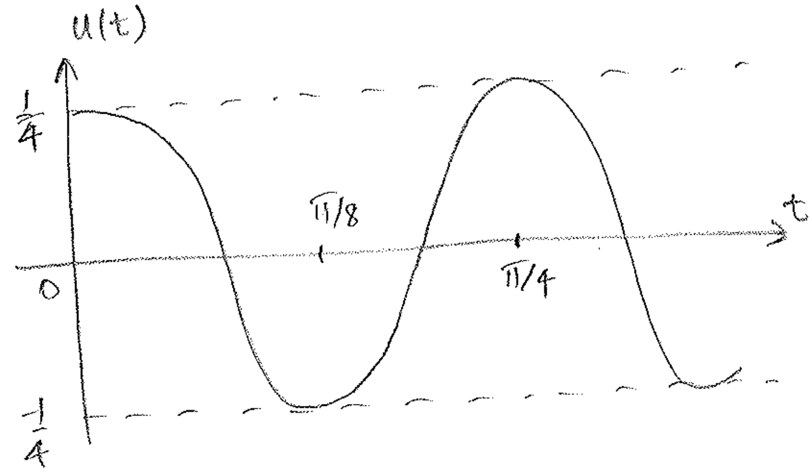
$\Rightarrow c_2 = 0$

$\therefore \quad \boxed{u(t) = \frac{1}{4}\cos(8t)}$ ft $= R\cos(\omega_0 t - \delta)$

frequency, $\omega_0 = 8$ rad/sec

Phase, $\delta = 0$

Amplitude $R = \frac{1}{4}$



Time period, $T = \frac{2\pi}{\omega_0}$

$= \frac{\pi}{4}$ sec.

Section 3.7 #11

First to find spring constant and damping constant :-

Find K: Force of 3N needed to stretch spring by 10 cms. $= 0.1$ m

$\Rightarrow 3 = K \times 0.1 \Rightarrow K = 30 \frac{N}{m} = 30 \frac{kg}{sec^2}$

Find γ: Damper exerts a force of 3N when velocity is 5 m/sec.

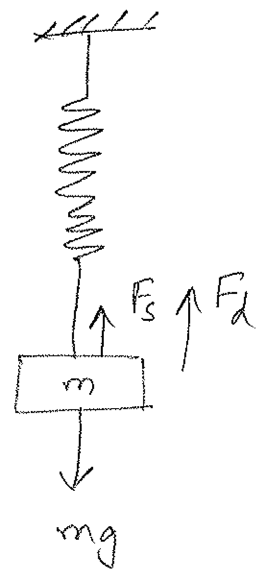
$$\Rightarrow \quad 3 = \gamma \underbrace{u'(t)}_{\text{velocity}}$$

$$\Rightarrow \quad \gamma = \frac{3}{5} \ \frac{N\ sec}{m}$$

$$= \frac{3}{5} \ \frac{Kg}{sec}$$

Sign convention:- Downward $\rightarrow$ positive

Upward $\rightarrow$ Negative



From equilibrium, initial displacement is 5 cm downward

$$\Rightarrow \quad u(0) = +5\ cm = 0.05\ m$$

Initial velocity $= 10$ cm/sec downward

$$\Rightarrow u'(0) = 0.1\ m/sec$$

Spring mass damper equation:- $\qquad m = 2\ kg$

$$mu'' + \gamma u' + Ku = 0$$

$$\Rightarrow \quad 2u'' + \frac{3}{5}u' + 30u = 0$$

$$\Rightarrow \quad 10u'' + 3u' + 150u = 0$$

Characteristic equation: $\quad 10r^2 + 3r + 150 = 0$

$$\Rightarrow r = \frac{-3 \pm \sqrt{9 - 4 \times 10 \times 150}}{20} = \frac{-3 \pm \sqrt{5991}}{20}$$

$\Rightarrow \quad r \approx -0.15 \pm i\,3.87$

$\lambda \pm i\mu$

General solution: $u(t) = e^{-0.15t}\left[c_1 \cos(3.87t) + c_2 \sin(3.87t)\right]$

Now $u(0) = 0.05$

$\Rightarrow \quad 0.05 = c_1$

$u'(0) = 0.1 \quad \Rightarrow \quad 0.1 = -0.15 \times c_1 + 3.87 \times c_2$

$\Rightarrow \quad 3.87\,c_2 = 0.1 + 0.15 \times 0.05$

$= 0.1075$

$\Rightarrow \quad c_2 \approx 0.027$

$\therefore \; u(t) = e^{-0.15t}\left[0.05 \cos(3.87t) + 0.027 \sin(3.87t)\right]$

$= e^{-0.15t} \cdot R\cos(\omega t - \delta)$

Now $\omega = 3.87$ rad/sec,

$$\left.\begin{aligned} R\cos\delta &= 0.05 \\ R\sin\delta &= 0.027 \end{aligned}\right\} \quad \begin{aligned} R^2 &= 0.05^2 + 0.027^2 \Rightarrow R = 0.05682 \\ \delta &= \tan^{-1}\left(\frac{0.027}{0.05}\right) = 0.4951 \end{aligned}$$

$\therefore \; u(t) = 0.05682\, e^{-0.15t} \cos\left[3.87t - 0.4951\right]$ m.

Quasi-frequency, $\omega = 3.87$ rad/sec.

Natural frequency, $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{30}{2}} \approx 3.8729$ rad/sec.

Ratio of quasi-frequency to natural frequency,

$$\frac{\omega}{\omega_0} \approx 0.999$$

Damping slightly reduces the frequency.

Section 3.7 #21

(a) Eq. 26: $u(t) = Re^{\frac{-\gamma t}{2m}} \cos(\mu t - \delta)$

$u(t)$ has a maximum when $\cos(\mu t - \delta)$ has a maximum. $\Rightarrow \mu t - \delta = 2n\pi$ where $n = 0, 1, 2, \ldots$

When $n=0$, $t_0 = \frac{\delta}{\mu}$

When $n=1$; $t_1 = \frac{2\pi}{\mu} + \frac{\delta}{\mu}$.

Time difference between successive maxima $= t_1 - t_0$
$= \frac{2\pi}{\mu}$.

(can also do it for t_n & t_{n+1}).

(b) At maxima, displacement

$$u_n(t) = Re^{-\frac{\gamma}{2m} t_n} \cos(2n\pi)$$

$$= Re^{-\frac{\gamma}{2m} t_n} \quad \text{where} \quad t_n = \frac{2n\pi + \delta}{\mu}$$

Next maxima occurs at $n+1$:

$$u_{n+1}(t) = Re^{-\frac{\gamma}{2m} t_{n+1}} \quad \text{where} \quad t_{n+1} = \frac{2(n+1)\pi + \delta}{\mu}$$

Ratio of successive displacement maxima:

$$\frac{u_n(t)}{u_{n+1}(t)} = \frac{Re^{-\frac{\gamma}{2m}t_n}}{Re^{-\frac{\gamma}{2m}t_{n+1}}} = e^{-\frac{\gamma}{2m}(t_n - t_{n+1})}$$

$$t_n - t_{n+1} = \frac{2n\pi + \cancel{\delta}}{\mu} - \frac{2(n+1)\pi + \cancel{\delta}}{\mu} = -\frac{2\pi}{\mu}$$

$$\therefore \quad \frac{u_n(t)}{u_{n+1}(t)} = e^{\frac{-\gamma}{\cancel{2}m} \times \frac{-\cancel{2}\pi}{\mu}} = e^{\frac{\gamma\pi}{\mu m}}$$

(c) Logarithmic decrement, $\Delta = \ln\left[\frac{u_n}{u_{n+1}}\right] = \frac{\gamma\pi}{\mu m}$

Section 3.8 #5

Find K first: Weight from mass, $m = 4\,lb$, stretches a spring by $L = 1.5$ in $= \frac{1.5}{12}$ ft.

$g = 32\ ft/sec^2$

$$\Rightarrow mg = F_s \quad \Rightarrow \quad mg = KL$$

$$\Rightarrow 4 \times 32 = K \times \frac{1.5}{12}$$

$$\Rightarrow K = 1024\ \frac{lb}{sec^2}$$

Since damping $\gamma = 0$, the equation for the spring-mass system becomes $mu'' + Ku = F(t)$.

The external forcing $F(t) = 2\cos(3t)$, Initial displacement = 2 in. downwards

$\Rightarrow \quad u(0) = +\frac{1}{6}$ ft

Initial velocity $= 0 \quad \Rightarrow \quad u'(0) = 0$

Equation becomes:

$$4u'' + 1024u = 2\cos(3t)$$

$$\Rightarrow \boxed{u'' + 256u = \frac{1}{2}\cos 3t} \quad \text{ft}$$

with $u(0) = \frac{1}{6}$,

$u'(0) = 0$.

Section 3.8 #12

Given that $K = 3$ N/m

$m = 2$ kg

Damping force, F_d = Magnitude of velocity, $u'(t)$

$\Rightarrow \quad F_d = u'(t) \quad \Rightarrow \quad \gamma = 1$

The equation of motion for the spring-mass-damper becomes

$$mu'' + \gamma u' + Ku = F(t).$$

where $F(t) = (3\cos 3t - 2\sin 3t)$ N

$$\Rightarrow \boxed{2u'' + u' + 3u = 3\cos(3t) - 2\sin(3t)}$$

General solution, $u(t) = u_c(t) + U(t)$

Here $U_c(t)$ is the solution of the homogeneous equation and $U(t)$ is the steady state response.

Using method of undetermined coefficients, we expect

$$U(t) = A\cos(3t) + B\sin(3t)$$

$$\Rightarrow U'(t) = -3A\sin(3t) + 3B\cos(3t)$$

$$U''(t) = -9A\cos(3t) - 9B\sin(3t)$$

Substituting $U(t)$ into the equation, we have

$$2[-9A\cos(3t) - 9B\sin(3t)] + [-3A\sin(3t) + 3B\cos(3t)] + 3[A\cos(3t) + B\sin(3t)] = 3\cos(3t) - 2\sin(3t)$$

$$\Rightarrow \quad -18A + 3B + 3A = 3 \Rightarrow -15A + 3B = 3 \Rightarrow -5A + B = 1$$

and $-18B - 3A + 3B = -2 \Rightarrow -3A - 15B = -2 \Rightarrow A + 5B = \frac{2}{3}$

$$\Rightarrow \left.\begin{aligned} A &= \frac{-1}{6} \\ B &= \frac{1}{6} \end{aligned}\right\} \Rightarrow U(t) = \frac{-1}{6}\cos(3t) + \frac{1}{6}\sin(3t).$$

↳ steady state response.

If $U(t) = R\cos(\omega t - \delta)$, we have

$$\left.\begin{aligned} R\cos\delta &= \frac{-1}{6} \\ R\sin\delta &= \frac{1}{6} \end{aligned}\right\} \quad \begin{aligned} R &= \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2} = \frac{\sqrt{2}}{6} \\ \tan\delta &= -1 \Rightarrow \delta = -\pi/4 \text{ or } \frac{3\pi}{4} \end{aligned}$$

$$\therefore U(t) = \frac{\sqrt{2}}{6}\cos\left(3t - \frac{3\pi}{4}\right) \text{ m}.$$

EXTRA PROBLEM - 1

$$y'' + y' + 2y = F_0 \cos(\omega t)$$

Let us consider the solution of the following equation:

$$\tilde{y}'' + \tilde{y}' + 2\tilde{y} = F_0 e^{i\omega t}$$

Since $e^{i\omega t} = \cos \omega t + i \sin \omega t$,

$$\cos(\omega t) = \underbrace{Re}_{\text{real part}}\left[e^{i\omega t}\right]$$

Therefore $Re[\tilde{y}] = y$.

Consider $\tilde{y}'' + \tilde{y}' + 2\tilde{y} = F_0 e^{i\omega t}$.

Particular solution = Steady state response.

Let $\tilde{y} = A_0 e^{i\omega t}$

Substituting such a guess, we have

$$(-\omega^2 + i\omega + 2) A_0 = F_0$$

So,
$$A_0 = \frac{F_0}{(2-\omega^2) + i\omega} = \frac{F_0}{(2-\omega^2) + i\omega} \times \frac{(2-\omega^2) - i\omega}{(2-\omega^2) - i\omega}$$

$$= \frac{F_0\left[(2-\omega^2) - i\omega\right]}{(2-\omega^2)^2 + \omega^2}$$

Thus
$$y_p = Re\left[A_0 e^{i\omega t}\right] = Re\left[\frac{F_0\left[(2-\omega^2) - i\omega\right]}{(2-\omega^2)^2 + \omega^2} \cdot \left[\cos \omega t + i \sin \omega t\right]\right]$$

$$= \frac{F_0}{(2-\omega^2)^2+\omega^2}\left[\underbrace{(2-\omega^2)\cos(\omega t) + \omega\sin(\omega t)}\right]$$

Let us reduce this to the form $A\cos(\omega t - \delta)$

$$= A\cos\delta\cos\omega t + A\sin\delta\sin\omega t$$

So $\left.\begin{aligned} A\cos\delta &= 2-\omega^2 \\ A\sin\delta &= \omega \end{aligned}\right\}$ $A = \sqrt{(2-\omega^2)^2+\omega^2}$, $\tan\delta = \frac{\omega}{2-\omega^2}$

Thus $y_p = R(\omega)\cos(\omega t - \delta)$

where Amplitude $R(\omega) = \frac{F_0}{(2-\omega^2)^2+\omega^2} \times A$

$$= \frac{F_0}{\left[(2-\omega^2)^2+\omega^2\right]^{1/2}}$$

Let us define $g(\omega) = \left[(2-\omega^2)^2+\omega^2\right]$, therefore

$$R(\omega) = F_0\left[g(\omega)\right]^{-1/2}$$

R has a maximum when $\frac{dR}{d\omega} = 0 \Rightarrow -\frac{F_0}{2}\left[g(\omega)\right]^{-3/2} g'(\omega) = 0$

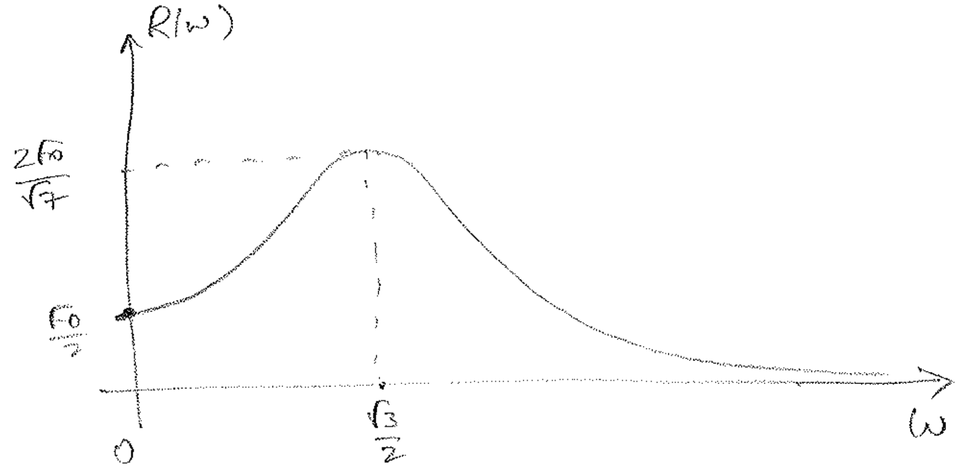
Now $g'(\omega) = 0$ when $2(2-\omega^2)(-2\omega) + 2\omega = 0$

$\Rightarrow$ $2(2-\omega^2) = 1$ or

$\omega = \sqrt{\frac{3}{2}}$

↳ Frequency when R is a maximum.

Now, at $\omega = \sqrt{\frac{3}{2}}$, $R\left(\frac{\sqrt{3}}{2}\right) = \frac{F_0}{\left[g\left(\frac{\sqrt{3}}{2}\right)\right]^{1/2}} = \frac{2F_0}{\sqrt{7}}$



<u>Extra Problem 2:</u>

\# (i) $y'''' - 4y = 0$

Put $y = e^{rt} \longrightarrow r^4 - 4 = 0$

Thus $r^4 = 4 \longrightarrow r^2 = 2$, $r^2 = -2$

$r = \pm\sqrt{2}$, $r = \pm i\sqrt{2}$

Thus we have four solutions:

$y_1 = e^{\sqrt{2}t}$, $y_2 = e^{-\sqrt{2}t}$, $\underbrace{y_3 = e^{\sqrt{2}it}, \quad y_4 = e^{-i\sqrt{2}t}}_{\text{can reduce this to } \sin(\sqrt{2}t) \text{ \& } \cos(\sqrt{2}t)}$

<u>General Solution:</u>

$$y(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + c_3\cos(\sqrt{2}t) + c_4\sin(\sqrt{2}t)$$

(ii) $y'''' - 4y'' + 3y = t^2$

Homogeneous Problem:- Put $y = e^{rt} \longrightarrow r^4 - 4r^2 + 3 = 0$

Let $\lambda = r^2 \Rightarrow \lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda-3)(\lambda-1) = 0$

$\Rightarrow \lambda = 1$ and $\lambda = 3$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$r^2 = 1$ $\qquad\qquad$ $r^2 = 3$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$r = \pm 1$ $\qquad\qquad$ $r = \pm\sqrt{3}$

Thus $y_1 = e^t$, $y_2 = e^{-t}$, $y_3 = e^{\sqrt{3}t}$, $y_4 = e^{-\sqrt{3}t}$

are the homogeneous solutions.

Particular Solution:- Put $y = A_0 + A_1 t + A_2 t^2$

Then $y'' = 2A_2$
$y'''' = 0$

$\Rightarrow \quad 0 - 4(2A_2) + 3[A_0 + A_1 t + A_2 t^2] = t^2$

$\Rightarrow 3A_2 = 1$, $A_1 = 0$ and $-8A_2 + 3A_0 = 0$

$\Rightarrow A_2 = 1/3$, $A_1 = 0$, $A_0 = \frac{8}{9}$

Thus $y_p = \frac{8}{9} + \frac{t^2}{3}$

Thus $y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{\sqrt{3}t} + c_4 e^{-\sqrt{3}t} + \left(\frac{8}{9} + \frac{t^2}{3}\right)$

$\hookrightarrow$ General Solution.