Section 3.4

(25) $t^2y'' + 3ty' + y = 0$; $t > 0$

Given $y_1(t) = \frac{1}{t}$ is a solution. Find $y_2(t)$.

We let $y = v(t)\, y_1(t) = \frac{v}{t}$

So $y' = -\frac{1}{t^2} v + t^{-1} v'$

$y'' = 2t^{-3} v - 2t^{-2} v' + t^{-1} v''$

Substitute to get

$$t^2 \left(t^{-1} v'' - 2t^{-2} v' + 2t^{-3} v \right) + 3t\left(t^{-1} v' - t^{-2} v \right) + t^{-1} v = 0$$

This simplifies to $t v'' + v'(-2+3) + v\left(\frac{2}{t} - \frac{3}{t} + \frac{1}{t}\right) = 0$

$$\Rightarrow \quad v'' + \frac{v'}{t} = 0$$

This is a second order equation in $v(t)$. But since $v(t)$ is itself absent, the above equation can also be interpreted as a first order equation in the variable $w(t) = v'(t)$. Therefore, we managed to reduced the order of the equation. Hence the name Reduction of Order.

The equation for $v'(t)$ becomes

$$w' + \frac{w}{t} = 0 \qquad \text{where} \qquad w = v'(t)$$

$$\Rightarrow \quad tw' + w = 0$$

$$\Rightarrow \quad (tw)' = 0 \quad \Rightarrow \quad tw = c_1$$

Thus $\quad w(t) = \frac{c_1}{t} = v'$

$$\Rightarrow \quad v(t) = c_1 \ln t + c_2$$

$$\therefore \quad y_2(t) = v(t)\, y_1(t) = (c_1 \ln t + c_2) \cdot \frac{1}{t}$$

$$= c_1 \frac{1}{t} \ln t + \frac{c_2}{t}$$

Notice that y_2 contains $\frac{1}{t}$ term which is identical to the first solution, $y_1(t)$. When we write the general solution

$$y(t) = Ay_1 + By_2$$

$$= \frac{A}{t} + B\left(\frac{c_1}{t} \ln t + \frac{c_2}{t}\right),$$

we can regroup the constants as

$$y(t) = \frac{(A + Bc_2)}{t} + Bc_1\left(\frac{1}{t} \ln t\right)$$

$$= A' y_1 + B' y_2$$

Therefore no new information is obtained by carrying forward c_1 & c_2. Therefore we set $c_1 = 1$ & $c_2 = 0$ $\Rightarrow$ $y_2(t) = \frac{1}{t} \ln t$

(28) $(x-1)y'' - xy' + y = 0$, $x>1$; $y_1(x) = e^x$

We put $y(t) = e^x v$

$\Rightarrow y' = e^x v + e^x v'$
$y'' = e^x v + 2e^x v' + e^x v''$

Substitution gives

$(x-1)[e^x v + 2e^x v' + e^x v''] - x[e^x v + e^x v'] + e^x v = 0$

We can cancel e^x throughout. Combining v'', v' & v terms,

we get

$(x-1)v'' + v'[2(x-1) - x] + v[(x-1) - x + 1] = 0$

Thus $(x-1)v'' + v'(x-2) = 0$

$\Rightarrow v'' + v'\left(\frac{x-2}{x-1}\right) = 0$

Thus $v'' + \left(1 - \frac{1}{x-1}\right)v' = 0$

Let $w = v'$ so that $w' + \left(1 - \frac{1}{x-1}\right)w = 0$

Integrating factor $\phi = e^{\int\left(1-\frac{1}{x-1}\right)dx} = e^{x - \ln(x-1)} = e^x \cdot e^{-\ln(x-1)}$

$= \frac{1}{x-1}e^x$

$\Rightarrow \frac{1}{x-1}e^x w' + \left(1 - \frac{1}{x-1}\right)\frac{1}{x-1}e^x = 0$

$\Rightarrow \left(\frac{1}{x-1}e^x w\right)' = 0 \Rightarrow$ Thus $w = c(x-1)e^{-x}$

We set $c = 1$ without loss of generality. Then

$$v'(x) = w(x) = (x-1)\,e^{-x}$$

~~Then v(t) =~~ $\cancel{\int^t t-1}$

Then $v(x) = \displaystyle\int^x (x-1)\,e^{-x}\,dx$

$= -xe^{-x}$; Using integration by parts.

$\therefore\; y_2(x) = e^x\, v(x) = e^x \times -xe^{-x}$

$= -x$

Therefore the second solution $y_2(x) = -x$

SECTION 3.5

(1) $y'' - 2y' - 3y = 3e^{2t}$

Homogeneous problem:- $y'' - 2y' - 3y = 0$

Let $y = e^{\lambda t}$ $\Rightarrow$ $\lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow$ $(\lambda - 3)(\lambda + 1) = 0$ $\Rightarrow$ $\lambda = -1, 3$

Thus $y_1(t) = e^{-t}$; $y_2(t) = e^{3t}$

Particular Solution:- Since $g(t) = e^{2t}$ which is not y_1 or y_2,

we set $y_p = Ae^{2t}$

So $y_p' = 2Ae^{2t}$; $y_p'' = 4Ae^{2t}$

$\Rightarrow$ $4Ae^{2t} - 2 \times 2Ae^{2t} - 3 \times Ae^{2t} = 3e^{2t}$

$\Rightarrow$ $(4A - 4A - 3A)e^{2t} = 3e^{2t}$ $\Rightarrow$ $A = -1$

$\therefore$ $y_p = -e^{2t}$

General Solution: $y(t) = c_1 y_1 + c_2 y_2 + y_p(t)$

$= c_1 e^{-t} + c_2 e^{3t} - e^{2t}$

Section 3.5 # 3

$y'' - 2y' - 3y = -3te^{-t}$

Homogeneous Problem:- $y'' - 2y - 3y = 0$: Same as previous problem.

$y_1(t) = e^{-t}$; $y_2 = e^{3t}$.

<u>Particular Solution</u>:- The nonhomogeneous term $g(t)$ is in the form $g(t) = -3t\,e^{\alpha t}$ where $\alpha = -1$.

Since $\alpha = -1$ is also one of the roots $\lambda = -1, 3$,

we set $y_p = t\,(At+B)\,e^{-t}$

appears because t is contained in $g(t)$

If α was not a root of the characteristic equation, we would have replaced this with just a single constant like Ae^{-t}. But in this case, $\alpha = -1$, which is a solution of the characteristic equation.

$$\Rightarrow \quad y_p = At^2 + Bt\,e^{-t}$$

$$y_p' = (2At+B)e^{-t} - (At^2+Bt)e^{-t}$$

$$y_p'' = 2Ae^{-t} - 2(2At+B)e^{-t} + (At^2+Bt)e^{-t}$$

So $y'' - 2y' - 3y = -3te^{-t}$ gives

$$2A - 4At - 2B + At^2 + Bt - 4At - 2B + 2At^2 + 2Bt - 3At^2 - 3Bt = -3t$$

So $t(-4A - 4A + B + 2B - 3B) + (-2B - 2B + 2A) = -3t$

$\Rightarrow -8A = -3 \Rightarrow A = \frac{3}{8}$ and $4B = 2A \Rightarrow B = \frac{3}{16}$

So $y = c_1 e^{-t} + c_2 e^{3t} + \left(\frac{3}{8}t^2 + \frac{3t}{16}\right)e^{-t}$: General Solution.

Section 3.5 #6 $\qquad y'' + 2y' + y = 2e^{-t} \longrightarrow g(t) = 2e^{\alpha t}, \ \alpha = -1$

Homogeneous Prob: $\quad y'' + 2y' + y = 0$

$y = e^{\lambda t} \Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda = -1$: Repeated root.

Thus $\quad y_1 = e^{-t} \ ; \quad y_2 = te^{-t}$

Particular Solution: Since $\alpha = -1$ is a repeated root of the characteristic polynomial, we should use

$$y_p = At^2e^{-t}$$

One t comes from the fact that we have a repeated root

Second t comes because te^{-t} also happens to be a solution of the homogeneous part.

TRY THIS: If we use only $y_p = Ate^{-t}$, LHS becomes zero. Hence we use $y_p = At^2e^{-t}$

$y_p' = 2Ate^{-t} - At^2e^{-t}$

$y_p'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$

Thus $y''+2y'+y = 2e^{-t}$ becomes

$$2A - 4At + At^2 + 2(2At - At^2) + At^2 = 2$$

Thus $2A = 2 \Rightarrow A = 1$

So $y_p = t^2 e^{-t}$

General Solution: $y(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}$

Section 3.5 # 10

$$u'' + \omega_0^2 u = \cos(\omega_0 t)$$

Homogeneous Problem:- $u'' + \omega_0^2 u = 0$

Let $u = e^{\lambda t} \Rightarrow \lambda^2 + \omega_0^2 = 0 \Rightarrow \lambda = \pm i\omega_0$

Thus $y_1 = \cos(\omega_0 t)$ & $y_2 = \sin(\omega_0 t)$ are homogeneous solution.

Particular Solution:- Since $\cos(\omega_0 t)$ is a solution, we cannot try $y_p = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ X

Instead, we let $u_p = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$

$$\Rightarrow u_p' = -At\omega_0\sin(\omega_0 t) + A\cos(\omega_0 t) + Bt\omega_0\cos(\omega_0 t) + B\cancel{t}\,\sin(\omega_0 t)$$

$$u_p'' = -At\omega_0^2\cos(\omega_0 t) - A\omega_0\sin(\omega_0 t) - A\omega_0\sin(\omega_0 t) - Bt\omega_0^2\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t) + B\omega_0\cos(\omega_0 t)$$

Substituting, we get

$$-At\omega_0^2 - Bt\omega_0^2$$

$$\left[\left(-At\,\omega_0^2 + 2B\,\omega_0\right)\cos(\omega_0 t) + \left(-Bt\,\omega_0^2 - 2A\,\omega_0\right)\sin(\omega_0 t)\right] + \left[At\cos(\omega_0 t) + Bt\sin(\omega_0 t)\right] = \cos(\omega_0 t)$$

$$\Rightarrow \left[-At\,\omega_0^2 + 2B\,\omega_0 + At\right]\cos(\omega_0 t) + \left[-Bt\,\omega_0^2 - 2A\omega_0 + Bt\right]\sin(\omega_0 t) = \cos(\omega_0 t)$$

Comparing $t\cos(\omega_0 t)$ terms: $-A\omega_0^2 + A = 0 \Rightarrow A(1-\omega_0^2) = 0 \Rightarrow A = 0$

Comparing $\cos(\omega_0 t)$ terms: $2B\omega_0 = 1 \Rightarrow B = \dfrac{1}{2\omega_0}$

$\therefore \quad u_p = \dfrac{t}{2\omega_0}\sin(\omega_0 t)$

$\therefore$ General solution is $u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \dfrac{t}{2\omega_0}\sin(\omega_0 t)$

#10 (Alternative approach)

$$u'' + \omega_0^2 u = \cos(\omega_0 t) = Re\left(e^{i\omega_0 t}\right)$$

where $e^{i\omega_0 t} = \cos(\omega_0 t) + i\sin(\omega_0 t)$ & Re denotes the real part.

Homogeneous problem: $u_1 = \cos(\omega_0 t)$; $u_2 = \sin(\omega_0 t)$

Particular Solution:-

Since $i\omega_0$ is a solution of the characteristic equation;

Then $\tilde{u}'' + \omega_0^2 \tilde{u} = e^{i\omega_0 t} \longrightarrow \tilde{u}_p = Ate^{i\omega_0 t}$

we solve for $\tilde{u}$ and take the real part of the solution to get $u_p(t)$

ie; $u_p(t) = Re[\tilde{u}(t)]$

$\tilde{u}_p = Ate^{i\omega_0 t}$

$\Rightarrow \tilde{u}_p' = Ae^{i\omega_0 t} + Ai\omega_0 te^{i\omega_0 t}$

$\tilde{u}_p'' = -A\omega_0^2 te^{i\omega_0 t} + 2Ai\omega_0 e^{i\omega_0 t}$

Substituting, we get $-A\omega_0^2 t + 2Ai\omega_0 + A\omega_0^2 t = 1$

$\Rightarrow 2Ai\omega_0 = 1 \Rightarrow A = \frac{1}{2i\omega_0} = \frac{i}{2i^2\omega_0} = \frac{-i}{2\omega_0}$

$\therefore \tilde{u}_p = \frac{-i}{2\omega_0} te^{i\omega_0 t} = \frac{i}{2\omega_0} t\,[\cos(\omega_0 t) + i\sin(\omega_0 t)]$

$\therefore u_p = Re[\tilde{u}_p] = \frac{1}{2\omega_0} t\sin(\omega_0 t)$

$\therefore$ General solution is $u = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{t}{2\omega_0}\sin(\omega_0 t)$

Section 3.5 # 13

$$y'' + y' - 2y = 2t \quad ; \qquad \begin{matrix} y(0) = 0 \\ y'(0) = 1 \end{matrix}$$

Homogeneous Problem:- $\quad y = e^{\lambda t} \Rightarrow \lambda^2 + \lambda - 2 = 0$

$$\Rightarrow (\lambda+2)(\lambda-1) = 0 \Rightarrow \lambda = 1, -2$$

Thus $\quad y_1(t) = e^t \quad ; \quad y_2(t) = e^{-2t}$

Particular Solution:- $\quad$ Let $y_p = At + B$

$$\Rightarrow y_p' = A$$

$$y_p'' = 0$$

$$\Rightarrow \quad 0 + A - 2(At+B) = 2t$$

$$\Rightarrow \quad A - 2B = 0 \quad \text{and} \quad -2A = 2$$

$$\Rightarrow A = -1 \quad ; \quad B = \frac{-1}{2}$$

$$\therefore \quad y_p = -t - \frac{1}{2}$$

$\therefore$ General Solution : $\quad y(t) = c_1 e^t + c_2 e^{-2t} - t - \frac{1}{2}$

Now $\quad y(0) = 0 \longrightarrow c_1 + c_2 - \frac{1}{2} = 0$

$\quad y'(0) = 1 \longrightarrow c_1 - 2c_2 - 1 = 1$

$\Big\}\quad c_1 = 1 \; ; \; c_2 = \frac{-1}{2}$

Thus $\quad y(t) = e^t - \frac{1}{2} e^{-2t} - t - \frac{1}{2}$

Section 3.5 # 30

$$y'' + 2y' + 5y = \begin{cases} 1 & 0 \le t \le \pi/2 \\ 0 & t > \pi/2 \end{cases} \qquad y(0) = y'(0) = 0$$

Homogeneous Problem:- $y'' + 2y' + 5y = 0$

Let $y = e^{\lambda t}$ $\rightarrow$ $\lambda^2 + 2\lambda + 5 = 0 \rightarrow (\lambda^2 + 2\lambda + 1) + 4 = 0$

$\Rightarrow (\lambda+1)^2 = -4 \Rightarrow \lambda = -1 \pm 2i$

Thus $y_1 = e^{-t}\cos(2t)$; $y_2 = e^{-t}\sin(2t)$

Particular Solution:- $y'' + 2y' + 5y = 1$ for $0 \le t \le \pi/2$

Let $y_p = A \Rightarrow 5A = 1 \Rightarrow A = 1/5$

For $t > \pi/2$, the equation is homogeneous.

So
$$y(t) = \begin{cases} c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \frac{1}{5} & 0 \le t \le \pi/2 \\ d_1 e^{-t}\cos(2t) + d_2 e^{-t}\sin(2t) & t \geq \pi/2 \end{cases}$$
(*)

We have four unknowns : c_1, c_2, d_1, d_2.

We set $y(0) = y'(0) = 0$ and y & y' continuous at $t = \pi/2$.

This gives us 4 equations for the 4 unknowns.

$$y'(t) = \begin{cases} -2c_1 e^{-t}\sin(2t) - c_1 e^{-t}\cos(2t) - c_2 e^{-t}\sin(2t) + 2c_2 e^{-t}\cos(2t) & 0 \le t \le \pi/2 \\ -2d_1 e^{-t}\sin(2t) - d_1 e^{-t}\cos(2t) - d_2 e^{-t}\sin(2t) + 2d_2 e^{-t}\cos(2t) & t \geq \pi/2 \end{cases}$$

Now $y(0) = 0 \longrightarrow c_1 + \frac{1}{5} = 0 \Rightarrow c_1 = -1/5$

$y'(0) = 0 \longrightarrow -c_1 + 2c_2 = 0 \Rightarrow c_2 = \frac{c_1}{2} = \frac{-1}{10}$

y continuous at $t = \pi/2$: $-c_1 e^{-\pi/2} + \frac{1}{5} = -d_1 e^{-\pi/2}$

y' continuous at $t = \pi/2$: $c_1 e^{-\pi/2} - 2c_2 e^{-\pi/2} = d_1 e^{-\pi/2} - 2d_2 e^{-\pi/2}$

Thus $d_1 = c_1 - \frac{1}{5} e^{\pi/2} = \frac{-1}{5}(1 + e^{\pi/2})$

And $d_2 = \frac{d_1}{2} + c_2 - \frac{c_1}{2} = \frac{d_1}{2} = \frac{-1}{10}(1 + e^{\pi/2})$

With c_1, c_2, d_1 & d_2 known, we obtain $y(t)$ from equation $(*)$.

Section 3.6 # 1

$y'' - 5y' + 6y = 2e^t$

Homogeneous Problem :- $y'' - 5y' + 6y = 0$

$y = e^{\lambda t} \Rightarrow \lambda^2 - 5y + 6 = 0 \longrightarrow (\lambda - 3)(\lambda - 2) = 0$

$\Rightarrow \lambda = 2, 3$

$\therefore y_1 = e^{2t}$; $y_2 = e^{3t}$

$y_{Homogeneous} = c_1 e^{2t} + c_2 e^{3t}$

Particular Solution:-

Let $y = u_1(t)\,e^{2t} + u_2(t)\,e^{3t}$

$$y' = 2e^{2t}u_1 + \underbrace{u_1' e^{2t} + u_2' e^{3t}}_{\text{set } = 0} + 3u_2 e^{3t}$$

We set $\boxed{u_1' e^{2t} + u_2' e^{3t} = 0}$ — (1)

$$\therefore \; y' = 2e^{2t}u_1 + 3u_2 e^{3t}$$

$$\Rightarrow y'' = 4e^{2t}u_1 + 2e^{2t}u_1' + 3u_2' e^{3t} + 9u_2 e^{9t}$$

Substituting, we get

$$(4e^{2t}u_1 + 2e^{2t}u_1' + 3u_2' e^{3t} + 9u_2 e^{3t}) - 5(2e^{2t}u_1 + 3u_2 e^{3t}) + 6(u_1 e^{2t} + u_2 e^{3t}) = 2e^t$$

$$\Rightarrow u_1\left[\cancelto{0}{4e^{2t} - 10e^{2t} + 6e^{2t}}\right] + u_2\left[\cancelto{0}{9e^{3t} - 15e^{3t} + 6e^{3t}}\right]$$

$$+ \; 2e^{2t}u_1' + 3e^{3t}u_2' = 2e^t$$

$$\Rightarrow \boxed{2e^{2t}u_1' + 3e^{3t}u_2' = 2e^t} \text{ — (2)}$$

$$\cancel{\Rightarrow \boxed{2e^{t}u_1' + 3e^{2t}u_2' = 2} \text{ — (2)}}$$

~~From equations (1) & (2), we solve for~~ $\cancel{u_1'}$ ~~&~~ $\cancel{u_2'}$.

From equations (1) & (2), we solve for u_1' & u_2': (8)

$$u_1' e^{2t} + u_2' e^{3t} = 0$$

$$2e^{2t} u_1' + 3e^{3t} u_2' = 2e^{t}$$

$$\Rightarrow \begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{t} \end{bmatrix}$$

$$\Rightarrow \quad u_1' = -2e^{-t} \quad \Rightarrow \quad u_1 = 2e^{-t} + c_1$$

$$u_2' = 2e^{-2t} \quad \Rightarrow \quad u_2 = -e^{-2t} + c_2$$

$$\therefore \; y(t) = (2e^{-t} + c_1) e^{2t} + (-e^{-2t} + c_2) e^{3t}$$

$$= c_1 e^{2t} + c_2 e^{3t} + 2e^{t} - e^{t}$$

$$y(t) = \underbrace{c_1 e^{2t} + c_2 e^{3t}}_{\text{Homogeneous solution}} + \underbrace{e^{t}}_{\text{Particular solution}}$$

Verification from method of undetermined coefficients:-

Let $y_p = Ae^{t} \Rightarrow y_p' = Ae^{t}$; $y_p'' = Ae^{t}$

$$\Rightarrow (A - 5A + 6A) e^{t} = 2e^{t} \Rightarrow A = 1$$

$$\therefore \; y_p = e^{t}$$

Section 3.6 # 13

$t^2 y'' - 2y = 3t^2 - 1 \;; \quad t > 0 \;; \quad y_1 = t^2 \;; \quad y_2 = \frac{1}{t}$

Verify y_1 & y_2 are solutions of $t^2 y'' - 2y = 0$:-

$y_1 = t^2 \;; \quad y_1' = 2t \;; \quad y_1'' = 2$

$\Rightarrow \quad t^2 \times 2 - 2t^2 = 0 \quad \Rightarrow y_1$ is a homogeneous solution.

$y_2 = \frac{1}{t} \;; \quad y_2' = \frac{-1}{t^2} \;; \quad y_2'' = \frac{2}{t^3}$

$\Rightarrow \quad t^2 \times \frac{2}{t^3} - 2 \cdot \frac{1}{t} = \frac{2}{t} - \frac{2}{t} = 0 \quad \Rightarrow y_2$ is also a homogeneous solution.

Homogeneous solution in general form:

$$y_H(t) = c_1 t^2 + \frac{c_2}{t}$$

Particular Solution :- Let $y = u_1(t)\, t^2 + u_2(t) \cdot \frac{1}{t}$

$$\therefore y' = 2t u_1 + \underbrace{u_1' t^2 + u_2' \frac{1}{t}}_{\text{set } = 0} - \frac{u_2}{t^2}$$

We set $\boxed{u_1' t^2 + \frac{u_2'}{t} = 0}$ — (1)

$$\therefore y' = 2t u_1 - \frac{u_2}{t^2}$$

$$\therefore y'' = 2t u_1' + 2u_1 - \frac{u_2'}{t^2} + \frac{2u_2}{t^3}$$

Substituting, we get

$$t^2\left[2tu_1' + 2u_1 - \frac{u_2'}{t^2} + \frac{2u_2}{t^3}\right] - 2\left[u_1t^2 + \frac{u_2}{t}\right] = 3t^2-1$$

$$\Rightarrow u_1\left[2t^2 - 2t^2\right] + u_2\left[\frac{2}{t} - \frac{2}{t}\right] + 2t^3u_1' - u_2' = 3t^2-1$$

$$\Rightarrow \boxed{2t^3u_1' - u_2' = 3t^2-1} \quad \text{—} \quad (2)$$

Solving for u_1' & u_2' from equations (1) & (2), we get

$$u_1' = \frac{1}{t} - \frac{1}{3t^3} \Rightarrow u_1 = \ln t + \frac{1}{6t^2} + c_1$$

$$u_2' = \frac{1}{3} - t^2 \Rightarrow u_2 = \frac{1}{3}(t-t^3) + c_2$$

$$\therefore y(t) = \left(\ln t + \frac{1}{6t^2} + c_1\right)t^2 + \left[\frac{1}{3}(t-t^3) + c_2\right]\frac{1}{t}$$

$$= c_1t^2 + \frac{c_2}{t} + \frac{1}{6} + \frac{1}{3}(1-t^2) + t^2\ln t$$

$$= \underbrace{\left(c_1 - \frac{1}{3}\right)}_{\text{call this } c_3} t^2 + \frac{c_2}{t} + \frac{1}{2} + t^2\ln t$$

$$= \underbrace{c_3t^2 + \frac{c_4}{t}}_{\text{homogeneous solution}} + \underbrace{\left(\frac{1}{2} + t^2\ln t\right)}_{\text{particular solution}} \qquad \text{where } c_4 = c_2$$

Section 3.6 # 16

$$(1-t)\,y'' + ty' - y = 2(t-1)^2 e^{-t}; \quad 0<t<1;$$

$$y_1 = e^t$$
$$y_2 = t$$

Homogeneous Solution: $\quad y_H = c_1 e^t + c_2 t$

Particular Solution: Let $y = u_1 e^t + u_2 t$

$$\Rightarrow y' = u_1 e^t + \underbrace{u_1' e^t + u_2' t}_{\text{set } = 0} + u_2$$

We set $\boxed{u_1' e^t + u_2' t = 0}$ — ①

$$\therefore y' = u_1 e^t + u_2$$

$$y'' = u_1 e^t + u_1' e^t + u_2'$$

Substituting, we get $\left(u_1 e^t + u_1' e^t + u_2'\right)(1-t) + t\left(u_1 e^t + u_2\right) - \left(u_1 e^t + u_2 t\right) = 2(t-1)^2 e^{-t}$

$$\Rightarrow u_1\left[e^t(1-t) + te^t - e^t\right] + u_2\left[t - t\right] + (1-t)e^t u_1' + (1-t)u_2' = 2(t-1)^2 e^{-t}$$

$$\Rightarrow (1-t)\left[e^t u_1' + u_2'\right] = 2(t-1)^2 e^{-t}$$

$$\Rightarrow \boxed{e^t u_1' + u_2' = 2(1-t)e^{-t}} \quad — ②$$

Solving for u_1' & u_2' from equations (1) and (2),

we get

$$u_1' = -2te^{-2t} \Rightarrow u_1 = \frac{1}{2}(1+2t)e^{-2t} + c_1$$

$$u_2' = 2e^{-t} \Rightarrow u_2 = -2e^{-t} + c_2$$

$$\therefore \quad y(t) = \left[\frac{1}{2}(1+2t)e^{-2t} + c_1\right]e^{t} + \left[-2e^{-t} + c_2\right] t$$

$$= \frac{1}{2}(1+2t)e^{-t} + c_1e^{t} - 2e^{-t}\cdot t + c_2 t$$

$$= c_1e^{t} + c_2 t + e^{-t}\left[\frac{1+2t}{2} - 2t\right]$$

$$= \underbrace{c_1e^{t} + c_2 t}_{\text{homogeneous solution}} + \underbrace{e^{-t}\left[\frac{1-2t}{2}\right]}_{\text{particular solution.}}$$