MATH 215/255 HW #5

SECTION 3.3

(9) $y'' + 2y' - 8y = 0$

Let $y = e^{rt}$ $\longrightarrow$ $r^2 + 2r - 8 = 0$

$$\Rightarrow r = \frac{-2 \pm \sqrt{4+32}}{2}$$

$$= -1 \pm 3$$

$$\Rightarrow r_1 = -4, \quad r_2 = 2$$

Thus $y = c_1 e^{-4t} + c_2 e^{2t}$

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(17) $y'' + 4y = 0$; $y(0) = 0$, $y'(0) = 1$

Put $y = e^{rt}$ $\Rightarrow$ $r^2 + 4 = 0$ $\Rightarrow$ $r = \pm 2i$

Thus $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$ are solutions.

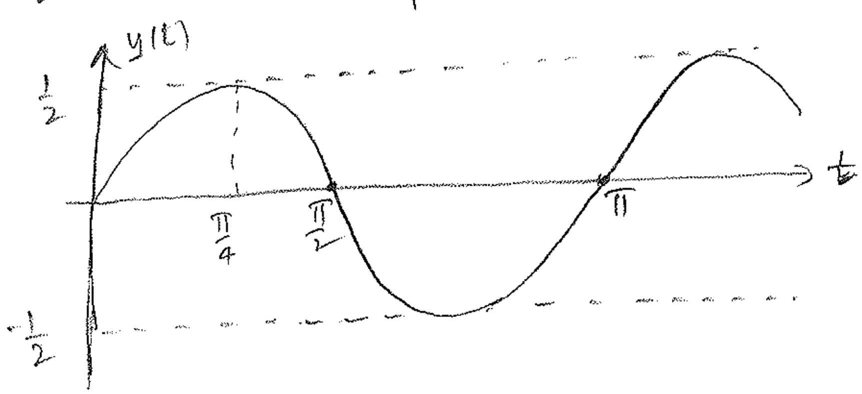
General solution $y(t) = c_1 \cos(2t) + c_2 \sin(2t)$

$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$

Now $y(0) = 0 \Rightarrow c_1 = 0$

$y'(0) = 1 \Rightarrow 2c_2 = 1 \Rightarrow c_2 = \frac{1}{2}$

Thus $y(t) = \frac{1}{2}\sin(2t)$: Oscillatory solution with period π.



(19) $y'' - 2y' + 5y = 0 \quad , \quad y(\frac{\pi}{2}) = 0 , \quad y'(\frac{\pi}{2}) = 2$

Let $y = e^{rt} \Rightarrow r^2 - 2r + 5 = 0$

$\Rightarrow r = \dfrac{+2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

Thus $y_1(t) = e^t \cos(2t) \quad , \quad y_2 = e^t \sin(2t)$

$\Rightarrow y(t) = e^t \left[c_1 \cos(2t) + c_2 \sin(2t) \right]$

Now $y(\frac{\pi}{2}) = 0 \Rightarrow 0 = c_1$

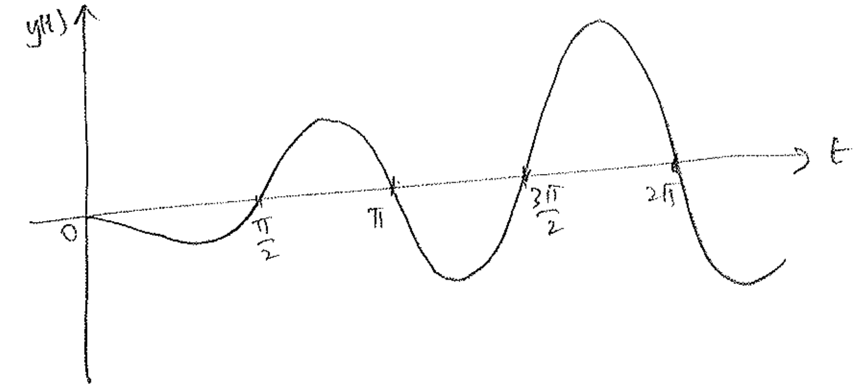
$y'(t) = e^t \left[-2c_1 \sin(2t) + 2c_2 \cos(2t) \right] + e^t \left[c_1 \cos(2t) + c_2 \sin(2t) \right]$

Now $y'(\frac{\pi}{2}) = 2 \Rightarrow 2 = e^{\pi/2} \left[-2c_2 \right] + e^{\pi/2} \left[c_1 \right]$

$\Rightarrow c_2 = -e^{-\pi/2}$

Thus $y(t) = -e^t \times e^{-\pi/2} \sin(2t) = -e^{(t-\frac{\pi}{2})} \sin(2t)$

Growing Oscillations



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$$y'' + 2y' + 6y = 0 \; ; \qquad \begin{aligned} y(0) &= 2 \\ y'(0) &= \alpha \geq 0 \end{aligned}$$

(a) General Solution:

Let $y = e^{rt} \longrightarrow r^2 + 2r + 6 = 0$

$$\longrightarrow r = \frac{-2 \pm \sqrt{4 - 24}}{2} = -1 \pm \sqrt{5}\, i$$

$$\Rightarrow y(t) = e^{-t}\left[c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)\right]$$

$$y'(t) = -e^{-t}\left[c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)\right]$$
$$+ e^{-t}\left[-\sqrt{5}c_1 \sin(\sqrt{5}t) + \sqrt{5}c_2 \cos(\sqrt{5}t)\right]$$

Now $y(0) = 2 \Rightarrow c_1 = 2$

$y'(0) = \alpha \Rightarrow -2 + \sqrt{5}c_2 = \alpha \Rightarrow c_2 = \dfrac{\alpha + 2}{\sqrt{5}}$

Thus $\boxed{y(t) = e^{-t}\left[2\cos(\sqrt{5}t) + \frac{(\alpha+2)}{\sqrt{5}} \sin(\sqrt{5}t)\right]}$

(b) Find α such that $y = 0$ at $t = 1$

$$\Rightarrow 0 = e^{-1}\left[2\cos(\sqrt{5}) + \frac{(\alpha+2)}{\sqrt{5}}\sin(\sqrt{5})\right]$$

$$\Rightarrow 2\cos(\sqrt{5}) = -\frac{(\alpha+2)}{\sqrt{5}}\sin(\sqrt{5})$$

$$\Rightarrow \alpha + 2 = \frac{-2\sqrt{5}}{\tan(\sqrt{5})} \Rightarrow \alpha = -2 - \frac{2\sqrt{5}}{\tan(\sqrt{5})}$$

$$\approx 1.50877$$

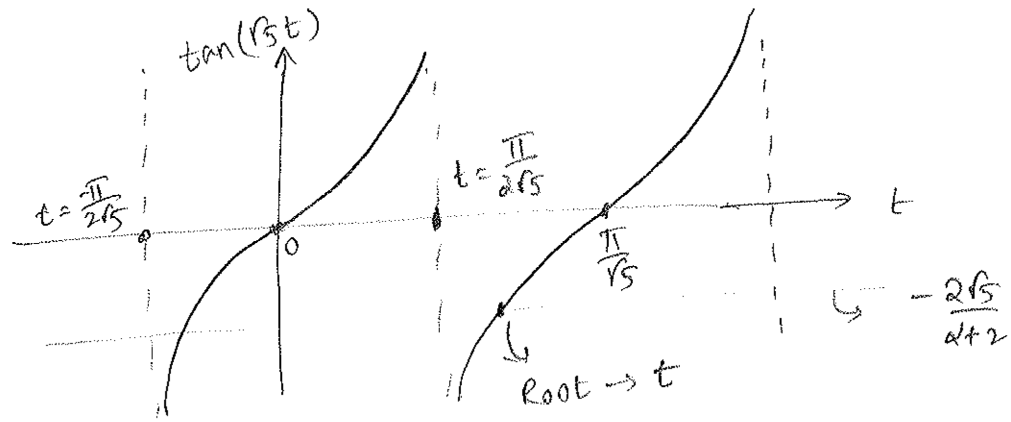
(c) Find smallest t so that $y = 0$.

$$y(t) = e^{-t}\left[2\cos(\sqrt{5}t) + \frac{(\alpha+2)}{\sqrt{5}}\sin(\sqrt{5}t)\right]$$

If $y = 0$, we have $2\cos(\sqrt{5}t) = -\frac{(\alpha+2)}{\sqrt{5}}\sin(\sqrt{5}t)$

$$\Rightarrow \boxed{\tan(\sqrt{5}t) = \frac{-2\sqrt{5}}{\alpha+2}}$$

Graphical Construction of Root!



Alternately, $-\tan(\sqrt{5}t) = \frac{2\sqrt{5}}{\alpha+2}$

$$\Rightarrow \tan(\pi - \sqrt{5}t) = \frac{2\sqrt{5}}{\alpha+2}$$

$$\Rightarrow \pi - \sqrt{5}t = \tan^{-1}\left[\frac{2\sqrt{5}}{\alpha+2}\right]$$

$$\Rightarrow \sqrt{5}t = \pi - \tan^{-1}\left[\frac{2\sqrt{5}}{\alpha+2}\right]$$

$$\Rightarrow \boxed{t = \frac{1}{\sqrt{5}}\left[\pi - \tan^{-1}\left(\frac{2\sqrt{5}}{\alpha+2}\right)\right]}$$

(d) As $\alpha \to \infty$, $t = \frac{\pi}{\sqrt{5}}$. This also agrees with our graphical picture above.

35) $t^2y'' + ty' + y = 0$

For the Euler's equation, put $x = \ln t$

$$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = \frac{dy}{dx}\cdot\frac{1}{t}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dx}{dt}$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\cdot\frac{1}{t}\right)\times\frac{1}{t} = \frac{1}{t}\left[\frac{d^2y}{dx^2}\cdot\frac{1}{t} + \frac{dy}{dx}\cdot\frac{d}{dx}\left(\frac{1}{t}\right)\right]$$

Now $\frac{d}{dx}\left(\frac{1}{t}\right) = \frac{-1}{t^2}\frac{dt}{dx} = \frac{-1}{t}$

$$\Rightarrow \frac{d^2y}{dt^2} = \frac{1}{t^2}\left[\frac{d^2y}{dx^2} - \frac{dy}{dx}\right] \Rightarrow t^2\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} - \frac{dy}{dx}$$

$\Rightarrow$ Equation becomes

$$\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + \frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Let $y = e^{rx}$ $\Rightarrow$ $r^2 + 1 = 0$ $\Rightarrow r = \pm i$

General solution: $y(x) = c_1\cos(x) + c_2\sin(x)$

$$\Rightarrow \boxed{y(t) = c_1\cos(\ln t) + c_2\sin(\ln t)}$$

35 (REPEAT) $t^2y'' + ty' + y = 0$

The above equation can also be solved by making the substitution $y = t^\lambda$.

$\Rightarrow \quad y' = \lambda t^{\lambda-1} \quad \Rightarrow \quad y'' = \lambda(\lambda-1)t^{\lambda-2}$

Thus $t^2\lambda(\lambda-1)t^{\lambda-2} + t\cdot\lambda t^{\lambda-1} + t^\lambda = 0$

$\Rightarrow \quad t^\lambda\left[\lambda(\lambda-1) + \lambda + 1\right] = 0$

Thus $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i$

Thus $y_1(t) = t^i = e^{i\ln t} = \cos(\ln t) + i\sin(\ln t)$

complex solution

Similarly, $y_2(t) = \cos(\ln t) - i\sin(\ln t)$

We can construct real solutions using $(y_1 + y_2)$ and $(y_1 - y_2)$.

This gives the general solution

$y(t) = c_1\cos(\ln t) + c_2\sin(\ln t)$

SECTION 3.4

(5) $y'' - 2y' + 10y = 0$

Put $y = e^{rt}$ $\Rightarrow$ $r^2 - 2r + 10 = 0$

$$\Rightarrow r = \frac{2 \pm \sqrt{4 - 40}}{2}$$

$$= 1 \pm 3i$$

General Solution: $y(t) = e^{t}\left[c_1 \cos(3t) + c_2 \sin(3t)\right]$

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(12) $y'' - 6y' + 9y = 0$; $y(0) = 0$; $y'(0) = 2$

Put $y = e^{rt}$ $\Rightarrow$ $r^2 - 6r + 9 = 0$

$\Rightarrow (r-3)^2 = 0 \Rightarrow r = 3$: Repeated root.

General Solution: $y(t) = c_1 e^{3t} + c_2 t e^{3t}$

$y'(t) = 3c_1 e^{3t} + 3c_2 t e^{3t} + c_2 e^{3t}$

Now $y(0) = 0 \Rightarrow c_1 = 0$

$y'(0) = 2 \Rightarrow c_2 = 2$

Thus $\boxed{y(t) = 2te^{3t}}$

As $t \to \infty$; $y \to \infty$.

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(14) $y'' + 4y' + 4y = 0$; $y(-1) = 2$; $y'(-1) = 1$

Put $y = e^{rt}$ $\Rightarrow$ $r^2 + 4r + 4 = 0$

$\Rightarrow (r+2)^2 \Rightarrow r = -2$: Repeated root

Thus $\quad y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

Now, $\quad y(-1) = 2 \Rightarrow \quad 2 = c_1 e^{2} - c_2 e^{2}$

$$\Rightarrow \quad c_1 - c_2 = 2e^{-2}$$

$$y'(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t}$$

Now $\quad y'(-1) = 1 \Rightarrow \quad 1 = -2c_1 e^{2} + 2c_2 e^{2} + c_2 e^{2}$

$$\Rightarrow \quad e^{-2} = -2c_1 + 2c_2 + c_2$$

$$= -2c_1 + 3c_2$$

$$\Rightarrow \quad -2c_1 + 3c_2 = e^{-2}$$

$$c_1 - c_2 = 2e^{-2} \Rightarrow 2c_1 - 2c_2 = 4e^{-2}$$

$$c_2 = 5e^{-2}$$

$$\Rightarrow \quad c_1 = c_2 + 2e^{-2} = 7e^{-2}$$

$$\Rightarrow \quad y(t) = 7e^{-2} e^{-2t} + 5e^{-2} t e^{-2t}$$

$$\boxed{y(t) = 7e^{-2(t+1)} + 5te^{-2(t+1)}}$$

As $t \to \infty$, the first term goes to zero.

$$\Rightarrow \quad y(t) \longrightarrow \lim_{t\to\infty} \frac{5t}{e^{2(t+1)}}$$

Using L'Hopital's rule, $\quad y(t) = \lim_{t\to\infty} \frac{5}{2e^{2(t+1)}} \longrightarrow 0$