MATH 215/255 HW#4 HARISH DIXIT

SECTION 2.5

(3) $\dfrac{dy}{dt} = y(y-1)(y-2)$; $y_0 \geq 0$

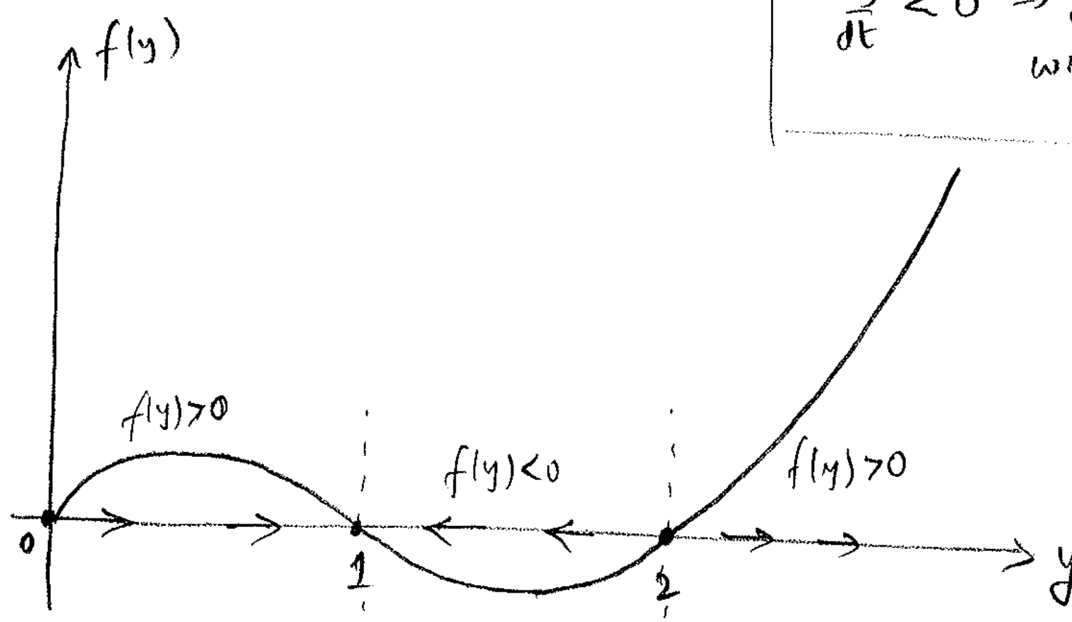
The above equation is of the form

$$\frac{dy}{dt} = f(y) \quad \text{where}$$

$$f(y) = y(y-1)(y-2)$$

If $f(y) > 0$;
$\dfrac{dy}{dt} > 0 \Rightarrow y$ increases with t.

If $f(y) < 0$;
$\dfrac{dy}{dt} < 0 \Rightarrow y$ decreases with t.



From the red arrows, we see that $y=1$ is stable and $y=0$ and $y=2$ are unstable.

To draw the solutions, we also need to know the concavity of the curves. This is obtained from $\dfrac{d^2y}{dt^2}$.

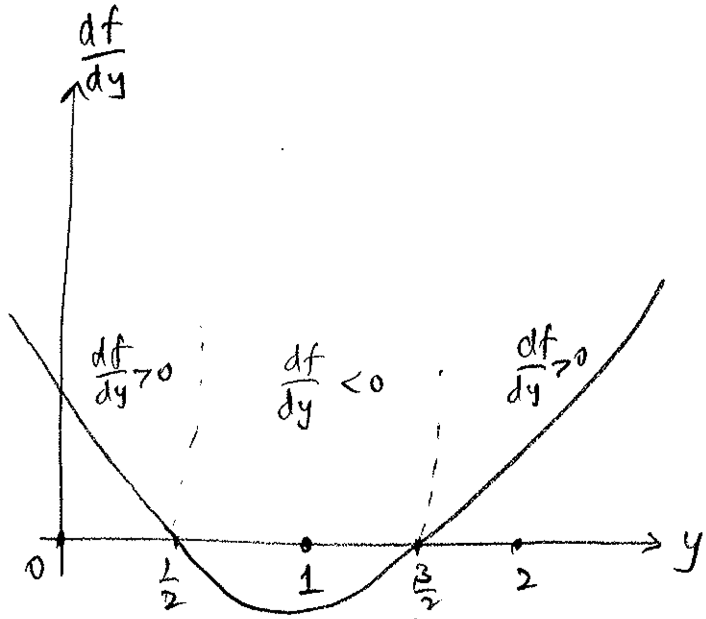
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(f(y)\right)$$

$$= \frac{d}{dy}\left(f(y)\right)\cdot\frac{dy}{dt}$$

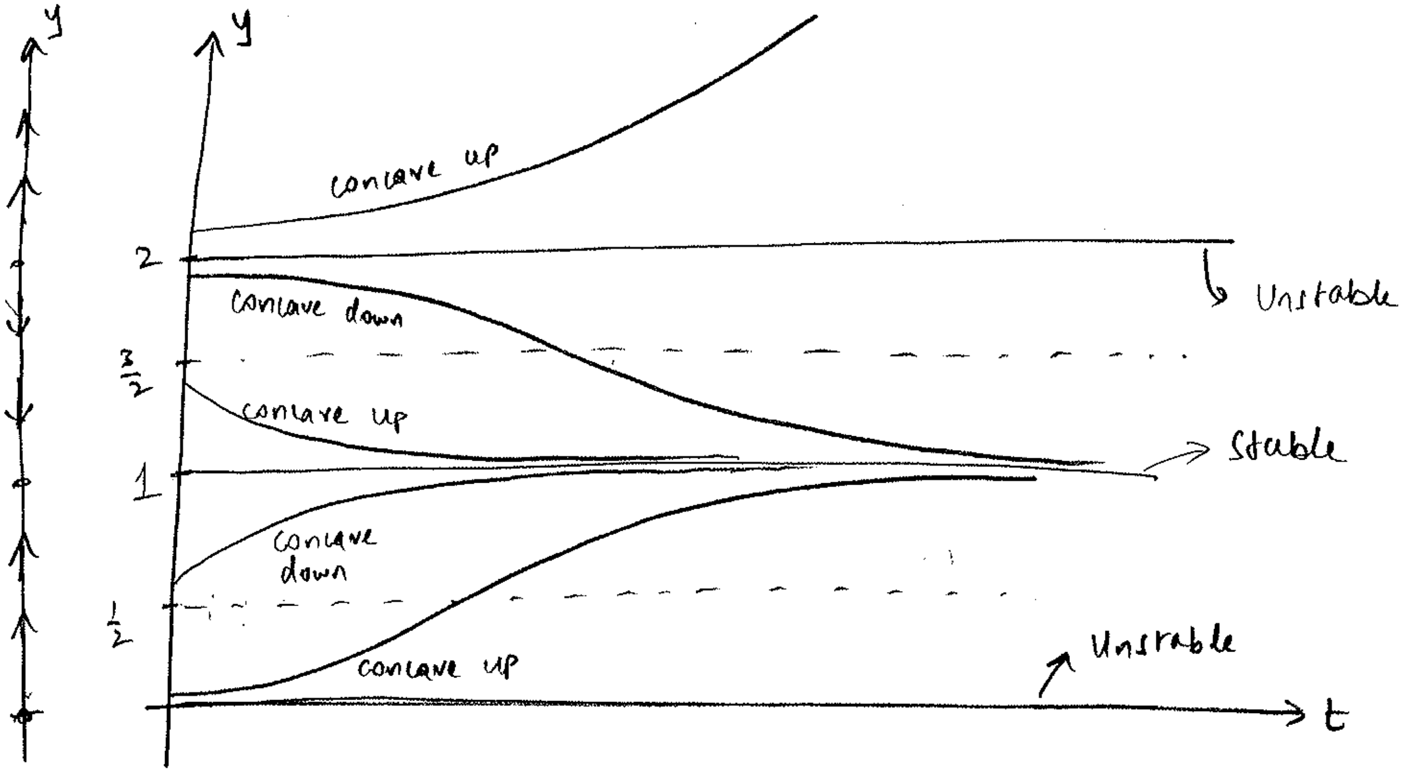
$$= \frac{df}{dy} \times f(y)$$

$$\frac{df}{dy} = (y-1)(y-2) + y(y-2) + y(y-1)$$

$$= 3y^2 - 6y + 2$$



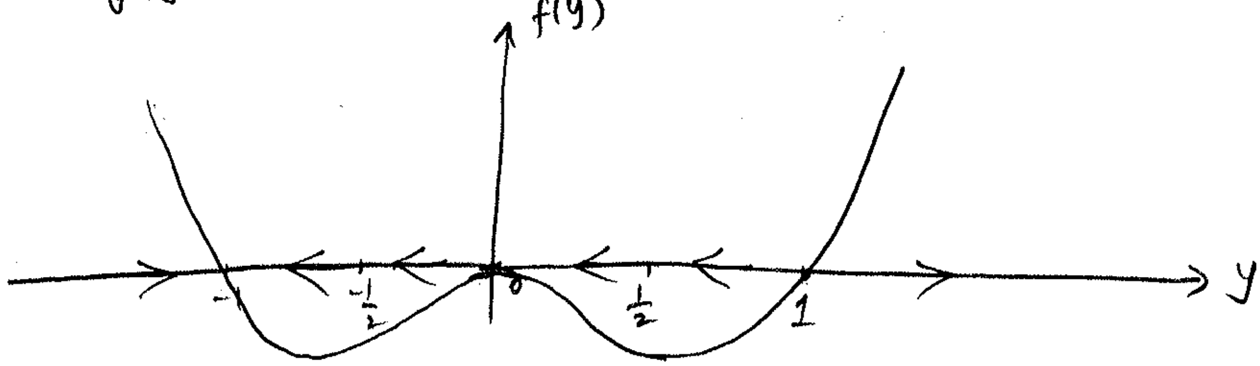
Interval	$\frac{dy}{dt} = f(y)$	$\frac{df}{dy}$	$\frac{d^2y}{dt^2} = \frac{df}{dy}\cdot f(y)$	Concavity
$(0, \frac{1}{2})$	>0	>0	>0	Concave up
$(\frac{1}{2}, 1)$	>0	<0	<0	Concave down
$(1, \frac{3}{2})$	<0	<0	>0	Concave up
$(\frac{3}{2}, 2)$	<0	>0	<0	Concave down
$(2, \infty)$	>0	>0	>0	Concave up

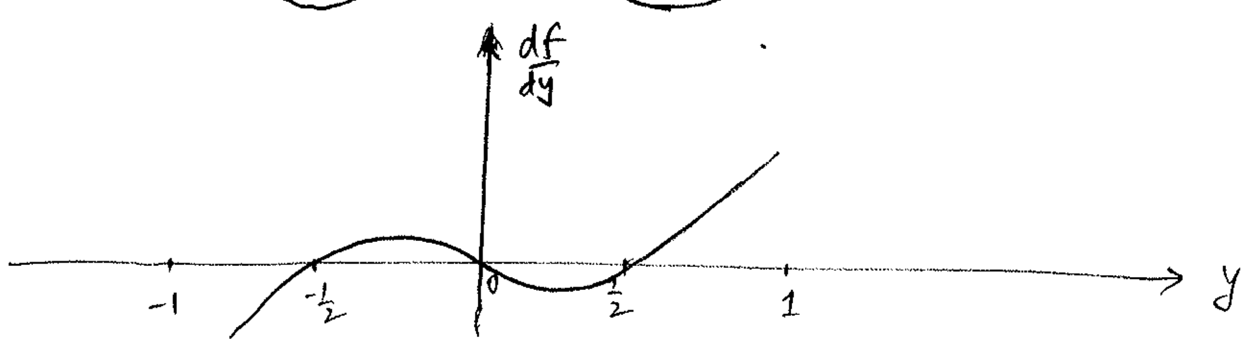


(9) $\dfrac{d^2y}{dt^2} = y^2(y^2-1)$; $-\infty < y_0 < \infty$

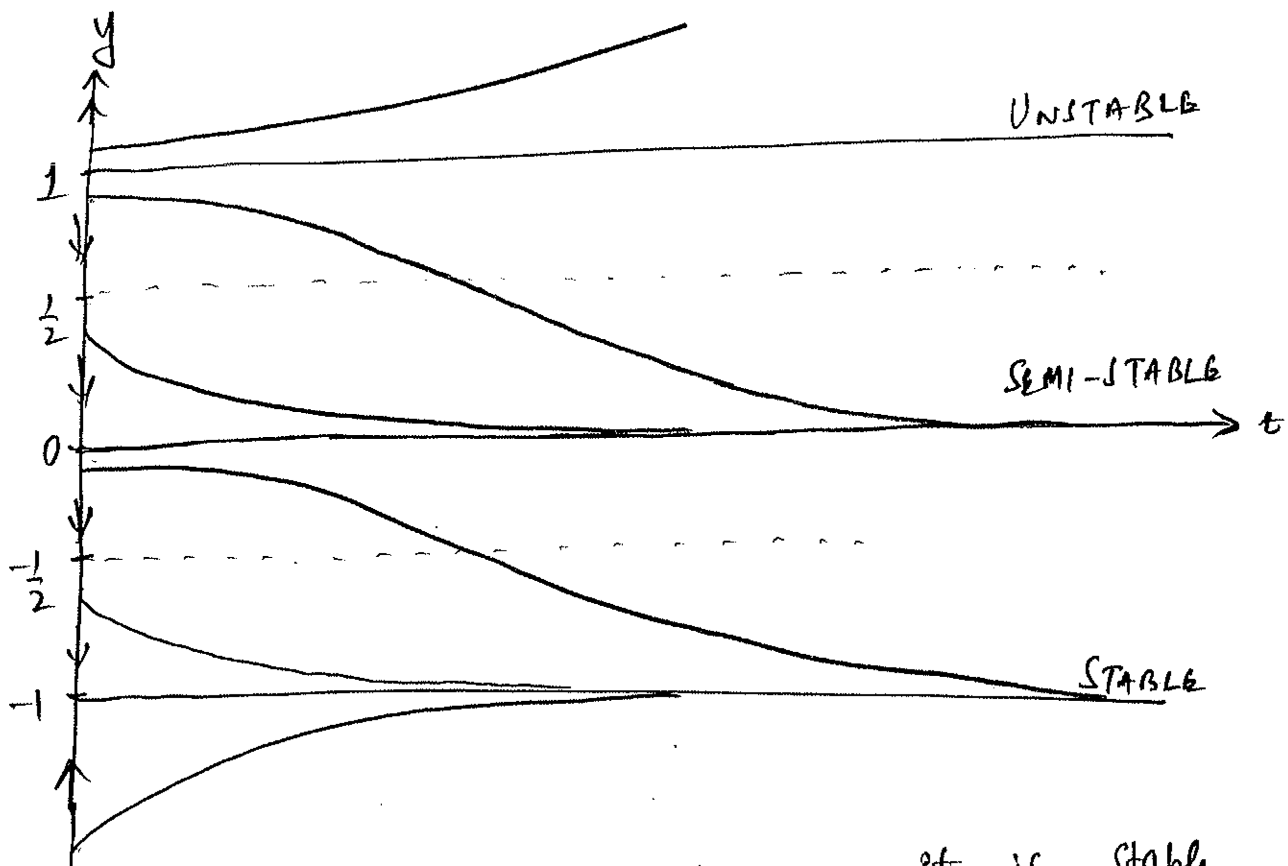
$f(y) = y^2(y^2-1)$

$\dfrac{df}{dy} = 2y(y^2-1) + y^2(2y) = 4y^3 - 2y$





Interval	$\frac{dy}{dt} = f(y)$	$\frac{df}{dy}$	$\frac{d^2y}{dt^2} = f(y) \times \frac{df}{dy}$
$(-\infty, -1)$	>0	<0	<0
$(-1, -\frac{1}{2})$	<0	<0	>0
$(-\frac{1}{2}, 0)$	<0	>0	<0
$(0, \frac{1}{2})$	<0	<0	>0
$(\frac{1}{2}, 1)$	<0	>0	<0
$(1, \infty)$	>0	>0	>0

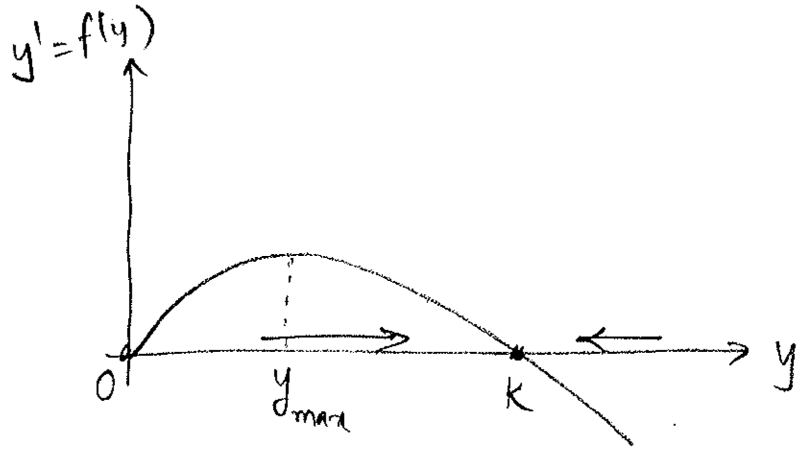


$y = 0$ is called semi-stable because it is stable on one side and unstable on the other side.

(16) $\dfrac{dy}{dt} = \underbrace{ry \cdot \ln\left(\dfrac{K}{y}\right)}_{f(y)}$, $r > 0$, $K > 0$

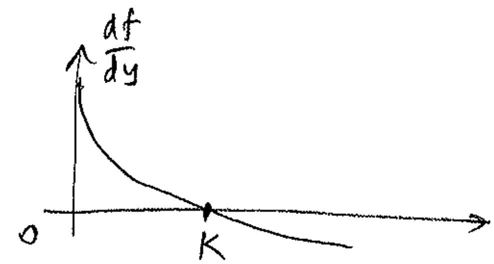
(a) Now, equilibrium solutions are

$y_e = 0$; $y_e = K$



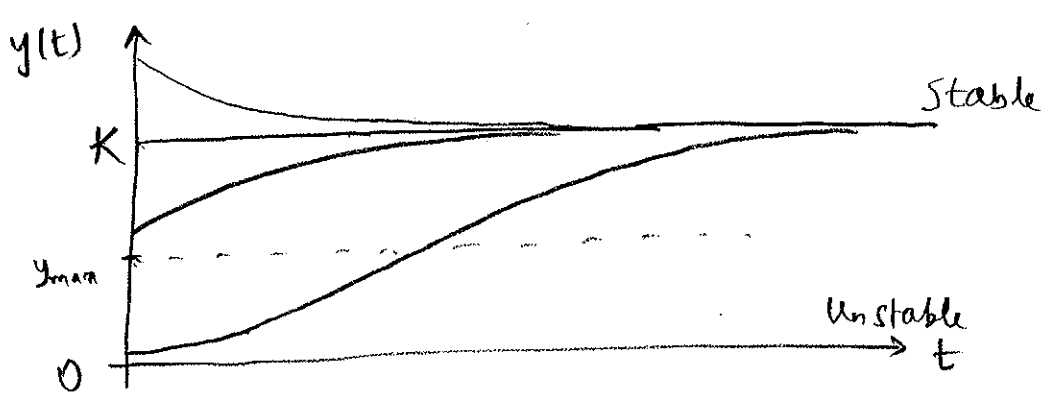
$y_e = 0$ is unstable

$y_e = K$ is stable



(b)

y_0	$y' = f(y)$	$y'' = f(y) \cdot f'(y)$
$0 < y_0 < y_{max}$	> 0	$(>0)\,(>0) = (>0)$: Concave up
$y_{max} < y_0 < K$	> 0	$(>0)\,(<0) = (<0)$: Concave down
$y_0 > 0$	< 0	$(<0)\,(<0) = (>0)$: Concave up



(c) Gompertz equation: $\left(\frac{dy}{dt}\right)_{Gom} = ry\ln\left(\frac{K}{y}\right)$

Logistic equation:- $\left(\frac{dy}{dt}\right)_{Log} = ry\left(1-\frac{y}{K}\right)$

Must show for $0<y\leq K$; $\left(\frac{dy}{dt}\right)_{Gom} \geqslant \left(\frac{dy}{dt}\right)_{Log}$ (To be shown).

We calculate

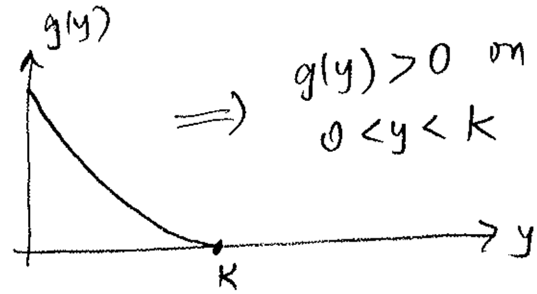
(*) $\left(\frac{dy}{dt}\right)_{Gom} - \left(\frac{dy}{dt}\right)_{Log} = ry\left[\ln\left(\frac{K}{y}\right) - \left(1-\frac{y}{K}\right)\right]$

Must show $g(y) = \ln\left(\frac{K}{y}\right) - 1 + \frac{y}{K} \geqslant 0$ on $0\leq y\leq K$

Notice $g(K)=0$; and $\lim_{y\to 0} g(y) = +\infty$

and $g'(y) = \frac{-1}{y} + \frac{1}{K} < 0$.

On $0<y<K$, thus we have



$\Longrightarrow$ $g(y)>0$ on $0<y<K$

Then (*) yields $\left(\frac{dy}{dt}\right)_{Gom} - \left(\frac{dy}{dt}\right)_{Log} = ry\,g(y) > 0$

on $0<y<K$

(20) <u>Schaefer model</u>

$$\frac{dy}{dt} = r\left(1-\frac{y}{K}\right)y - Ey \; ; \qquad y \to \text{fish population}$$

(a) Now equilibrium solutions are :

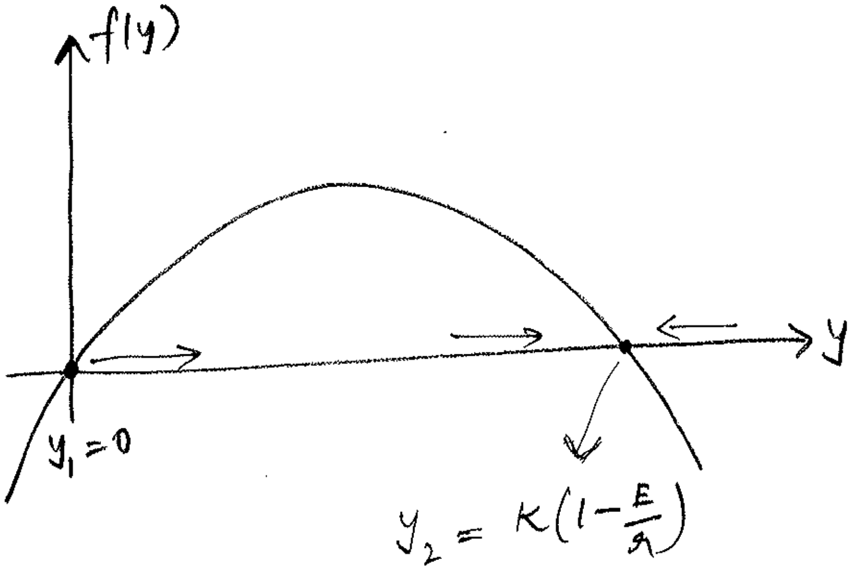
$$\frac{dy}{dt} = 0 \implies \left[r\left(1-\frac{y}{K}\right) - E\right] y = 0$$

$$\implies y = 0 \quad \text{and} \quad y = K\left(1-\frac{E}{r}\right)$$

If $E > r$, $y < 0$ and this is unphysical.

If $E < r$; $y = K\left(1-\frac{E}{r}\right) > 0$.

(b) Plotting $f(y)$, we have



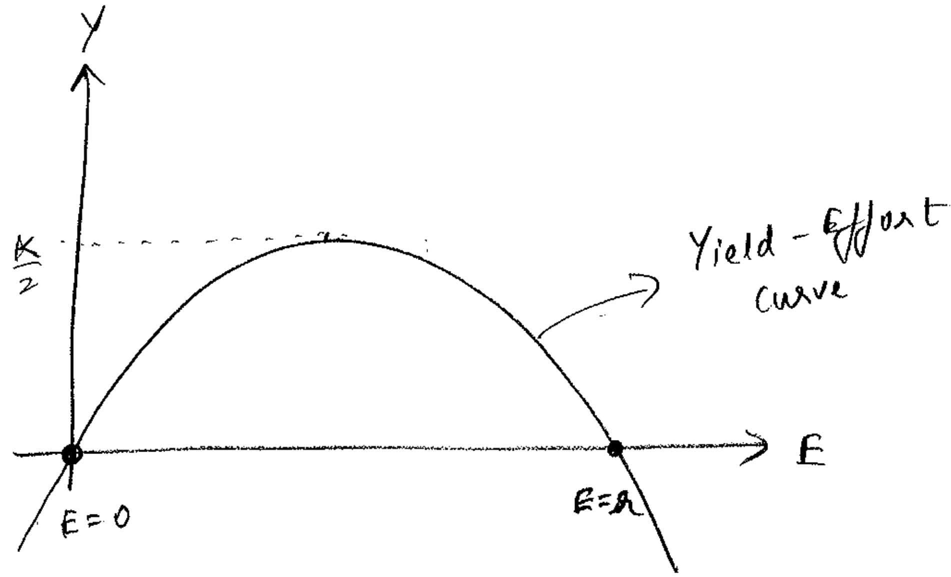
$y_1 = 0$ is unstable

$y_2 = K\left(1-\frac{E}{r}\right)$ is stable.

(c) Given that Y = Effort $\times$ Stable population

$$\Rightarrow \quad Y = E \times y_2$$

$$\boxed{Y = EK\left(1 - \frac{E}{r}\right)}$$



Section 3.1

(1) Find general solution:

$$y'' + 2y' - 3y = 0$$

This is an equation with constant coefficients.

$\Rightarrow$ Let $y = e^{rt}$

$\Rightarrow$ $y' = re^{rt}$

$y'' = r^2 e^{rt}$

Substituting y, y' & y'', we get

$$e^{rt}(r^2 + 2r - 3) = 0$$

Since $e^{rt} \neq 0$, we have $\boxed{r^2 + 2r - 3 = 0}$

Characteristic equation.

Roots:

$$r = \frac{-2 \pm \sqrt{4 + 4 \times 3}}{2}$$

$$= \frac{-2 \pm 4}{2}$$

$$= -3, 1$$

The two solutions are

$$y_1(t) = e^{-3t},$$

$$y_2(t) = e^{t}$$

General Solution: $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$\Rightarrow \boxed{y(t) = c_1 e^{-3t} + c_2 e^{t}}$$

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(9) Find $y(t)$ in the following initial value problem (IVP)?

$$y'' + y' - 2y = 0 \;; \qquad \begin{aligned} y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

Let $y = e^{rt}$. We get

$$(r^2 + r - 2)\, e^{rt} = 0$$

$$\Rightarrow \quad r^2 + r - 2 = 0$$

$$\Rightarrow \quad r^2 + 2r - r - 2 = 0 \quad \Rightarrow \quad (r+2)(r-1) = 0$$

$$\Rightarrow \quad r_1 = -2 \;; \qquad r_2 = 1$$

Therefore,

$$y_1(t) = e^{-2t}$$

$$y_2(t) = e^{t}$$

General Solution:- $y(t) = c_1 e^{-2t} + c_2 e^{t}$

Now $y(0) = 1 \quad \Rightarrow \quad 1 = c_1 + c_2$

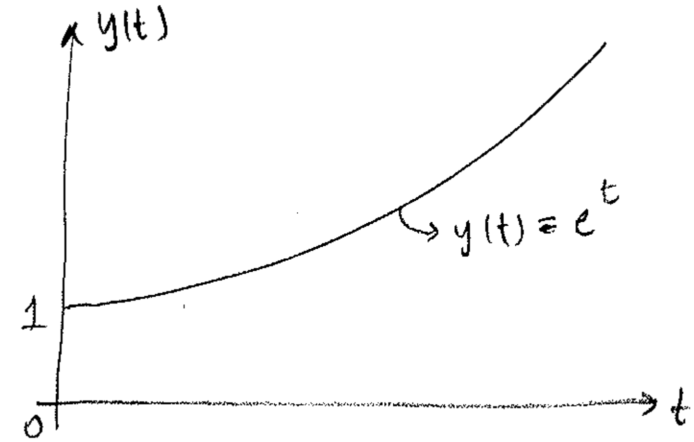
$y'(0) = 1 \quad \Rightarrow \quad 1 = -2c_1 + c_2$

$$\Rightarrow \quad c_1 = 0 \;; \quad c_2 = 1$$

Exact Solution: $\boxed{y(t) = e^{t}}$

As $t \to \infty$;

$y(t) \to \infty$



(11) $6y'' - 5y' + y = 0$; $y(0) = 4$, $y'(0) = 0$

Find $y(t)$?

Let again use $y = e^{rt}$

Characteristic equation : $(6r^2 - 5r + 1) = 0$

$r_1 = \frac{1}{3}$; $r_2 = \frac{1}{2}$

$y_1(t) = e^{t/3}$; $y_2 = e^{t/2}$

General solution: $y(t) = C_1 e^{t/3} + C_2 e^{t/2}$

Now $y(0) = 4 \Rightarrow 4 = C_1 + C_2$

$y'(0) = 0 \Rightarrow 0 = \frac{C_1}{3} + \frac{C_2}{2}$

$$\left.\begin{aligned} 4 &= C_1 + C_2 \\ 0 &= \frac{C_1}{3} + \frac{C_2}{2} \end{aligned}\right\} \quad \begin{aligned} C_1 + C_2 &= 4 \\ 2C_1 + 3C_2 &= 0 \end{aligned}$$

$\therefore$ $C_1 = 12$
$C_2 = -8$

Therefore $\boxed{y(t) = 12e^{t/3} - 8e^{t/2}}$

$y'(t) = 4e^{t/3} - 4e^{t/2}$

As $t \to \infty$; $e^{t/2}$ grows much faster than $e^{t/3}$.

Therefore $y(t)$ becomes large and is negative.

Moreover, $y'(t) < 0$ as t becomes large.

$\Rightarrow$ As $t \to \infty$; $y \to -\infty$.

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(15) $y'' + 8y' - 9y = 0$; $y(1) = 1$, $y'(1) = 0$

Characteristic equation:-

$r^2 + 8r - 9 = 0$

$\Rightarrow$ $r_1 = -9$

and $r_2 = 1$

Therefore, $y_1(t) = e^{-9t}$
$y_2(t) = e^{t}$

General Solution:- $y(t) = c_1 e^{-9t} + c_2 e^t$

Now $y(1) = 1$ $\Rightarrow$ $1 = c_1 e^{-9} + c_2 e$

$y'(1) = 0$ $\Rightarrow$ $0 = -9c_1 e^{-9} + c_2 e$

$\Rightarrow$ $c_1 = \dfrac{e^9}{10}$; $c_2 = \dfrac{9}{10e}$

Exact Solution:-

$$y(t) = \frac{e^9}{10} e^{-9t} + \frac{9}{10} e^{-1} \cdot e^{t}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{10} e^{-9(t-1)} + \frac{9}{10} e^{t-1}}$$

As $t \to \infty$, the first term becomes negligible, but the second term becomes large & is positive.

$\Rightarrow$ As $t \to \infty$; $y(t) \to \infty$

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(17) Find the differential equation with general solution

$$y(t) = c_1 e^{2t} + c_2 e^{-3t}$$

$$\frac{dy}{dt} = 2c_1 e^{2t} - 3c_2 e^{-3t}$$

$$\frac{d^2y}{dt^2} = 4c_1 e^{2t} + 9c_2 e^{-3t}$$

Solving for c_1 & c_2 in terms of $y(t)$ and $\frac{dy}{dt}$:

$$c_1 e^{2t} = \frac{1}{5}\left(3y + \frac{dy}{dt}\right)$$

$$c_2 e^{-3t} = \frac{1}{5}\left(2y - \frac{dy}{dt}\right)$$

Substituting c_1 & c_2 into $\frac{d^2y}{dt^2}$, we get

$$\frac{d^2y}{dt^2} = 4 \cdot \frac{1}{5}\left(3y + \frac{dy}{dt}\right) + \frac{9}{5}\left(2y - \frac{dy}{dt}\right)$$

$$= \frac{1}{5}\left[12y + 4\frac{dy}{dt} + 18y - 9\frac{dy}{dt}\right]$$

$$= \frac{1}{5}\left(30y - 5\frac{dy}{dt}\right) = 6y - \frac{dy}{dt}$$

$$\Rightarrow \boxed{\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0}$$

Alternate approach:

Since $y(t) = c_1 y_1 + c_2 y_2$

where $y_1 = e^{2t}$; $y_2 = e^{-3t}$

There we have $r_1 = 2$, $r_2 = -3$

characteristic equation: $(r-2)(r+3) = 0$

$\Rightarrow r^2 + r - 6 = 0$

$\Rightarrow$ Governing differential equation is

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0$$

(22) $4y'' - y = 0$

$y(0) = 2$
$y'(0) = \beta$

find β so that $y \to 0$ as $t \to \infty$.

Let $y = e^{rt}$

$\Rightarrow 4r^2 - 1 = 0 \Rightarrow r_1 = \frac{1}{2}, \quad r_2 = \frac{-1}{2}$

Thus $y = c_1 e^{t/2} + c_2 e^{-t/2} \Rightarrow y(0) = 2 \longrightarrow c_1 + c_2 = 2$

$y' = \frac{c_1}{2} e^{t/2} - \frac{c_2}{2} e^{-t/2} \Rightarrow y'(0) = \beta \rightarrow \frac{c_1}{2} - \frac{c_2}{2} = \beta$

The growing solution as $t \to \infty$ is $e^{t/2}$.

Therefore, we require $c_1 = 0. \Rightarrow c_2 = 2$

$\therefore \boxed{\beta = -1}$ for $y \to 0$ as $t \to \infty$.

Section 3.2

(1) Given $y_1 = e^{2t}$, $y_2 = e^{-3t/2}$, find the Wronskian.

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2} e^{-3t/2} \end{vmatrix}$$

$$= \frac{-3}{2} e^{t/2} - 2e^{t/2} = \frac{-7}{2} e^{t/2}$$

(12) Find interval of existence:

$$(x-2)y'' + y' + (x-2)(\tan x)\, y = 0 \qquad y(3) = 1, \quad y'(3) = 2$$

Writing the equation in the form

$y'' + p(t)\, y' + q(t) = 0$, we have

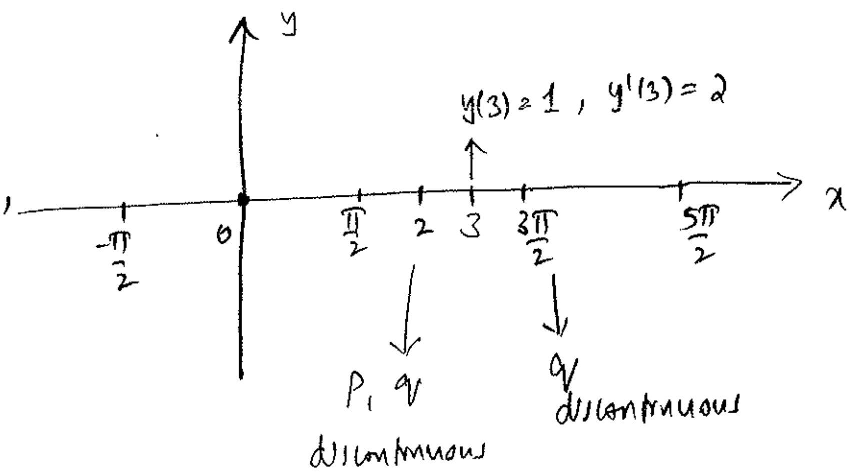
$$p(t) = \frac{1}{x-2} \quad ; \quad q(t) = \frac{\tan x}{x-2}$$

$p(t)$ becomes discontinuous at $x = 2$. and when

$q(t)$ becomes discontinuous at $x = 2$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Since the initial condition is prescribed at $x = 3$,



the largest extent around $x = 3$ for which both $p(t)$ and $q(t)$ are continuous is

$$2 < x < \frac{3\pi}{2}$$

(26) $x^2y'' - x(x+2)\,y' + (x+2)\,y = 0, \quad x > 0;$

$y_1(x) = x$

$y_2(x) = xe^x$

Verify that y_1 & y_2 are solutions:

For y_1:

$y_1 = x$

$y_1' = 1$

$y_1'' = 0$

Substituting, we have $x^2(0) - x(x+2)\cdot 1 + (x+2)\,x = 0$

$\Rightarrow \quad 0 = 0$

$\Rightarrow \quad y_1$ is a solution.

For y_2:

$y_2 = xe^x$

$y_2' = xe^x + e^x$

$y_2'' = xe^x + 2e^x$

$\Rightarrow (xe^x + 2e^x)\,x^2 - x(x+2)(xe^x + e^x) + (x+2)\cdot xe^x$

$\Rightarrow \quad e^x\left[(x+2)x^2 - x(x+2)(x+1) + x(x+2)\right]$

$\Rightarrow e^x\left[(x+2)(x^2+x) - x(x+1)(x+2)\right]$

$= 0$

$\Rightarrow \quad y_2 = e^x \cdot x$ is a solution.

For $y_1(t)$ and $y_2(t)$ to form a fundamental set,

we require $W[y_1, y_2](t) \neq 0$

$$W[y_1, y_2] = \begin{vmatrix} x & xe^x \\ 1 & (x+1)e^x \end{vmatrix}$$

$$= (x+1)xe^x - xe^x$$

$$= x^2 e^x \neq 0 \quad \text{for } x > 0.$$

$\Rightarrow$ y_1 & y_2 are fundamental solutions.

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(31) $x^2 y'' + xy' + (x^2 - \nu^2)\, y = 0$

Writing the equation in the standard form

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x) = \dfrac{1}{x}$; $q(x) = \dfrac{x^2 - \nu^2}{x}$

Using Abel's formula, we have

$$W = ce^{-\int p(x)\,dx} = ce^{-\int \frac{1}{x}\,dx}$$

$$= ce^{-\ln x} = \frac{c}{x}$$