MATH 215/255 HW #3 HARISH DIXIT

SECTION 2.6

(1) $(2x+3) + (2y-2)y' = 0$

The above equation is in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \text{where}$$

$$M(x,y) = 2x+3$$

$$N(x,y) = 2y-2$$

$$\frac{\partial M}{\partial y} = 0 \quad ; \quad \frac{\partial N}{\partial x} = 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; the equation is exact.

Let $\psi(x,y)$ be a function such that

$$\frac{\partial \psi}{\partial x} = M \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N$$

Consider $\frac{d}{dx}\psi(x, y(x)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\cdot\frac{dy}{dx}$

$$= M + N\frac{dy}{dx}$$

$$= 0 \quad \text{(from the equation)}$$

Integrating on both sides, we have $\boxed{\psi(x,y) = \text{Constant}}$

Therefore, our goal is to find $\psi(x,y)$.

Since $\frac{\partial \psi}{\partial x} = M$, we have

$$\frac{\partial \psi}{\partial x} = 2x+3 \Rightarrow \psi = x^2+3x+\underbrace{h(y)}_{\text{Integration constant}}$$

But $\frac{\partial \psi}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left(x^2+3x+h(y)\right) = 2y-2$

$$\Rightarrow \frac{dh}{dy} = 2y-2 \Rightarrow h = y^2-2y+C_0$$

$$\therefore \psi(x,y) = x^2+3x+y^2-2y+C_1$$

$\therefore \psi = \text{const}$ becomes

$$\boxed{x^2+3x+y^2-2y = C}$$: Solution.

---

(2) $(2x+4y)+(2x-2y)\frac{dy}{dx} = 0$

Here $M(x,y) = 2x+4y \Rightarrow \frac{\partial M}{\partial y} = 4$

$N(x,y) = 2x-2y \Rightarrow \frac{\partial N}{\partial x} = 2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

(13) $\quad (2x-y)\,dx + (2y-x)\,dy = 0 \quad ; \quad y(1) = 3$

The equation can be written as

$$(2x-y) + (2y-x)\frac{dy}{dx} = 0$$

Here $\quad M(x,y) = 2x-y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = -1$

and $\quad N(x,y) = 2y-x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -1$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Let $\frac{\partial \psi}{\partial x} = M \; ; \; \frac{\partial \psi}{\partial y} = N.$ $\quad$ <u>To find ψ :</u>

First, $\frac{\partial \psi}{\partial x} = 2x-y \quad \Rightarrow \quad \psi = x^2 - xy + h(y)$

Since $\frac{\partial \psi}{\partial y} = N$, we have $\quad -\cancel{x} + h'(y) = 2y - \cancel{x}$

$$\Rightarrow \quad h'(y) = 2y \quad \Rightarrow \quad h(y) = y^2 + C_1$$

$\therefore \quad \psi(x,y) = x^2 - xy + y^2 + C_1$

$\therefore$ Solution is $\quad \psi = C$

$$\Rightarrow \quad x^2 - xy + y^2 = C$$

Now $\quad y(1) = 3 \quad \Rightarrow \quad 1 - 3 + 9 = C \quad \Rightarrow \quad C = 7$

$\therefore \quad x^2 - xy + y^2 = 7$

or, $y^2 - xy + (x^2 - 7) = 0$

$$\Rightarrow \quad y = \frac{x \pm \sqrt{x^2 - 4(x^2 - 7)}}{2}$$

$$= \frac{x \pm \sqrt{28 - 3x^2}}{2}$$

We have two solutions. We now check which of these solutions actually satisfies the initial condition.

Since $y(1) = 3$, we have

$$y_{(\text{at } x=1)} = \frac{1 \pm \sqrt{28 - 3}}{2}$$

$$= \frac{1 \pm 5}{2}$$

Clearly, the solution with positive sign satisfies the initial condition. Therefore, the correct solution is

$$\boxed{y = \frac{x + \sqrt{28 - 3x^2}}{2}}$$

Solution valid when $28 - 3x^2 \geq 0 \Rightarrow 3x^2 - 28 \leq 0$

$$\Rightarrow \boxed{|x| \leq \sqrt{\frac{28}{3}}}$$

(15) $(xy^2 + bx^2y) + (x+y)x^2 \frac{dy}{dx} = 0$

$$M(x,y) = xy^2 + bx^2y \implies \frac{\partial M}{\partial y} = 2xy + bx^2$$

$$N(x,y) = x^3 + yx^2 \implies \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

for the equation to be exact, we require

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies b = 3$$

Now, $M(x,y) = xy^2 + 3x^2y$

$N(x,y) = x^3 + yx^2$

Let $\left.\begin{array}{l}\frac{\partial \psi}{\partial x} = M \\ \frac{\partial \psi}{\partial y} = N\end{array}\right\} \begin{array}{l}\implies \psi(x,y) = \frac{x^2y^2}{2} + x^3y + h(y) \\ \implies \cancel{x^2y} + \cancel{x^3} + \frac{dh}{dy} = \cancel{x^3 + yx^2}\end{array}$

$$\frac{dh}{dy} = 0 \implies h = C_1$$

$$\therefore \psi(x,y) = \frac{x^2y^2}{2} + x^3y + C_1$$

Solution is $\psi =$ constant $\implies \boxed{\frac{x^2y^2}{2} + x^3y = C}$

Section 2.4

(1) $(t-3)\,y' + (\ln t)\,y = 2t \quad ; \quad y(1) = 2$

Writing the equation in the standard form $y' + p(t)\,y = q(t)$, we have

$$p(t) = \frac{\ln t}{t-3} \quad ; \quad q(t) = \frac{2t}{t-3}$$

$p(t)$ becomes discontinuous at $t=0$ and $t=3$

$q(t)$ becomes discontinuous at $t=3$.

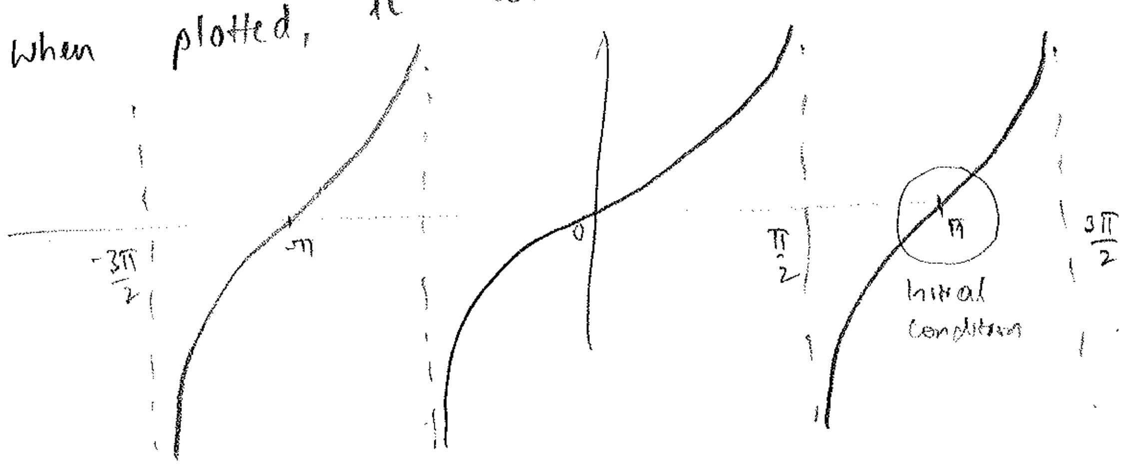
Since the initial condition is specified at $t=1$, the interval of existence is $\underline{0<t<3}$

(3) $y' + (\tan t)\,y = \sin t \quad ; \quad y(\pi) = 0$

$p(t) = \tan t$

$q(t) = \sin t$

The function $\tan t$ is discontinuous at odd multiples of $\frac{\pi}{2}$. When plotted, it looks like this



Since initial condition is at $t = \pi$, we have the following interval of existence: (4)

$$\frac{\pi}{2} < t < \frac{3\pi}{2}$$

---

(28) $t^2 y' + 2ty - y^3 = 0 \ ; \quad t > 0$

Rewriting the above equation in the form (Bernoulli equation),

$$y' + p(t)\, y = q(t)\, y^n$$

we have

$$p(t) = \frac{2}{t} \ ; \quad q(t) = \frac{1}{t^2} \ ; \quad n = 3$$

Use the substitution (as given in the book),

$$v = y^{1-n} = y^{-2} \Rightarrow v = \frac{1}{y^2}$$

$$\therefore \frac{dv}{dt} = \frac{-2}{y^3} \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{-y^3}{2} \frac{dv}{dt}$$

$$\Rightarrow -\frac{y^3}{2} \frac{dv}{dt} + \frac{2}{t} \cdot y = \frac{1}{t^2} y^3$$

$$\Rightarrow \frac{dv}{dt} - \frac{4}{t} \frac{1}{y^2} = \frac{-2}{t^2}$$

But $\frac{1}{y^2} = v \Rightarrow \boxed{\frac{dv}{dt} - \frac{4v}{t} = \frac{-2}{t^2}}$

Linear equation.

Solve for v:

Integrating factor $\phi(t) = e^{\int -\frac{4}{t}\,dt} = e^{-4\ln t}$

$= \frac{1}{t^4}$

$$\therefore \frac{1}{t^4}\frac{dv}{dt} - \frac{4}{t^5}v = \frac{-2}{t^2}\cdot\frac{1}{t^4}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{t^4}v\right) = \frac{-2}{t^6}$$

$$\Rightarrow \frac{v}{t^4} = -2\cdot\frac{t^{-5}}{-5} + C$$

$$\Rightarrow v = \frac{2}{5}\frac{1}{t} + Ct^4$$

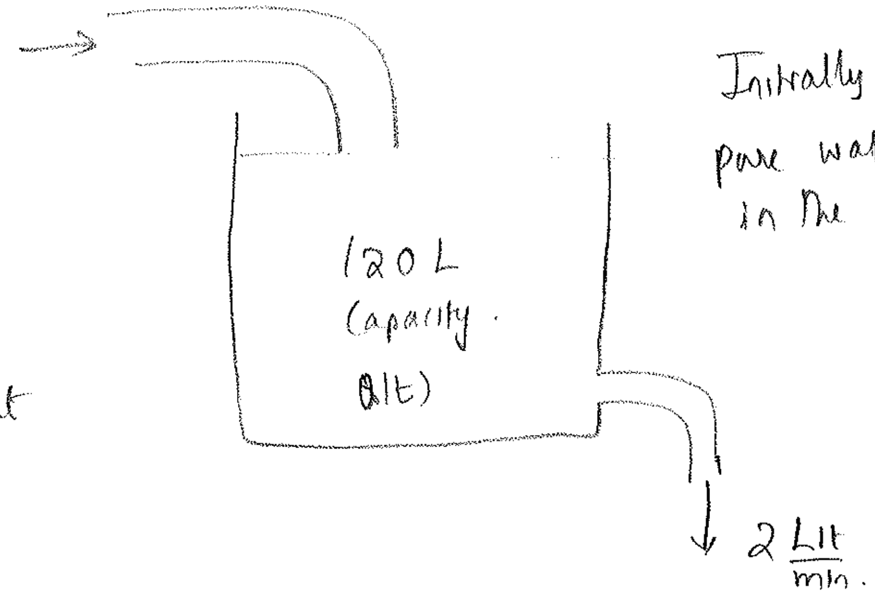
Or, in terms of y, we have

$$\frac{1}{y^2} = \frac{2}{5t} + Ct^4 = \frac{2+5Ct^5}{5t}$$

$$\Rightarrow \boxed{y = \pm\left[\frac{5t}{2+5Ct^5}\right]^{1/2}}$$

Section 2.3

(2) $\gamma \frac{gm}{Lit}$, $2 \frac{Lit}{min}$



Initially, pure water in the tank

Find amount of salt in tank at any time t ?

Let $Q(t)$ be the amount of salt in the tank. $\to$ (in gms)

Rate of change of salt = Rate in − Rate out

$$\Rightarrow \frac{dQ}{dt} = \gamma \frac{gm}{Lit} \times 2 \frac{lit}{min} - \frac{Q(t)}{120} \frac{gm}{Lit} \times 2 \frac{Lit}{min}.$$

$\hookrightarrow$ Units : $\frac{gm}{min}$.

$$\Rightarrow \boxed{\frac{dQ}{dt} = 2\gamma - \frac{Q}{60}}$$

At $t=0$; $Q=0$

Writing the equation in standard form,

$$\boxed{\frac{dQ}{dt} + \frac{Q}{60} = 2\gamma}$$

Integrating factor, $\phi = e^{\int \frac{1}{60} dt} = e^{\frac{t}{60}}$

$$\Rightarrow \quad e^{\frac{t}{60}} \frac{dQ}{dt} + e^{\frac{t}{60}} \frac{Q}{60} = 2\gamma e^{\frac{t}{60}}$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{\frac{t}{60}} Q\right) = 2\gamma e^{\frac{t}{60}}$$

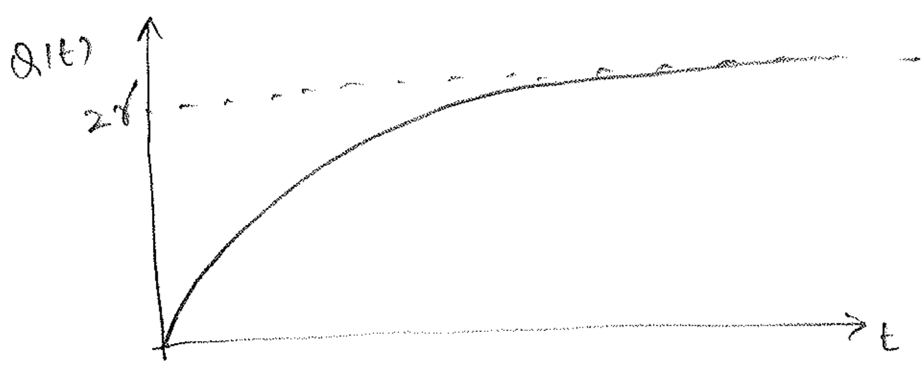
$$\Rightarrow \quad e^{\frac{t}{60}} \cdot Q = 2\gamma e^{\frac{t}{60}} + C$$

At $t = 0$; $Q = 0$

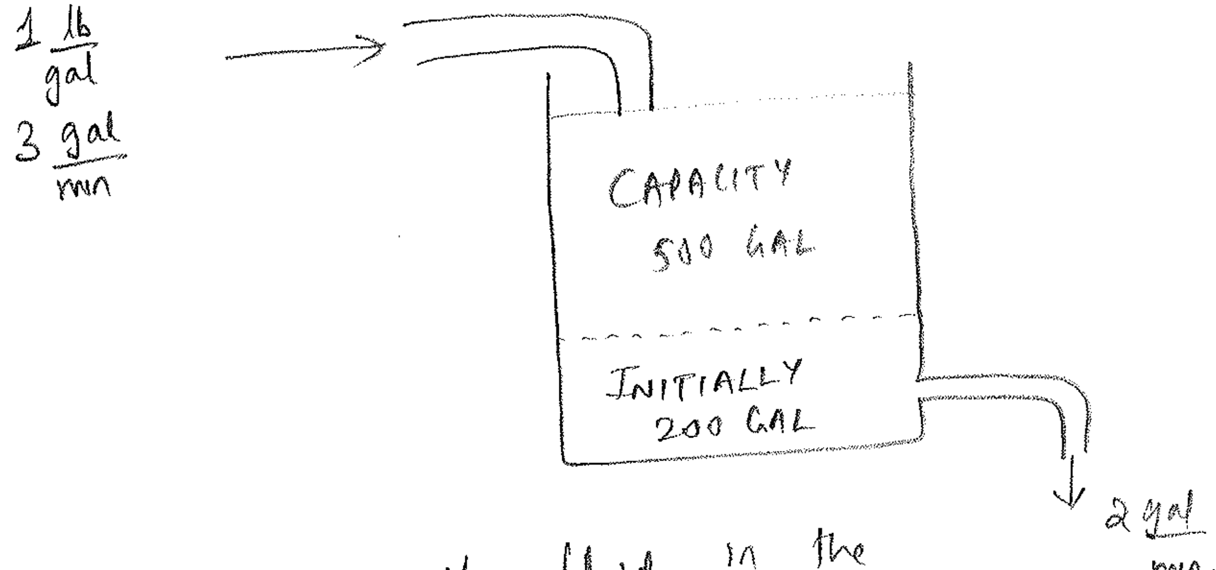
$$\Rightarrow \quad 0 = 2\gamma + C \quad \Rightarrow \quad C = -2\gamma$$

$$\Rightarrow \quad Q = 2\gamma - 2\gamma e^{-\frac{t}{60}} = 2\gamma\left(1 - e^{-\frac{t}{60}}\right)$$

$\therefore$ As $t \to \infty$; $Q \to 2\gamma$



(4)



At $t=0$; Volume of fluid in the tank, $V_0 = 200$ GAL.

Fluid volume increase at the rate of $1 \frac{gal}{min}$

(Inlet $3 \frac{gal}{min}$; Outlet $2 \frac{gal}{min}$
$\Rightarrow$ Increase $= (3-2) = 1 \frac{gal}{min}$)

Therefore, volume of fluid after t minutes $= (200+t)$ gal.

Let $Q(t)$ be amount of salt (in lb) in the tank.

$$\frac{dQ}{dt} = \text{Rate in} - \text{Rate Out}.$$

Rate In: $\quad 1 \frac{lb}{gal} \times 3 \frac{gal}{min} = 3 \frac{lb}{min}.$

Rate Out: $\quad \frac{Q(t)}{V(t)} \frac{lb}{gal} \times 2 \frac{gal}{min} = \frac{Q(t)}{200+t} \times 2$

$$\Rightarrow \quad \frac{dQ}{dt} = 3 - \frac{2Q}{200+t}$$

$$\Rightarrow \quad \frac{dQ}{dt} + \left(\frac{2}{200+t}\right) Q = 3$$

Integrating factor, $\phi = e^{\int \frac{2}{(200+t)} dt}$

$$= e^{2 \ln|200+t|}$$

$$= (200+t)^2$$

$$\Rightarrow \quad (200+t)^2 \frac{dQ}{dt} + 2(200+t) Q = 3 \cdot (200+t)^2$$

$$\Rightarrow \quad \frac{d}{dt}\left[(200+t)^2 Q\right] = 3 \cdot (200+t)^2$$

$$\Rightarrow \quad (200+t)^2 Q = (200+t)^3 + C$$

At $t = 0$; $Q(t=0) = 100$ lb

$$\Rightarrow (200)^2 \times 100 = (200)^3 + C \quad \Rightarrow C = -100 \times (200)^2$$

$$\Rightarrow \quad \boxed{Q(t) = (200+t) - \frac{100 \times (200)^2}{(200+t)^2}} \quad \text{for } t < 300 \text{ min.}$$

At $t = 300$ min ; $V = 500$ gal.

For $t > 300$; tank overflows.

Salt concentration at $t = 300$ min :

$$Q(300) = 500 - \frac{100 \times (200)^2}{(500)^2}$$

$$= 484 \text{ lb.}$$

Or, concentration $= \dfrac{Q(300)}{V(300)} = \dfrac{484}{500} \dfrac{\text{lb}}{\text{gal}} = \dfrac{121}{125} \dfrac{\text{lb}}{\text{gal}}$.

Infinite tank :-

As $t \to \infty$; $Q(t) \to \infty$, $V(t) \to \infty$.

But salt concentration ,

$$C = \lim_{t \to \infty} \frac{Q(t)}{V(t)} = \lim_{t \to \infty} \frac{(200+t) - \dfrac{100 \times (200)^2}{(200+t)^2}}{(200+t)}$$

$$= \lim_{t \to \infty} \left\{ 1 - \frac{100 \times (200)^2}{(200+t)^3} \right\}$$

$$= 1 \frac{\text{lb}}{\text{gal}}.$$

(18) $$\frac{du}{dt} = -K[u - T(t)]$$

where $T(t) = T_0 + T_1 \cos(\omega t)$

(a) Calculate solution in general form:

Now $$\frac{du}{dt} = -K[u - T_0 - T_1 \cos(\omega t)]$$

$$\Rightarrow \quad \frac{du}{dt} + Ku = K[T_0 + T_1 \cos(\omega t)]$$

Integrating factor $\phi = e^{\int K\,dt} = e^{Kt}$

$$\Rightarrow \quad \frac{d}{dt}(e^{Kt} \cdot u) = K T_0 e^{Kt} + K T_1 e^{Kt} \cos(\omega t)$$

Thus $$u e^{Kt} = T_0 e^{Kt} + K T_1 \int e^{Kt} \cos(\omega t)\,dt$$

We must calculate $I = \int e^{Kt} \cos(\omega t)\,dt$

We can use integration by parts, or use a clever trick.

Using the Euler formula; (see Wikipedia)

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

$\therefore \cos(\omega t) = Re\{e^{i\omega t}\}$ where Re = Real Part.

Now,

$$I = Re\left[\int e^{Kt} e^{i\omega t}\, dt\right] = Re\left[\int e^{(K+i\omega)t}\, dt\right]$$

$$= Re\left[\frac{1}{K+i\omega} e^{(K+i\omega)t}\right] = Re\left[\frac{K-i\omega}{K^2+\omega^2} \cdot e^{(K+i\omega)t}\right]$$

$$\Rightarrow I = \frac{e^{Kt}}{K^2+\omega^2} Re\left[(K-i\omega)\left[\cos(\omega t) + i\sin(\omega t)\right]\right]$$

$$= \frac{e^{Kt}}{K^2+\omega^2}\left[K\cos(\omega t) + \omega\sin(\omega t)\right]$$

Therefore, $$u e^{Kt} = T_0 e^{Kt} + \frac{KT_1 e^{Kt}}{K^2+\omega^2}\left[K\cos(\omega t) + \omega\sin(\omega t)\right] + C$$

Thus $$u(t) = \underbrace{Ce^{-Kt}}_{\substack{\text{Transient} \\ \text{part}}} + \underbrace{T_0 + \frac{KT_1}{K^2+\omega^2}\left[K\cos(\omega t) + \omega\sin(\omega t)\right]}_{\text{Steady State Part.}}$$

decays as $t \to \infty$

(b) $T_0 = 60°F$; $T_1 = 15°F$; $K = 0.2\ hr^{-1}$; $\omega = \frac{\pi}{12}$

Let's first rewrite $u(t)$ as follows:

$$u(t) = Ce^{-Kt} + T_0 + \frac{T_1 K}{\sqrt{K^2+\omega^2}}\left[\frac{K}{\sqrt{K^2+\omega^2}}\cos(\omega t) + \frac{\omega}{\sqrt{K^2+\omega^2}}\sin(\omega t)\right]$$

$$= Ce^{-Kt} + T_0 + \frac{T_1 K}{\sqrt{K^2+\omega^2}} \cos(\omega t - \delta)$$

where $\cos\delta = \dfrac{K}{\sqrt{K^2+\omega^2}}$

$\sin\delta = \dfrac{\omega}{\sqrt{K^2+\omega^2}}$

δ is the phase angle.

$\therefore$ $u(t) = Ce^{-Kt} + T_0 + R\cos(\omega t - \delta)$

where amplitude $R = \dfrac{T_1 K}{\sqrt{K^2+\omega^2}}$ (units of temperature)

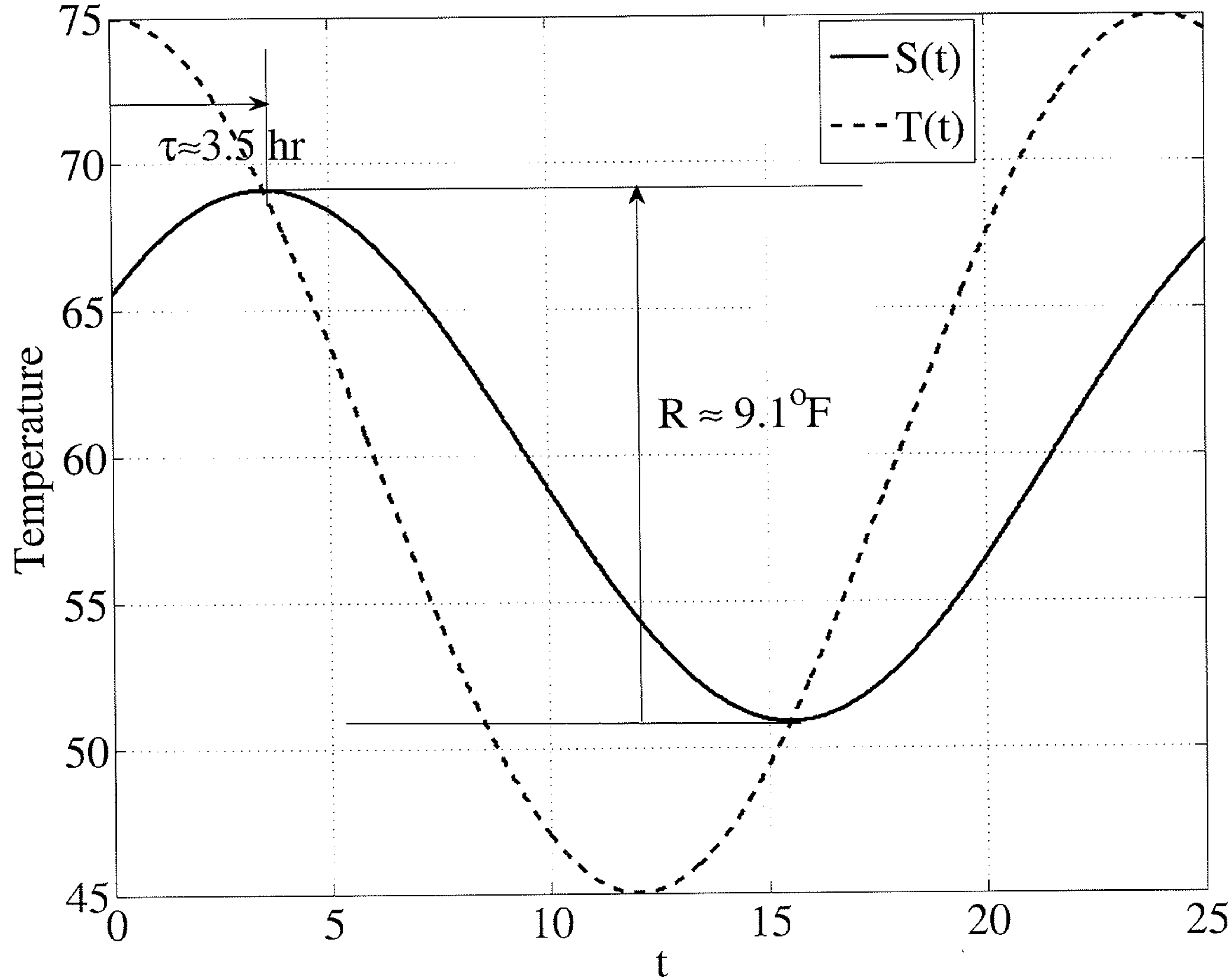
Using the numerical values, we get $R = 9.1062\,^\circ F$

The ambient temperature oscillates as

$T(t) = T_0 + T_1\cos(\omega t)$, whereas the

temperature of the box oscillates as

$u = Ce^{-Kt} + T_0 + R\cos[\omega(t - \tau)]$



75
70
65
60
55
50
45
0
5
10
15
20
25
Temperature
t
S(t)
T(t)
τ≈3.5 hr
R ≈ 9.1°F

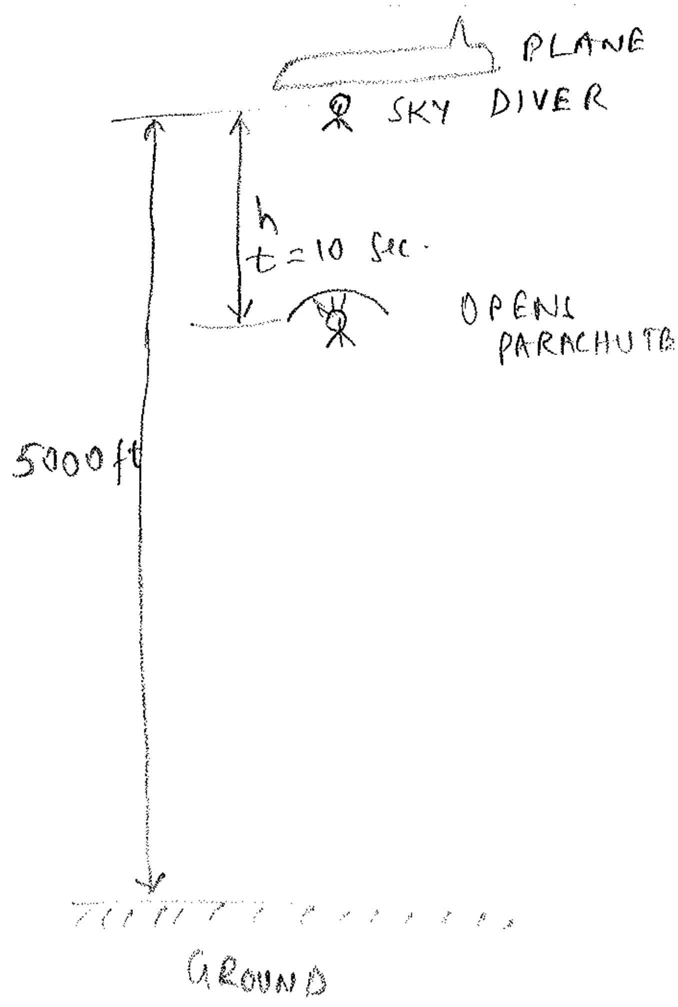
The phase lag $\tau = \frac{\delta}{\omega}$

Using the numerical values, we get

$$\tau = 3.508 \text{ hrs.}$$

(Notice that ω is in units of hr^{-1}).

R and τ can also be evaluated from a plot. See attached plot.



(23) $m = 180$ lb

Before $t = 10$ sec:

$$m\frac{dv}{dt} = mg - 0.75v$$

$$\Rightarrow \boxed{\begin{array}{l} \frac{dv}{dt} + \frac{0.75}{m}v = g \\ v(0) = 0 \end{array}}$$

After $t = 10$ sec.:

$$\boxed{\begin{array}{l} \frac{dv}{dt} + \frac{12}{m}v = g \\ v(10) = ? \end{array}}$$

For $t < 10$ sec:

$$\frac{dv}{dt} + \frac{0.75}{180} v = g$$

$$\Rightarrow \text{Integrating factor, } \phi = e^{\int \frac{0.75}{180} dt}$$

$$= e^{\frac{0.75}{180} t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{\frac{0.75}{180} t} v\right) = g e^{\frac{0.75}{180} t}$$

$$\Rightarrow e^{\frac{0.75}{180} t} v = g e^{\frac{0.75 t}{180}} \times \frac{180}{0.75} + c$$

At $t = 0$; $v = 0$ $\Rightarrow$ $0 = g \times \frac{180}{0.75} + c$

$$\Rightarrow c = -\frac{180 g}{0.75}$$

$$\Rightarrow \boxed{v = \frac{180 g}{0.75}\left(1 - e^{-\frac{0.75}{180} t}\right)}$$

Let us take $g \approx 32.174 \frac{ft^2}{sec}$ (Source: Wikipedia "Standard Gravity")

$$\Rightarrow v = 240 g \left(1 - e^{-\frac{t}{240}}\right)$$

At $t = 10$ sec., $v_{10} = 9.79\, g$

With $g = 32.174 \ \frac{ft}{sec^2}$,

$$\boxed{V_{10} = 314.983 \ ft/sec}$$

(b) Since $\frac{dh}{dt} = V \Rightarrow h = \int_0^t v\,dt \approx 1600 \ ft.$

(c) After 10 seconds:

Now the equation becomes

$$\frac{dv}{dt} + \frac{12}{180} v = g$$

$$\Rightarrow \boxed{\frac{dv}{dt} + \frac{v}{15} = g}$$

We can reset the time to get

$$v(0) = 9.79\,g$$

$\Rightarrow$ Integration factor $\phi = e^{\frac{t}{15}}$

$$\Rightarrow \quad v e^{t/15} = 15 g e^{t/15} + c$$

$$\Rightarrow \quad v = 15g + ce^{-t/15}$$

Now $v(0) = 9.79\,g \Rightarrow 9.79\,g = 15\,g + C$

$$\Rightarrow C = -5.21\,g$$

$$\Rightarrow \boxed{v = 15g - 5.21\,g\,e^{-t/15}}$$

(c) As $t \to \infty$; $V \to V_L$.

$$\Rightarrow V_L = 15g$$

$$= 482.61 \text{ ft/sec}$$

Extra Problem:

Equation: $$\frac{du}{dt} = -K(u-T)$$

$T = 15°C$

$u(0) = 50°C$

$K = 0.1 \text{ sec}^{-1}$

$$\frac{du}{dt} + Ku = KT \Rightarrow \frac{du}{dt} + 0.1u = 1.5$$

Integrating factor, $\phi = e^{0.1t}$

$$\Rightarrow \frac{d}{dt}\left(e^{0.1t} u\right) = 1.5\, e^{0.1t}$$

$$\Rightarrow e^{0.1t} u = \frac{1.5}{0.1} e^{0.1t} + C$$

$$\Rightarrow \boxed{u = 15 + Ce^{-0.1t}}$$

$u(0) = 50 \Rightarrow 50 = 15 + C \Rightarrow C = 35$

$$\Rightarrow \boxed{u = 15 + 35\, e^{-0.1t}}$$

Let τ be the time when $u = 25°C$.

$$\Rightarrow \quad 25 = 15 + 35e^{-0.1\tau}$$

$$\Rightarrow \quad 10 = 35e^{-0.1\tau}$$

$$\Rightarrow \quad e^{-0.1\tau} = \frac{10}{35}$$

$$\Rightarrow \quad -0.1\tau = \ln\left(\frac{10}{35}\right)$$

$$\Rightarrow \quad \boxed{\tau = 12.52 \text{ sec}}$$