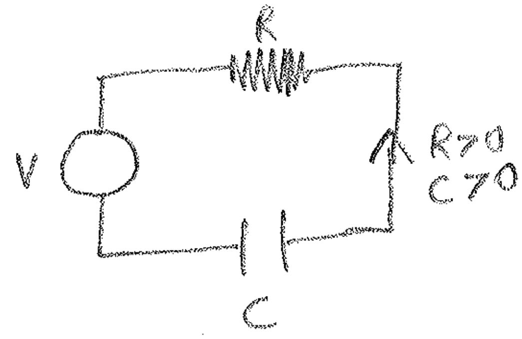
MATH 215/255 HW #2 HARISH DIXIT ①

SECTION 1.2 #17

$$\frac{dQ}{dt} + \frac{Q}{RC} = \frac{V}{R}$$



(i) Integrating factor:

$$\phi = e^{\int^t \frac{1}{RC}\,dt} = e^{\frac{t}{RC}}$$

Multiplying by ϕ: $$e^{\frac{t}{RC}} \frac{dQ}{dt} + e^{\frac{t}{RC}} \cdot \frac{Q}{RC} = \frac{V}{R} e^{\frac{t}{RC}}$$

$$\Rightarrow \frac{d}{dt}\left(e^{\frac{t}{RC}} \cdot Q\right) = \frac{V}{R} e^{\frac{t}{RC}}$$

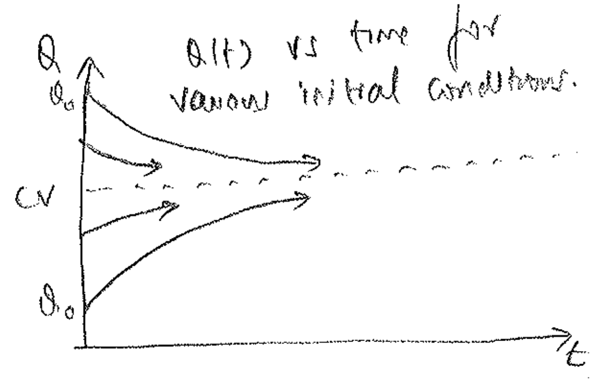
$$\Rightarrow e^{\frac{t}{RC}} Q = CV e^{\frac{t}{RC}} + A$$

↳ integration constant

$$\Rightarrow Q = CV + Ae^{-\frac{t}{RC}}$$

Now $Q(0) = Q_0 \Rightarrow Q_0 = CV + A \Rightarrow A = Q_0 - CV$

$$\therefore \boxed{Q(t) = CV + (Q_0 - CV)e^{-t/RC}}$$



For $Q_0 = 0$;

$$Q(t) = CV\left[1 - e^{-t/RC}\right]$$

(ii) As $t \to \infty$; $Q_L = \lim_{t\to\infty} Q(t) = CV$

(ii) Now suppose that battery is removed for $t \geq t_1$

The problem is

$$Q' + \frac{Q}{RC} = \begin{cases} \frac{V}{R} & 0 \leq t \leq t_1 \\ 0 & t > t_1 \end{cases}$$

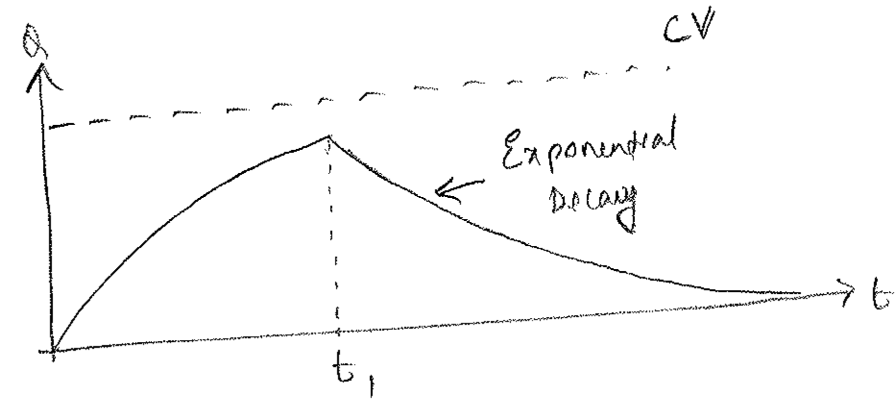
$Q(0) = Q_0$ and continuous across $t = t_1$.

Solving for Q in each case, we have the solutions :

$$Q(t) = \begin{cases} CV + (Q_0 - CV)\, e^{-t/RC} & ; \quad 0 \leq t \leq t_1 \\ Q(t_1)\, e^{-t/RC} & \quad t \geq t_1 \end{cases}$$

where $Q(t_1) = CV + (Q_0 - CV)\, e^{-t_1/RC}$

Thus



SECTION 2.1 # 2

2) $y' - 2y = t^2 e^{2t}$

The integrating factor is $\phi(t) = e^{\int -2\, dt}$

$= e^{-2t}$

Multiplying the equation by ϕ:

$e^{-2t} y' - e^{-2t} \cdot 2y = t^2 e^{2t} \cdot e^{-2t}$

$\Rightarrow \frac{d}{dt}(e^{-2t} y) = t^2$

$\Rightarrow e^{-2t} y = \frac{t^3}{3} + C \Rightarrow \boxed{y = \frac{t^3}{3} e^{2t} + C e^{2t}}$

General Solution

Expect $y \to \infty$ as $t \to \infty$.

5) $y' - 2y = 3e^t$

The integrating factor is $\phi(t) = e^{-\int 2\, dt} = e^{-2t}$

Thus, $e^{-2t} y' - 2y e^{-2t} = 3e^t e^{-2t}$

$\Rightarrow \frac{d}{dt}(e^{-2t} y) = 3e^{-t}$

$\Rightarrow e^{-2t} y = -3e^{-t} + C$

$\Rightarrow \boxed{y = -3e^t + Ce^{2t}}$

Now $y \to \infty$ as $t \to \infty$ if $c > 0$.

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(14) $y' + 2y = te^{-2t}$; $y(1) = 0$

Integrating factor is $\phi(t) = e^{\int 2\,dt} = e^{2t}$

Thus $e^{2t} y' + 2y e^{2t} = te^{-2t} \cdot e^{2t}$

$\Rightarrow \frac{d}{dt}(e^{2t} y) = t$

$\Rightarrow e^{2t} y = \frac{t^2}{2} + c$

$\Rightarrow \boxed{y = \frac{t^2}{2} e^{-2t} + ce^{-2t}}$

Now $y(1) = 0 \Rightarrow 0 = \frac{1}{2} e^{-2} + ce^{-2}$

$\Rightarrow c = \frac{-1}{2}$

$\Rightarrow y = \frac{t^2}{2} e^{-2t} - \frac{1}{2} e^{-2t}$

$$\boxed{y(t) = \left(\frac{t^2-1}{2}\right) e^{-2t}}$$

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(16) $y' + \left(\frac{2}{t}\right) y = \frac{\cos t}{t^2}$; $y(\pi) = 0$; $t > 0$

Integrating factor is $\phi = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = t^2$

Thus, $t^2y' + 2ty = \cos t$

$\Rightarrow \frac{d}{dt}(t^2 y) = \cos t$

$\Rightarrow t^2 y = \sin t + C$

$\Rightarrow \boxed{y = \frac{\sin t}{t^2} + \frac{C}{t^2}}$: General solution.

Now $y(\pi) = 0 \Rightarrow 0 = 0 + \frac{C}{\pi^2} \Rightarrow C = 0$

$\therefore \boxed{y(t) = \frac{\sin t}{t^2}}$: Particular Solution

(30) $y' - y = 1 + 3\sin t$; $y(0) = y_0$

To find a y_0 such that $y(t)$ is finite as $t \to \infty$.

The integrating factor is $\phi(t) = e^{\int -dt}$

$= e^{-t}$

Thus, $e^{-t}y' - e^{-t}y = e^{-t}(1 + 3\sin t)$

$\Rightarrow \frac{d}{dt}(e^{-t}y) = e^{-t} + 3\sin t\, e^{-t}$

$\Rightarrow e^{-t}y = -e^{-t} + 3\int \sin t\, e^{-t}\, dt$

To evaluate $\int \sin t \, e^{-t} \, dt$:

Let $I = \int e^{-t} \sin t \, dt$

Using integration by parts, we get

$$I = \sin t \left(\frac{e^{-t}}{-1}\right) - \int \cos t \left(\frac{e^{-t}}{-1}\right) dt$$

$$= -\sin t \, e^{-t} + \int \cos t \, e^{-t} \, dt + c_1$$

$$= -\sin t \, e^{-t} + \left\{ \cos t \left(\frac{e^{-t}}{-1}\right) - \int -\sin t \cdot \frac{e^{-t}}{-1} \, dt \right\} + c_1$$

$$= -\sin t \, e^{-t} + \left\{ -\cos t \, e^{-t} - \underbrace{\int \sin t \, e^{-t} dt}_{I} \right\} + c_1$$

$$\Rightarrow 2I = -\sin t \, e^{-t} - \cos t \, e^{-t} + c_1$$

$$\Rightarrow I = -\frac{(\sin t + \cos t)}{2} e^{-t} + c_1$$

$$\therefore e^{-t} y = -e^{-t} + 3 \left\{ -\frac{(\sin t + \cos t)}{2} e^{-t} + c_1 \right\}$$

$$= -e^{-t} - \frac{3}{2} (\sin t + \cos t) e^{-t} + 3c_1$$

$$\Rightarrow \boxed{y(t) = -1 - \frac{3}{2} (\sin t + \cos t) + 3c_1 e^{t}}$$

Now, $y(0) = y_0$

$$\Rightarrow y_0 = -1 - \frac{3}{2}(0+1) + 3C_1$$

$$= -1 - \frac{3}{2} + 3C_1$$

$$\Rightarrow 3C_1 = y_0 + \frac{5}{2}$$

$$\therefore y(t) = -1 - \frac{3}{2}(\sin t + \cos t) + \left(y_0 + \frac{5}{2}\right)e^t$$

As $t \to \infty$; $\sin t$ & $\cos t$ are bounded. Therefore, for $y(t)$ to be bounded, the coefficient of e^t has to vanish $\Longrightarrow$ $y_0 + \frac{5}{2} = 0$

$$\Rightarrow \boxed{y_0 = \frac{-5}{2}}$$

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(32) To show that all solutions of $2y' + ty = 2$ approach a limit as $t \to \infty$.

And find the limit.

Standard form: $y' + \left(\frac{t}{2}\right)y = 1$

Integrating factor is $\mu(t) = e^{\int \frac{t}{2}\,dt} = e^{t^2/4}$

Thus, $e^{t^2/4}y' + \left(\frac{t}{2}\right)e^{t^2/4}y = e^{t^2/4}$

$$\Rightarrow \frac{d}{dt}\left(y\,e^{t^2/4}\right) = e^{t^2/4}$$

$$\Rightarrow \quad y e^{t^2/4} = \int e^{t^2/4}\, dt + C$$

Now, use $y(0) = y_0$

$$\Rightarrow \quad y e^{t^2/4} = \int_0^t e^{\frac{s^2}{4}}\, ds + y_0$$

↳ s is a dummy variable.

↳ Integral vanishes when $t = 0$ $\quad \int_0^0 (\ ) = 0$

We can verify that when $t = 0$; $y = y_0$ is satisfied.

∴ Exact solution is

$$y(t) = e^{-t^2/4} \int_0^t e^{s^2/4}\, ds + y_0 e^{-t^2/4}$$

<u>What happens</u> when <u>$t \to \infty$?</u>

L'Hopital's rule! (First term)

$$\lim_{t \to \infty} \frac{\int_0^t e^{\frac{s^2}{4}}\, ds}{e^{t^2/4}} = \lim_{t \to \infty} \frac{e^{t^2/4}}{\frac{t}{2} e^{t^2/4}} =$$

$$= \lim_{t \to \infty} \frac{2}{t} \to 0$$

Therefore, $y \to 0$ as $t \to \infty$.

In general, as $t >> 1$, $\quad y \approx \frac{2}{t}$

Section 2.2

(1) $y' = \frac{x^2}{y}$

This is a nonlinear, but separable equation.

$$y\,dy = x^2\,dx$$

$$\Rightarrow \int y\,dy = \int x^2\,dx + C_1$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + C_1$$

$$\Rightarrow y^2 = \frac{2}{3}x^2 + C \quad \text{where} \quad C = 2C_1$$

$$\Rightarrow \boxed{y = \pm\left[\frac{2x^2}{3} + C\right]^{1/2}}$$

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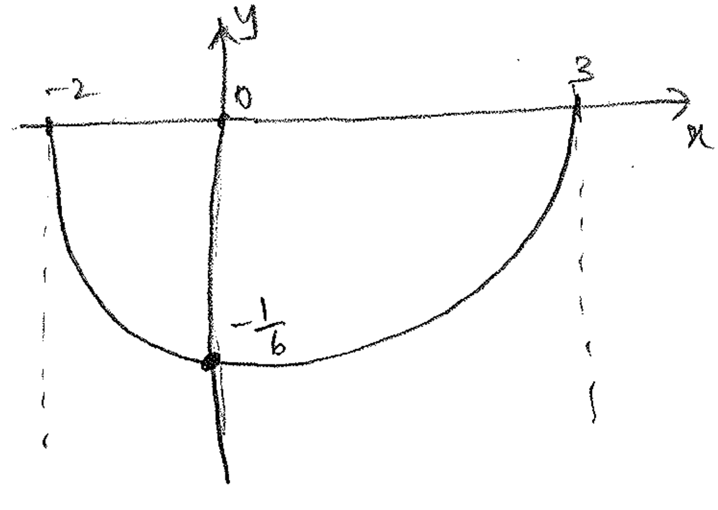
(9) $y' = (1-2x)\,y^2$; $y(0) = \frac{-1}{6}$

$$\frac{dy}{y^2} = (1-2x)\,dx \Rightarrow \frac{-1}{y} = x - x^2 + c$$

Now $y(0) = \frac{-1}{6} \Rightarrow 6 = c \Rightarrow y = \frac{1}{x^2 - x - 6}$

Thus $\boxed{y = \frac{1}{(x-3)(x+2)}}$

Defined for $-2 < x < 3$
(NOT OUTSIDE)



(15) $$y' = \frac{2x}{1+2y} \;; \quad y(2) = 0$$

Now $(1+2y)\,dy = 2x\,dx \Rightarrow y + y^2 = x^2 + C$

Now $y(2) = 0 \Rightarrow 0 = 4 + C \Rightarrow C = -4$

Thus $y^2 + y = x^2 - 4$

$$\Rightarrow y^2 + y + (4 - x^2) = 0$$

$$\Rightarrow y = \frac{-1 \pm \sqrt{1 - 4(4 - x^2)}}{2}$$

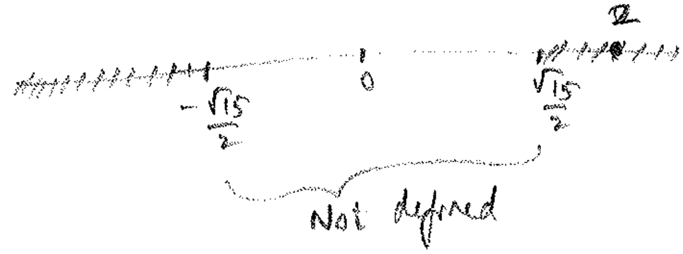
$$y = \frac{-1 \pm \sqrt{4x^2 - 15}}{2}$$

Of the two solutions, only one of them satisfies the initial condition. Check for yourself that $+$ sign is needed.

$$\therefore \; y(x) = \frac{-1 + \sqrt{4x^2 - 15}}{2}$$

Solution exists for $4x^2 - 15 > 0 \Rightarrow \boxed{|x| \geq \frac{\sqrt{15}}{2}}$

However, the initial condition is defined at $x = 2$



Therefore, the only interval which also satisfies the initial condition is : $x > \frac{\sqrt{15}}{2}$

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(32) $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$

(a) Notice that $\frac{dy}{dx} = \frac{x}{2y} + \frac{3}{2}\frac{y}{x}$

$$= \frac{1}{2\left(\frac{y}{x}\right)} + \frac{3}{2}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Thus equation is in Homogeneous form.

(b) Let $V = \frac{y}{x}$.

So $y = Vx \Rightarrow y' = xV' + V$

So $xV' + V = \frac{1}{2V} + \frac{3V}{2} \Rightarrow xV' = \frac{1}{2V} + \frac{V}{2}$

Then $xV' = \frac{1}{2V}(1+V^2) \Rightarrow x\frac{dv}{dx} = \frac{1+V^2}{2V}$

Then $\frac{2V}{1+V^2}\,dV = \frac{dx}{x} \Rightarrow \ln|1+V^2| = \ln|x| + \ln C$

$$= \ln\{|x|C\}$$

Therefore, $\quad |1+V^2| = c|x|$

We substitute $V = \frac{y}{x}$ now.

$$\left|1+\frac{y^2}{x^2}\right| = c|x|$$

$$\Rightarrow \quad |x^2+y^2| = c|x|^3$$

Thus $\quad x^2+y^2 = cx^3 \quad$ for $x>0$ and $c>0$

and $\quad x^2+y^2 = -cx^3 \quad$ for $x<0$ and $c>0$

(34) $$\frac{dy}{dx} = \frac{-(4x+3y)}{2x+y}$$

(a) Dividing the numerator and denominator by x, we have

$$\frac{dy}{dx} = -\left[\frac{4+3\left(\frac{y}{x}\right)}{2+\left(\frac{y}{x}\right)}\right] = f\left(\frac{y}{x}\right)$$

Thus the equation is in homogeneous form.

(b) Let $\quad V = \frac{y}{x} \Rightarrow y = Vx$

$$\Rightarrow y' = xV' + V$$

So $\quad xV' + V = -\left[\frac{4+3V}{2+V}\right] \Rightarrow xV' = \frac{-4-3V}{2+V} - V$

$$\Rightarrow \quad x v' = \frac{-4-3v-2v-v^2}{2+v}$$

$$\Rightarrow \quad x \frac{dv}{dx} = -\frac{(v^2+5v+4)}{2+v}$$

$$\Rightarrow \quad \frac{2+v}{v^2+5v+4}\, dv = \frac{-dx}{x}$$

(Use $v^2+5v+4 = (v+1)(v+4)$)

$$\Rightarrow \quad \frac{2+v}{(v+1)(v+4)}\, dv = -\frac{dx}{x}$$

To integrate left hand side, we need to use partial fractions.

Recall $$\frac{2+v}{(v+1)(v+4)} = \frac{A}{v+1} + \frac{B}{v+4}$$

To find A & B:

$$\Rightarrow \quad 2+v = A(v+4) + B(v+1)$$

$$\therefore \quad \left.\begin{array}{l} A+B = 1 \\ 4A+B = 2 \end{array}\right\} \quad \begin{array}{l} A = \frac{1}{3} \\ B = \frac{2}{3} \end{array}$$

$$\Rightarrow \int \frac{2+v}{(v+1)(v+4)}\, dv = \frac{1}{3}\int \frac{dv}{v+1} + \frac{2}{3}\int \frac{dv}{v+4}$$

$$= \frac{1}{3}\ln|v+1| + \frac{2}{3}\ln|v+4|$$

$$= \frac{1}{3}\left[\ln|v+1| + \ln|v+4|^2\right]$$

$$= \frac{1}{3}\left[\ln\left(|v+1|\,|v+4|^2\right)\right]$$

We therefore get

$$\frac{1}{3}\ln\left[|v+1||v+4|^2\right] = -\ln|x| + \ln c_1$$

$$= \ln\left(\frac{c_1}{|x|}\right)$$

$$\Rightarrow \ln\left\{|v+1||v+4|^2\right\} = \ln\left(\frac{c_1^3}{|x|^3}\right)$$

Substituting $v = \frac{y}{x}$, we have

where $c = c_1^3$

$$\left|\frac{y}{x}+1\right|\left|\frac{y}{x}+4\right|^2 = \frac{c}{|x|^3}$$

$$\Rightarrow \boxed{|x+y||y+4x|^2 = c}$$