MATH 215/255 HW #11 H. DINIT

Section 7.8 #1

$$\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x} \qquad A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

Let $\vec{x} = \vec{\xi} e^{\lambda t} \Rightarrow \vec{x}' = \vec{\xi} \lambda e^{\lambda t}$

$$\Rightarrow \vec{\xi} \lambda e^{\lambda t} = A\vec{x}$$
$$= A\vec{\xi} e^{\lambda t}$$

$$\Rightarrow (A\vec{\xi} - \lambda \vec{\xi}) e^{\lambda t} = 0$$

$$\Rightarrow (A - \lambda I)\vec{\xi} = 0$$

Eigenvalues: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (3-\lambda)(-1-\lambda) + 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)^2 = 0$$

$\Rightarrow \lambda_1 = \lambda_2 = 1$: Repeated root.

Eigenvectors:
NOW $(A - \lambda_1 I)\vec{\xi}^{(1)} = 0 \Rightarrow (A - I)\vec{\xi}^{(1)} = 0$

$$\Rightarrow \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \vec{\xi}^{(1)} = 0 \Rightarrow \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \vec{\xi} = 0$$

$$\Rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus $\vec{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t}$ is the first solution.

Now put for the second solution

$$\vec{x} = \vec{\xi}^{(1)} te^t + \vec{\eta} e^t$$

Then $\vec{x}' = \vec{\xi}^{(1)} te^t + \vec{\xi}^{(1)} e^t + \vec{\eta} e^t$

Thus $\vec{\xi}^{(1)} te^t + \vec{\xi}^{(1)} e^t + \vec{\eta} e^t = A\left(\vec{\xi}^{(1)} te^t + \vec{\eta} e^t\right)$

Comparing similar terms on both sides:

$O(te^t)$: $\quad A\vec{\xi}^{(1)} = \vec{\xi}^{(1)}$: same as the first solution.

$O(e^t)$: $\quad A\vec{\eta} - \vec{\eta} = \vec{\xi}^{(1)}$

$$\Rightarrow \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Row operation: $R_2 = R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\longrightarrow \eta_1 - 2\eta_2 = 1$$

Let $\eta_2 = k \Rightarrow \eta_1 = 1 + 2k$

$$\Rightarrow \vec{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1+2k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

same as $\vec{\xi}^{(1)}$

Thus $\vec{x}^{(2)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} e^t$ is second solution.

Without loss of generality, we can set $k = 0$.

Then, the general solution is

$$\vec{X} = c_1\begin{bmatrix}2\\1\end{bmatrix}e^t + c_2\left\{\begin{bmatrix}2\\1\end{bmatrix}te^t + \begin{bmatrix}1\\0\end{bmatrix}e^t\right\}$$

Section 7-8 #2 $\quad \vec{X}' = \begin{bmatrix}4 & -2\\8 & -4\end{bmatrix}\vec{X}$

Eigenvalues: $|A-\lambda I| = 0 \Rightarrow (4-\lambda)(-4-\lambda)+16=0 \rightarrow \lambda^2=0$
$\rightarrow \lambda_1 = 0$

Eigenvector: $(A-\lambda_1 I)\vec{\xi} = 0 \longrightarrow \begin{bmatrix}4 & -2\\8 & -4\end{bmatrix}\vec{\xi}^{(1)} = 0 \rightarrow \vec{\xi} = \begin{bmatrix}1\\2\end{bmatrix}$

Second Solution: Put $\vec{X} = \vec{\xi}^{(1)}te^{\lambda_1 t} + \vec{\eta}e^{\lambda_1 t}$
$= \vec{\xi}^{(1)}t + \vec{\eta}$

$\Rightarrow \vec{X}' = \vec{\xi}^{(1)} \longrightarrow$ ~~$A\vec{\xi}^{(1)} + \vec{\xi}t + \vec{\eta} = A($~~

$$\vec{\xi}^{(1)} = A(\vec{\xi}^{(1)}t + \vec{\eta})$$

$$\longrightarrow A\vec{\eta} = \vec{\xi}^{(1)} \Rightarrow \begin{bmatrix}4 & -2\\8 & -4\end{bmatrix}\begin{bmatrix}\eta_1\\\eta_2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix}4 & -2\\0 & 0\end{bmatrix}\begin{bmatrix}\eta_1\\\eta_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \quad \text{— } (*)$$

$$\longrightarrow 4\eta_1 - 2\eta_2 = 0$$

$$\longrightarrow \begin{bmatrix}\eta_1\\\eta_2\end{bmatrix} = \begin{bmatrix}1/4\\0\end{bmatrix} + k\begin{bmatrix}1/2\\1\end{bmatrix}$$

If we set $k=0$, we have

$$\vec{X} = \begin{bmatrix}1\\2\end{bmatrix}t + \begin{bmatrix}1/4\\0\end{bmatrix}$$

So general solution is

$$\vec{X} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\left[\begin{pmatrix}1\\2\end{pmatrix}t + \begin{pmatrix}1/4\\0\end{pmatrix}\right]$$

Remark: In solving (*), we can also make $\eta_1 = 0$ and $\eta_2 = -1/2$ or any other choice that satisfies (*).

So $\vec{X} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\left[\begin{pmatrix}1\\2\end{pmatrix}t + \begin{pmatrix}0\\-1/2\end{pmatrix}\right]$ also works.

Section 7.8 # 7 $\quad \vec{X}' = \begin{bmatrix}1 & -4\\4 & -7\end{bmatrix}\vec{X}$; $\vec{X}(0) = \begin{pmatrix}3\\2\end{pmatrix}$

Now $|A-\lambda I| = 0 \rightarrow (1-\lambda)(-7-\lambda)+16 = 0 \rightarrow (\lambda+3)^2 = 0$

$\rightarrow \lambda^2 + 6\lambda + 9 = 0$

$\rightarrow \lambda_1 = -3$

Then $(A+3I)\vec{\xi}^{(1)} = 0 \rightarrow \begin{bmatrix}4 & -4\\4 & -4\end{bmatrix}\vec{\xi}^{(1)} = 0 \rightarrow \vec{\xi} = \begin{bmatrix}1\\1\end{bmatrix}$

Now put $\vec{X} = \vec{\xi}^{(1)} t e^{-3t} + \vec{\eta} e^{-3t}$ so that

$(A+3I)\vec{\eta} = \vec{\xi}^{(1)} \rightarrow \begin{bmatrix}4 & -4\\4 & -4\end{bmatrix}\vec{\eta} = \begin{bmatrix}1\\1\end{bmatrix}$

$\rightarrow \begin{bmatrix}4 & -4\\0 & 0\end{bmatrix}\begin{bmatrix}\eta_1\\\eta_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$\rightarrow 4\eta_1 - 4\eta_2 = 1 \rightarrow \eta_1 = \eta_2 + 1/4$

$\therefore \vec{\eta} = \begin{bmatrix}1/4\\0\end{bmatrix} + k\begin{bmatrix}1\\1\end{bmatrix}$. We can set $k=0$ without loss of generality.

Hence, $\vec{X}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$

Thus general solution is

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right\}$$

Now $\vec{X}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$

So $c_1 = 2$

$c_1 + \frac{c_2}{4} = 3 \rightarrow c_2 = 4$

$$\text{So } \vec{X} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + 4 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} \right]$$

$$= 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + e^{-3t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix} e^{-3t}$$

Section 7.9 # 1

$$\vec{X}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{X} + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Homogeneous Problem:- $\vec{X}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{X}$

$|A - \lambda I| = 0 \rightarrow (2-\lambda)(-2-\lambda) + 3 = 0 \rightarrow \lambda^2 - 1 = 0$

$\rightarrow \lambda = \pm 1$

$\rightarrow \lambda_1 = 1, \quad \lambda_2 = -1$

Now $(A - \lambda_1 I)\vec{\xi}^{(1)} = 0 \rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \vec{\xi}^{(1)} = 0 \longrightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$(A-\lambda_2 I)\vec{\xi}^{(2)} = 0 \longrightarrow \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \vec{\xi}^{(2)} = 0 \longrightarrow \vec{\xi} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\therefore \quad \vec{X}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t \quad ; \quad \vec{X}^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

$$\therefore \quad \vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} = \vec{\Psi}\vec{c}$$: Homogeneous Solution

where $\vec{\Psi} = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$: Fundamental matrix

& $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Particular Solution:- Let $\vec{X} = \vec{\Psi}\vec{u}$ where $\vec{u} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$

$$\Rightarrow \vec{X}' = \vec{\Psi}'\vec{u} + \vec{\Psi}\vec{u}'$$ where $\vec{g}(t) = \begin{bmatrix} e^t \\ t \end{bmatrix}$

$$\Rightarrow \vec{\Psi}'\vec{u} + \vec{\Psi}\vec{u}' = A\vec{\Psi}\vec{u} + \vec{g}(t)$$

$$\Rightarrow \underbrace{(\vec{\Psi}' - A\vec{\Psi})}_{=0}\vec{u} + \vec{\Psi}\vec{u}' = \vec{g}(t)$$

$$\rightarrow \vec{\Psi}\vec{u}' = \vec{g}$$

$$\rightarrow \vec{u}' = \vec{\Psi}^{-1}\vec{g} = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}^{-1} \begin{bmatrix} e^t \\ t \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix}}{[3-1]} \begin{bmatrix} e^t \\ t \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 - te^{-t} \\ -e^{2t} + te^t \end{bmatrix}$$

$$\Rightarrow u_1' = \frac{1}{2}(3 - te^{-t}) \longrightarrow u_1 = c_1 + \frac{1}{2}\left[3t + (t+1)e^{-t}\right]$$

$$u_2' = \frac{1}{2}\left[-e^{2t} + te^t\right] \longrightarrow u_2 = c_2 - \frac{1}{4}\left[-2te^t + e^{2t} + 2e^{t}\right]$$
$$= c_2 - \frac{1}{4}\left[e^{2t} + 2(1-t)e^t\right]$$

$$\because \vec{X} = \vec{\Psi}\cdot\vec{u}$$

$$= \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \begin{bmatrix} c_1 + \frac{3}{2}t + \frac{1}{2}(t+1)e^{-t} \\ c_2 - \frac{1}{4}e^{2t} - \frac{1}{2}(1-t)e^t \end{bmatrix}$$

$$= \begin{bmatrix} \left\{c_1e^t + \frac{3}{2}te^t + \frac{1}{2}(t+1)\right\} + \left\{c_2e^{-t} - \frac{1}{4}e^t - \frac{1}{2}(1-t)\right\} \\ \left\{c_1e^t + \frac{3}{2}te^t + \frac{1}{2}(t+1)\right\} + \left\{3c_2e^{-t} - \frac{3}{4}e^t - \frac{3}{2}(1-t)\right\} \end{bmatrix}$$

$$= c_1\begin{bmatrix}1\\1\end{bmatrix}e^t + c_2\begin{bmatrix}1\\3\end{bmatrix}e^{-t} + \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}te^t + \frac{1}{2}(t+1)\begin{bmatrix}1\\1\end{bmatrix}$$
$$- \frac{1}{4}\begin{bmatrix}1\\3\end{bmatrix}e^t - \frac{1}{2}(1-t)\begin{bmatrix}1\\3\end{bmatrix}$$

$$= c_1\begin{bmatrix}1\\1\end{bmatrix}e^t + c_2\begin{bmatrix}1\\3\end{bmatrix}e^{-t} + \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}te^t - \frac{1}{4}\begin{bmatrix}1\\3\end{bmatrix}e^t + t\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}$$

Section 7.9 #4

$$\vec{X}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{X} + \begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix}$$

Let us first solve this problem using variation of parameters.

Homogeneous Problem:- $|A - \lambda I| = 0 \rightarrow (1-\lambda)(-2-\lambda) - 4 = 0$

$$\rightarrow \lambda^2 + \lambda - 6 = 0$$
$$\rightarrow (\lambda+3)(\lambda-2) = 0$$
$$\rightarrow \lambda_1 = 2, \quad \lambda_2 = -3$$

- $(A-2I)\vec{\xi}^{(1)} = 0 \longrightarrow \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \vec{\xi}^{(1)} = 0 \longrightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\longrightarrow \vec{X}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

- $(A+3I)\vec{\xi}^{(2)} = 0 \longrightarrow \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \vec{\xi}^{(2)} = 0 \longrightarrow \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$$\longrightarrow \vec{X}^{(2)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$$

$$\vec{X}_H = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} = \vec{\Psi} \cdot \vec{c}$$

where $\vec{\Psi} = \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix}$

$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Particular Solution:. Let $\vec{X} = \vec{\Psi} \cdot \vec{u}$ where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\longrightarrow \vec{\Psi}\,\vec{u}\,' = \vec{g}$ where $\vec{g} = \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$

$\longrightarrow \vec{u}\,' = \vec{\Psi}^{-1}\vec{g} = \cancel{\begin{bmatrix} e^{2t} & e^{-3t} \\ & -4e^{-3t} \end{bmatrix}}$

$$= \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix}^{-1} \cdot \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4e^{-2t} & e^{-2t} \\ e^{3t} & -e^{3t} \end{bmatrix} \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 4e^{-4t} - 2e^{-t} \\ e^{t} + 2e^{4t} \end{bmatrix}$$

$$\therefore \quad u_1' = \frac{1}{5}\left(4e^{-4t} - 2e^{-t}\right) \longrightarrow u_1 = -\frac{1}{5}\left(e^{-4t} - 2e^{-t}\right) + c_1$$

$$u_2' = \frac{1}{5}\left(e^{t} + 2e^{4t}\right) \longrightarrow u_2 = \frac{1}{5}\left(e^{t} + \frac{1}{2}e^{4t}\right) + c_2$$

$$\therefore \vec{X} = \vec{\Psi} \cdot \vec{u} = \begin{bmatrix} c_1 e^{2t} + c_2 e^{-3t} + \frac{1}{2} e^{t} \\ c_1 e^{2t} - 4c_2 e^{-3t} - e^{-2t} \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

Method of Undetermined Coefficients :-

Rewrite the equation as $\qquad A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

$$\vec{X}' = A\vec{X} + \underbrace{e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{g_1(t)} + \underbrace{e^{t} \begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{g_2(t)}$$

• With $g_1(t)$: Neglect g_2 & find the particular solution.

Let $\vec{X}_p = \vec{V} e^{-2t}$

$\Rightarrow \vec{X}_p' = -2\vec{V} e^{-2t} \Rightarrow -2\vec{V} e^{-2t} = A\vec{V} e^{-2t} + e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\rightarrow (A + 2I)\vec{V} \cancel{e^{-2t}} = -\cancel{e^{-2t}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} 3V_1 + V_2 = -1 \\ 4V_1 = 0 \end{matrix}$

$\Rightarrow V_1 = 0 \rightarrow V_2 = -1 \Rightarrow \vec{V} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\Rightarrow \vec{X}_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-2t}$

• With $g_2(t)$: Neglect g_1 & find the particular solution.

Let $\vec{X}_p = \vec{V} e^{t}$

$\Rightarrow \vec{V} e^{t} = A\vec{V} e^{t} + e^{t} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$\Rightarrow \quad (A-I)\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\rightarrow \quad v_2 = 0$

$\quad 4v_1 - 3v_2 = 2 \quad \Rightarrow v_1 = 1/2$

$\therefore \vec{v} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$\therefore \vec{X}_p = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^t$

$\therefore$ Particular solution $= \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^t$

: Adding the two particular solutions.

General solution:

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^t$$

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$$\vec{X}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{X} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

We will use undetermined coefficients for the particular solution.

Homogeneous Problem:-

$|A - \lambda I| = 0 \rightarrow (2-\lambda)(-2-\lambda) + 3 = 0 \rightarrow \lambda^2 - 1 = 0$
$\rightarrow \lambda_1 = 1, \ \lambda_2 = -1$

• $(A - \lambda_1 I)\vec{\xi}^{(1)} = 0 \rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \vec{\xi}^{(1)} = 0 \rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- $(A-\lambda_2 I)\vec{\xi}^{(2)}=0 \longrightarrow \begin{bmatrix}3 & -1\\ 3 & -1\end{bmatrix}\vec{\xi}^{(2)}=0 \longrightarrow \vec{\xi}^{(2)}=\begin{bmatrix}1\\3\end{bmatrix}$

Homogeneous Solution : $\vec{X}_H = c_1\begin{bmatrix}1\\1\end{bmatrix}e^t + c_2\begin{bmatrix}1\\3\end{bmatrix}e^{-t}$

Particular Solution:- Since $\vec{g}(t)=e^t\begin{bmatrix}1\\-1\end{bmatrix}$, & since e^t is part of the homogeneous solution, we let

$$\vec{X}_p = \vec{v}te^t + \vec{\eta}e^t$$

So $\vec{X}_p' = \vec{v}e^t + \vec{v}te^t + \vec{\eta}e^t$

$$\Rightarrow \vec{v}e^t + \vec{v}te^t + \vec{\eta}e^t = A(\vec{v}te^t + \vec{\eta}e^t) + \begin{bmatrix}1\\-1\end{bmatrix}e^t$$

Comparing similar terms:-

$O(te^t)$: $A\vec{v} = \vec{v} \longrightarrow \begin{bmatrix}1 & -1\\ 3 & -3\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=0 \Rightarrow v_1=v_2$

Let $v_1=\mu \Rightarrow v_2=\mu$

$\Rightarrow \vec{v}=\mu\begin{bmatrix}1\\1\end{bmatrix}$

$O(e^t)$: $A\vec{\eta}-\vec{\eta} = \vec{v}-\begin{bmatrix}1\\-1\end{bmatrix}$ where $\vec{v}=\mu\begin{bmatrix}1\\1\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}1 & -1\\ 3 & -3\end{bmatrix}\vec{\eta} = \begin{bmatrix}\mu-1\\ \mu+1\end{bmatrix}$$

Now row reduce : $R_2 \to R_2 - 3R_1$

$$\longrightarrow \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}\vec{\eta} = \begin{bmatrix}\mu-1\\ -2\mu+4\end{bmatrix}$$

Set $-2\mu+4=0 \longrightarrow \mu=2$

$\therefore \eta_1-\eta_2 = \mu-1 = 1$.

If $\eta_2=k$, $\eta_1=1+k$

$\Rightarrow \vec{\eta}=\begin{bmatrix}1+k\\k\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}+k\begin{bmatrix}1\\1\end{bmatrix}$

Without loss of generality, let $k = 0$

$$\Rightarrow \vec{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then $\vec{X}_p = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$: Particular Solution

Now adding the homogeneous solution, the general solution is

$$\vec{X} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

<u>Section 7.9 #11</u> $\vec{X}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$

Consider the homogeneous problem:

$|A - \lambda I| = 0 \rightarrow (2-\lambda)(-2-\lambda) + 5 = 0 \rightarrow \lambda^2 + 1 = 0$

$\rightarrow \lambda_{\pm} = \pm i$

Now $(A - \lambda_+ I)\vec{\xi} = 0 \rightarrow \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \vec{\xi} = \vec{0} \rightarrow \begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix} \vec{\xi} = \vec{0}$

Now $(2-i)\xi_1 = 5\xi_2$. Let $\xi_1 = 5$ $\Rightarrow \xi_2 = 2-i$ $\Rightarrow \vec{\xi} = \begin{bmatrix} 5 \\ 2-i \end{bmatrix}$

$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\therefore \vec{X} = \left\{\begin{bmatrix} 5 \\ 2 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} e^{it}$: Complex first Solution.

We now separate the real & imaginary part of the above solution:

$$\vec{X}_R = \mathrm{Re}\left[\left\{\begin{bmatrix}5\\2\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}\right\}(\cos t + i\sin t)\right] = \begin{bmatrix}5\\2\end{bmatrix}\cos t + \begin{bmatrix}0\\1\end{bmatrix}\sin t$$

$$\vec{X}_I = \mathrm{Im}\left[\left\{\begin{bmatrix}5\\2\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}\right\}(\cos t + i\sin t)\right] = -\begin{bmatrix}0\\1\end{bmatrix}\cos t + \begin{bmatrix}5\\2\end{bmatrix}\sin t$$

Thus $\vec{X}_R = \begin{bmatrix}5\cos t\\ 2\cos t + \sin t\end{bmatrix}$; $\vec{X}_I = \begin{bmatrix}5\sin t\\ 2\sin t - \cos t\end{bmatrix}$

The solution to the homogeneous problem is

$$\vec{X} = c_1 \vec{X}_R + c_2 \vec{X}_I$$

Particular Solution:- We consider solution to $\vec{y}$ that satisfies

$$\vec{y}\,' = A\vec{y} + e^{it}\begin{bmatrix}0\\1\end{bmatrix}.$$

Since $\cos t = \mathrm{Re}[e^{it}]$, we should have $\vec{x} = \mathrm{Re}\{\vec{y}\}$.

Now since i is one of the two roots $\lambda_\pm$, we must guess

$$\vec{y} = \vec{v}\, t e^{it} + \vec{\eta}\, e^{it}$$

Then $\vec{y}\,' = \vec{v} e^{it} + it\,\vec{v} e^{it} + i\vec{\eta} e^{it}$

Substituting, we have

$$it\,\vec{v} e^{it} + \vec{v} e^{it} + i\vec{\eta} e^{it} = A\left(\vec{v} t e^{it} + \vec{\eta} e^{it}\right) + e^{it}\begin{bmatrix}0\\1\end{bmatrix}$$

Comparing similar terms on both sides:-

$O(te^{it})$: $A\vec{v} = i\vec{v} \longrightarrow (A - iI)\vec{v} = 0 \rightarrow \vec{v} = \mu\begin{bmatrix}5\\2-i\end{bmatrix}$

where μ is a scalar.

$O(e^{it})$:

$$\vec{v} + i\vec{\eta} = A\vec{\eta} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad (A - iI)\vec{\eta} = \vec{v} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \mu \begin{bmatrix} 5 \\ 2-i \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \vec{\eta} = \begin{bmatrix} 5\mu \\ \mu(2-i) - 1 \end{bmatrix}$

Row operation: $\quad R_2 \to R_2 - \frac{R_1}{2-i}$

$$\Rightarrow \begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix} \vec{\eta} = \begin{bmatrix} 5\mu \\ \mu(2-i) - 1 - \frac{5\mu}{2-i} \end{bmatrix}$$

We must set $\quad \mu(2-i) - 1 = \frac{5\mu}{2-i} = \frac{5\mu(2+i)}{5} = \mu(2+i)$

So $\quad 2\mu - i\mu - 1 = 2\mu + i\mu \quad \to \quad 2\mu = i$

$\to \quad 2i\mu = -1 \quad \to \quad \mu = i/2$

Now we must solve

$$\begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix} \vec{\eta} = \begin{bmatrix} 5\mu \\ 0 \end{bmatrix}$$

$\to \quad (2-i)\eta_1 - 5\eta_2 = 5\mu = \frac{5i}{2}$

Let $\eta_1 = 2+i$. Then $\quad 5 - 5\eta_2 = \frac{5i}{2}$

$$\to \eta_2 = \frac{1}{5}\left(5 - \frac{5i}{2}\right) = \frac{1}{5}\left(\frac{10-5i}{2}\right)$$

$$\Rightarrow \vec{\eta} = \begin{bmatrix} 2+i \\ 1 - i/2 \end{bmatrix}$$

Now $\vec{y} = \frac{i}{2}\begin{bmatrix} 5 \\ 2-i \end{bmatrix} te^{it} + \begin{bmatrix} 2+i \\ 1-i/2 \end{bmatrix} e^{it}$

Now $\vec{X}_p = Re[\vec{y}] = Re\left\{ \frac{i}{2}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) t(\cos t + i\sin t)\right\}$

$$+ Re\left\{\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}\right)(\cos t + i\sin t)\right\}$$

$\Rightarrow \vec{X}_p = -\frac{1}{2}\begin{bmatrix} 5 \\ 2 \end{bmatrix} t\sin t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} t\cos t + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}\sin t$

$$= \begin{pmatrix} -5/2 \sin t \\ -\sin t + \frac{1}{2}\cos t \end{pmatrix} t + \begin{pmatrix} 2\cos t - \sin t \\ \cos t + \frac{1}{2}\sin t \end{pmatrix}$$

This yields the general solution:

$$\vec{X} = c_1\begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2\begin{bmatrix} 5\sin t \\ 2\sin t - \cos t \end{bmatrix} + \begin{bmatrix} -5/2 \sin t \\ 1/2 \cos t - \sin t \end{bmatrix} t$$

$$+ \begin{bmatrix} 2\cos t - \sin t \\ \cos t + \frac{1}{2}\sin t \end{bmatrix}$$

Section 9.1 # 1

$\vec{X}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\vec{X}$ $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$

(a) $|A - \lambda I| = 0 \rightarrow (3-\lambda)(-2-\lambda) + 4 = 0$

$\rightarrow \lambda^2 - \lambda - 6 + 4 = 0 \rightarrow \lambda^2 - \lambda - 2 = 0$

$\rightarrow (\lambda - 2)(\lambda + 1) = 0$

$\rightarrow \lambda_1 = 2, \quad \lambda_2 = -1$

- $(A-\lambda_1 I)\vec{\xi}=0$ $\rightarrow$ $\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\vec{\xi}=0$ $\longrightarrow$ $\vec{\xi}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

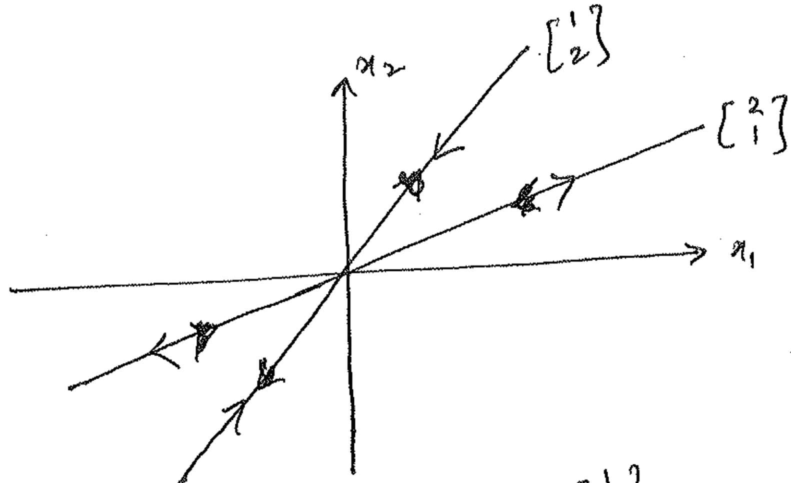
  $\lambda_1=2$

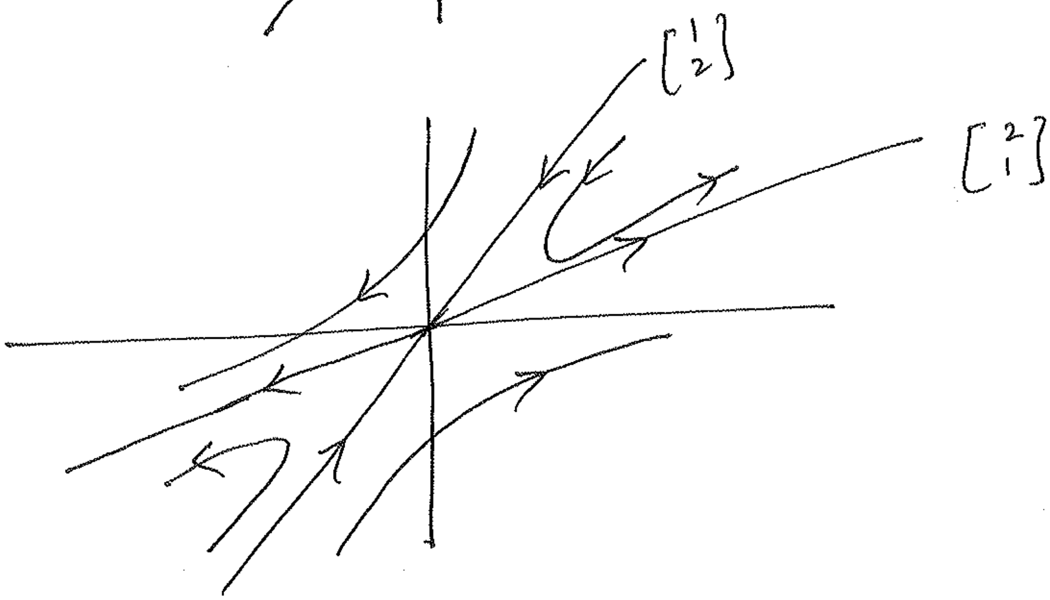
- $(A-\lambda_2 I)\vec{\xi}=0$ $\rightarrow$ $\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}\vec{\xi}=0$ $\rightarrow$ $\vec{\xi}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  $\lambda_2=-1$

(b) Since $\lambda_1 > 0$ & $\lambda_2 < 0$, critical point $(0,0)$ is a saddle.

(c)





Section 9.1 #2

$$\vec{x}\,' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x} \qquad A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$$

(a) $|A - \lambda I| = 0 \rightarrow (5-\lambda)(1-\lambda) + 3 = 0 \rightarrow \lambda^2 - 6\lambda + 8 = 0$

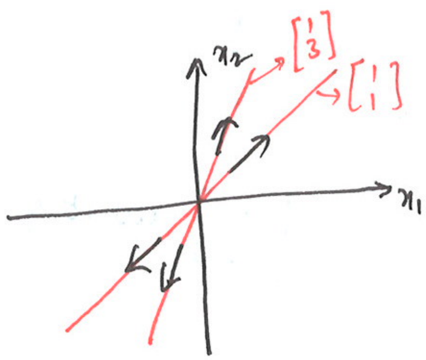
$\rightarrow \lambda = \dfrac{6 \pm \sqrt{36 - 4 \cdot 8}}{2}$

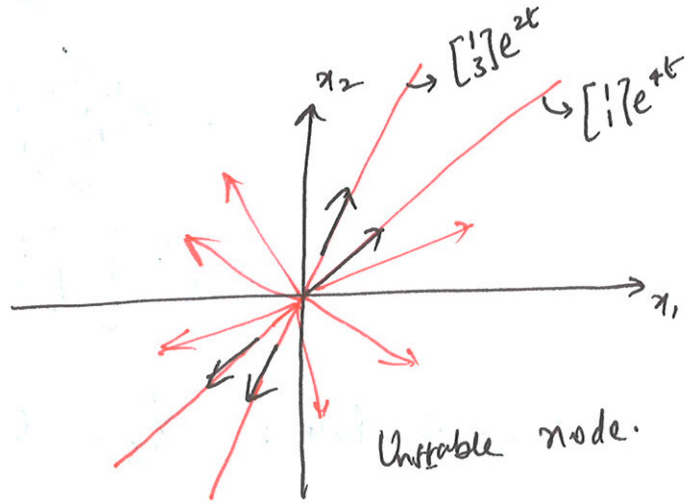
$= \dfrac{6 \pm 2}{2} \rightarrow \lambda_1 = 2, \ \lambda_2 = 4$

- $(A - \lambda_1 I)\vec{\xi}^{(1)} = 0 \rightarrow \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \vec{\xi}^{(1)} = \vec{0} \rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
- $(A - \lambda_2 I)\vec{\xi}^{(2)} = 0 \rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \vec{\xi}^{(2)} = \vec{0} \rightarrow \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) Since $\lambda_1 > 0$ & $\lambda_2 > 0$, the critical point $(0,0)$ is an unstable node.

(c)





Section 9.1 #7

$$\vec{x}\,' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \vec{x} \qquad A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

(a) $|A - \lambda I| = 0 \rightarrow (3-\lambda)(-1-\lambda) + 8 = 0 \rightarrow \lambda^2 - 2\lambda + 5 = 0$

$\rightarrow \lambda = \dfrac{2 \pm \sqrt{4 - 20}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

$$\bullet \quad (A-\lambda_1 I)\vec{\xi} = 0 \longrightarrow \left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1+2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{\xi} = 0$$

$$\longrightarrow \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}\vec{\xi} = 0$$

$$\longrightarrow \begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix}\vec{\xi} = 0 \longrightarrow \begin{matrix} (1-i)\xi_1 - \xi_2 = 0 \\ 2\xi_1 - (1+i)\xi_2 = 0 \end{matrix}$$

$$\longrightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \vec{X} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^{(1+2i)t}$$

Real Part : $$\vec{X}_R = Re\left\{\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^{t} \cdot e^{2it}\right\}$$

$$= e^{t} Re\left\{\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)(\cos 2t + i\sin 2t)\right\}$$

$$= e^{t}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 2t\right\}$$

Imaginary Part : $$\vec{X}_I = e^{t}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos 2t\right\}$$

$$\therefore \vec{X}_R = e^{t}\begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix}$$

$$\vec{X}_I = e^{t}\begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$$

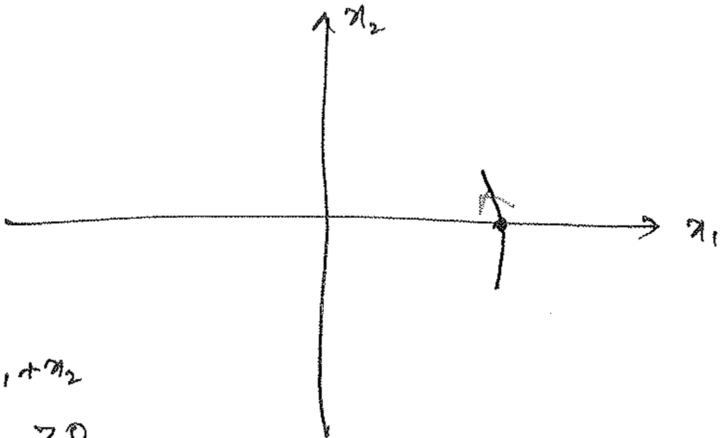
Since λ_1 & λ_2 are complex conjugates, the critical point is a spiral. Since $Re[\lambda_\pm] > 0$, it is an unstable spiral.

$x_1' = 5x_1 - x_2$

$x_2' = 3x_1 + x_2$

Choose a point

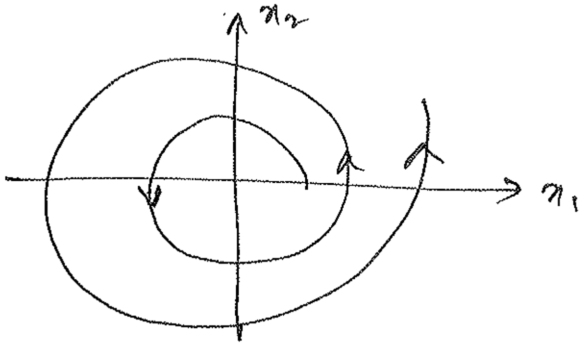
$x_2 = 0,\ x_1 > 0$



Since $x_2' = 3x_1 + x_2$

$= 3x_1 > 0$

$\Rightarrow$ x_2 increases with time.



EXTRA PROBLEM

$$\vec{X}' = \begin{bmatrix} \alpha & 1 \\ -2 & -3 \end{bmatrix} \vec{X} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$ $\hookrightarrow$ constant vector

(i) Let $\vec{X}_p = \vec{\xi}$

$$\Rightarrow \vec{X}_p' = 0 \qquad \Rightarrow \vec{\xi} = -A^{-1} \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\rightarrow 0 = A\vec{\xi} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\Rightarrow \vec{\xi} = - \frac{\begin{bmatrix} -3 & -1 \\ 2 & \alpha \end{bmatrix}}{-3\alpha + 2} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \frac{1}{3\alpha - 2} \begin{bmatrix} -25 \\ 10 + 10\alpha \end{bmatrix}$$

General Solution : $\vec{X} = c_1 \vec{V}_1 e^{\lambda_1 t} + c_2 \vec{V}_2 e^{\lambda_2 t} + \vec{\xi}$

where $\vec{\xi} = \frac{1}{3\alpha - 2} \begin{bmatrix} -25 \\ 10 + 10\alpha \end{bmatrix}$

This formula is valid provided that $\alpha \neq \frac{2}{3}$ ie;

we need $|A| \neq 0$ since $|A| = \frac{1}{(2-3\alpha)}\begin{bmatrix} -3 & -1 \\ 2 & \alpha \end{bmatrix}$

(ii) Recall that

sum of eigenvalues = Trace

Product of eigenvalues = Determinant.

We require $\lim_{t\to\infty} \vec{X}(t) = \vec{\xi}$

In other words, we want the homogeneous part to vanish.

$\Rightarrow \quad Re(\lambda_1) < 0 \quad \& \quad Re(\lambda_2) < 0.$

Therefore, we need

Trace $A < 0$ & $\operatorname{Det}(A) > 0$

$-3\alpha + 2 > 0$

$\Rightarrow \quad -3 + \alpha < 0 \quad , \quad \alpha < \frac{2}{3}$

$\rightarrow \quad \alpha < 3 \quad ,$

These are both satisfied when $\alpha < \frac{2}{3}$.

Now, suppose $\alpha = \frac{2}{3}$, then what is the particular solution?

we write $\vec{X} = \vec{v}t + \vec{\eta} \quad \rightarrow \quad \vec{v} = (A\vec{v})t + A\vec{\eta} + \vec{b}$

$\vec{b} \downarrow \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

Now set $A\vec{v} = 0 \quad \rightarrow \quad \vec{v} = \mu\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ μ is any scalar

Thus $A\vec{\eta} + \vec{b} = \mu\begin{bmatrix} -3 \\ 2 \end{bmatrix} \rightarrow A\vec{\eta} = \mu\begin{bmatrix} -3 \\ 2 \end{bmatrix} - \vec{b}$

So $\begin{bmatrix} 2/3 & 1 \\ -2 & -3 \end{bmatrix}\vec{\eta} = \begin{bmatrix} -5-3\mu \\ -10+2\mu \end{bmatrix}$

Now use row reduction

$$\begin{bmatrix} 2/3 & 1 \\ 0 & 0 \end{bmatrix} \vec{\eta} = \begin{bmatrix} -5-3\mu \\ -25-7\mu \end{bmatrix}$$

Now, we must set $\mu = -25/7$ to solve the system.

Hence $\frac{2}{3}\eta_1 + \eta_2 = -5 + \frac{25}{7} \cdot 3 = \frac{40}{7}$

Set $\eta_1 = 0 \Rightarrow \eta_2 = \frac{40}{7}$ (To find any solution).

Then, $\vec{X}_p = \frac{-25}{7}\begin{bmatrix} -3 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ \frac{40}{7} \end{bmatrix}$

So $\vec{X}_p = \begin{bmatrix} 75/7 \\ -50/7 \end{bmatrix} t + \begin{bmatrix} 0 \\ 40/7 \end{bmatrix}$