MATH 215/255 HOMEWORK #10

Section 7.5 #3

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} \qquad A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

(a) Put $\vec{x} = \vec{\xi} e^{\lambda t}$

$$\Rightarrow \vec{x}' = \vec{\xi} \lambda e^{\lambda t}$$

Substituting in $\vec{x}' = A\vec{x}$, we get $\vec{\xi} \lambda e^{\lambda t} = A \vec{\xi} e^{\lambda t}$

$$\Rightarrow (A - \lambda I)\vec{\xi} = 0$$

Eigenvalues: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)(-2-\lambda) + 3 = 0$$

$$\Rightarrow -(4-\lambda^2) + 3 = 0 \Rightarrow \lambda^2 - 1 = 0$$

So $\lambda = \pm 1$. Let $\lambda_1 = 1$, $\lambda_2 = -1$

Eigenvectors: • with $\lambda_1 = 1$, we have $(A - \lambda_1 I)\vec{\xi}^{(1)} = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} \xi_1 - \xi_2 &= 0 \\ \& \ 3\xi_1 - 3\xi_2 &= 0 \end{aligned}\right\} \xi_1 = \xi_2$$

$$\therefore \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

• with $\lambda_2 = -1$, we get $(A + I)\vec{\xi}^{(2)} = 0$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \left.\begin{aligned} 3\xi_1 - \xi_2 &= 0 \\ 3\xi_1 - \xi_2 &= 0 \end{aligned}\right\} \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

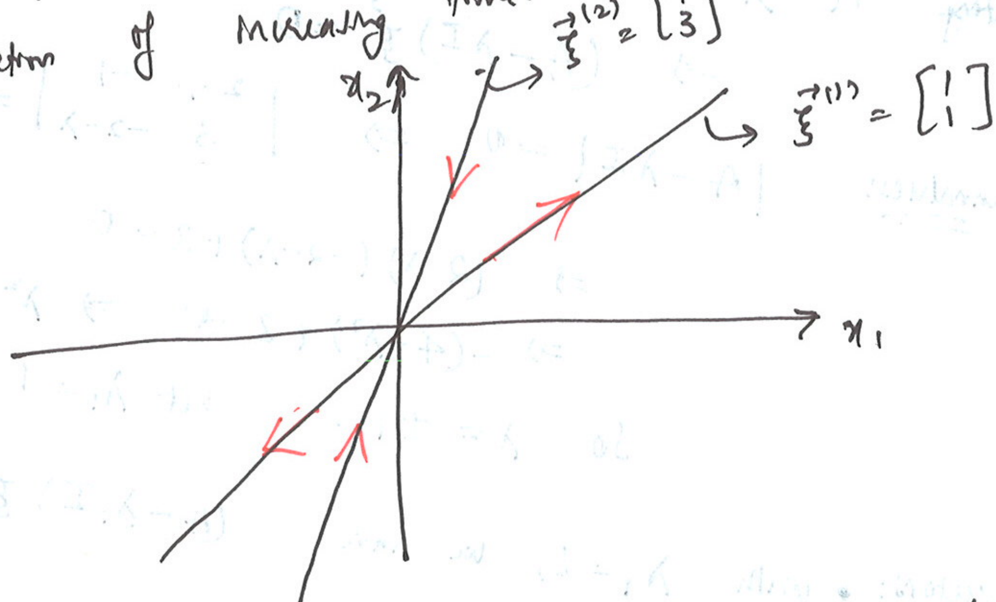
$$\therefore \quad \vec{X}^{(1)} = \vec{\xi}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} \quad : \text{Growing solution}$$

$$\text{and} \quad \vec{X}^{(2)} = \vec{\xi}^{(2)} e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} \quad : \text{Decaying solution.}$$

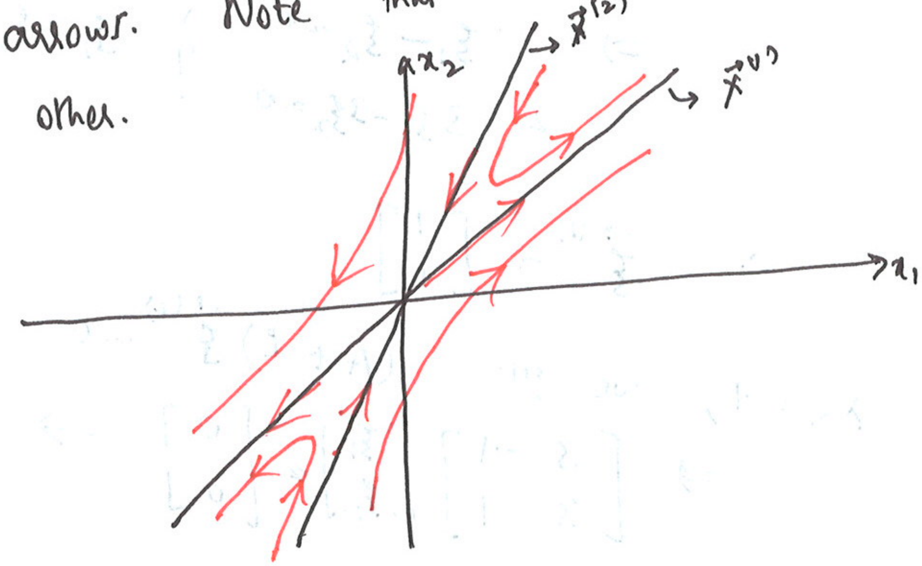
General solution:- $\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$

(b) Plotting the solution:

- First plot the eigenvectors & show arrows showing the direction of increasing time.



- Now fill in the plot with smooth curves obeying the arrows. Note that no two curves can cross each other.



As $t \to \infty$; we conclude that

$$\vec{X} \sim c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t \quad \text{when } c_1 \neq 0.$$

$$\to \infty$$

Section 7.5 #7

$$\vec{X}' = A\vec{X} \quad \text{with} \quad A = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$$

(a) Put $\vec{X} = \vec{\xi} e^{\lambda t} \longrightarrow (A - \lambda I)\vec{\xi} = 0$

Eigenvalues: $|A - \lambda I| = 0$

so $\begin{vmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{vmatrix} = 0$

so $(4-\lambda)(-6-\lambda) + 24 = 0 \Rightarrow \lambda^2 + 2\lambda = 0$

$\longrightarrow \lambda_1 = 0, \quad \lambda_2 = -2$

- $(A - \lambda_1 I)\vec{\xi}^{(1)} = 0 \Rightarrow \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \left.\begin{matrix} 4\xi_1 - 3\xi_2 = 0 \\ 8\xi_1 - 6\xi_2 = 0 \end{matrix}\right\} 4\xi_1 - 3\xi_2 = 0$$

$$\therefore \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \vec{\xi}^{(1)} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

- $(A - \lambda_2 I)\vec{\xi}^{(2)} = 0 \Rightarrow \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$

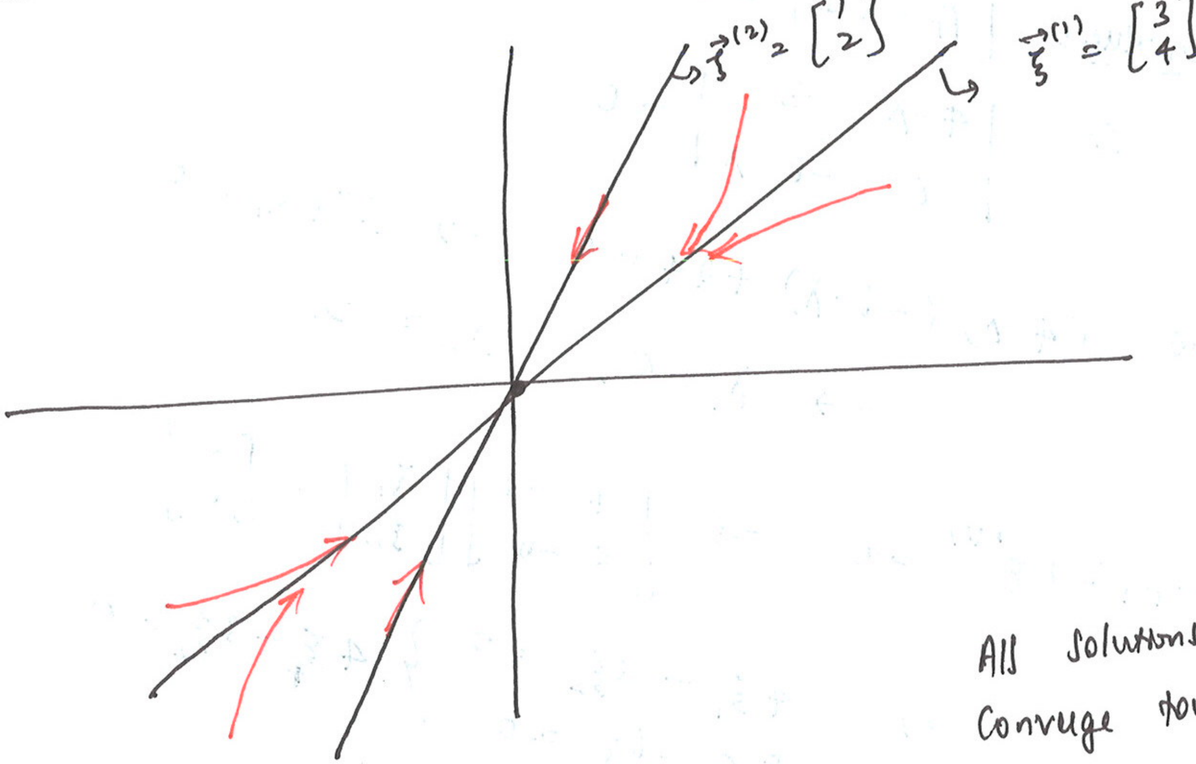
$$\Rightarrow \quad \left.\begin{aligned} 6\xi_1 - 3\xi_2 &= 0 \\ 8\xi_1 - 4\xi_2 &= 0 \end{aligned}\right\} \; 2\xi_1 = \xi_2$$

$$\therefore \quad \vec{\xi}^{(2)} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

<u>General</u> <u>Solution</u>:-
$$\vec{X} = c_1 e^{\lambda_1 t} \vec{\xi}^{(1)} + c_2 e^{\lambda_2 t} \vec{\xi}^{(2)}$$

$$= c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$$

(b) As $t \to \infty$; $\vec{X} \to c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ unless $c_1 = 0$



All solutions (curves) converge toward $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Section 7.5 # 16

$\vec{x}' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \vec{x}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Put $\vec{x} = \vec{\xi} e^{\lambda t}$. We get $(A - \lambda I)\vec{\xi} = 0$

So $|A - \lambda I| = 0 \to (-2-\lambda)(4-\lambda) + 5 = 0$

$\to \lambda^2 - 2\lambda - 3 = 0$

So $(\lambda - 3)(\lambda + 1) = 0 \to \lambda_1 = 3, \lambda_2 = -1$

Eigenvectors:

- $\lambda_1 = 3$: $(A - \lambda_1 I)\vec{\xi}^{(1)} = 0 \to \begin{bmatrix} -5 & 1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$

  So $\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

- $\lambda_2 = -1$: $(A - \lambda_2 I)\vec{\xi}^{(2)} = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$

  $\Rightarrow \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

General Solution is

$$\vec{x} = c_1 \vec{\xi}^{(1)} e^{\lambda_1 t} + c_2 \vec{\xi}^{(2)} e^{\lambda_2 t}$$

Hence $\boxed{\vec{x} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}}$

Given that $\vec{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Recall

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{(ad-bc)}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \frac{-1}{4} \begin{bmatrix} 1 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \frac{-1}{4} \begin{bmatrix} 1-3 \\ -5+3 \end{bmatrix} = \frac{-1}{4} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\therefore c_1 = c_2 = \frac{1}{2}$$

Thus $\vec{X}(t) = \frac{1}{2} \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

As $t \to \infty$; $\vec{X}(t) \to \frac{1}{2} \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t}$

Section 7.5 #29

$ay'' + by' + cy = 0$, $a \neq 0$, a, b & c constants

(a) Let $x_1 = y$, $x_2 = y'$. $\Rightarrow x_1' = x_2$

$\therefore x_1' = y'$. But $y' = x_2$

& $x_2' = y''$ from the equation, $y'' = -\frac{b}{a} y' - \frac{c}{a} y$

$\Rightarrow x_2' = \frac{-b}{a} x_2 - \frac{c}{a} x_1$

$$\therefore \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} & ? & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The elements of the matrix can be obtained from the two equations:

$$x_1' = x_2$$
$$x_2' = -\frac{b}{a} x_2 - \frac{c}{a} x_1$$

$$\therefore \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}$$

(b) Now Eigenvalues of A are obtained from

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)\left(-\frac{b}{a} - \lambda\right) + \frac{c}{a} = 0 \Rightarrow \lambda^2 + \lambda \frac{b}{a} + \frac{c}{a} = 0$$

So that $a\lambda^2 + b\lambda + c = 0$

This ~~E~~ is same as the characteristic equation associated with putting $y = e^{\lambda t}$ in $ay'' + by' + cy = 0$

Section 7.6 # 1

$$\vec{x}' = A\vec{x} \quad \text{with} \quad A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

(a) Put $\vec{x} = e^{\lambda t}\vec{\xi}$.

Then $(A - \lambda I)\vec{\xi} = 0 \implies |A - \lambda I| = 0$

This yields $(3-\lambda)(-1-\lambda) + 8 = 0$

$\Rightarrow \lambda^2 - 2\lambda + 5 = 0 \longrightarrow \lambda^2 - 2\lambda + 1 = -4$

Thus $\lambda_\pm = 1 \pm 2i$

- Eigenvectors: $(A - \lambda_+ I)\vec{\xi} = 0$

$$\Rightarrow \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}\vec{\xi} = 0 \longrightarrow \begin{bmatrix} 1-i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$$

Thus $(1-i)\xi_1 = \xi_2$.

Let $\xi_1 = 1$, then $\xi_2 = 1-i$

$$\text{So} \quad \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{is a complex eigenvector.}$$

$$\therefore \vec{x}^{(1)} = \vec{\xi}^{(1)} e^{\lambda_+ t} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(1+2i)t}$$

- Let us now split $\vec{x}^{(1)}$ into real & imaginary parts:

Real part: $$\vec{x}_R = Re\left\{\vec{x}^{(1)}\right\} = Re\left\{\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)e^{(1+2i)t}\right\}$$

Recall $e^{(1+2i)t} = e^{t} \cdot e^{2it}$

$$= e^{t} \cdot (\cos 2t + i \sin 2t)$$

$$\therefore \vec{X}_R = e^{t} \cdot \mathrm{Re}\left\{ \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) (\cos 2t + i \sin 2t) \right\}$$

$$= e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos 2t - e^{t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(2t)$$

$$= e^{t} \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix}$$

Similarly imaginary part:

$$\vec{X}_I = \mathrm{Im}\left\{ \vec{X}^{(1)} \right\} = \mathrm{Im}\left\{ \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{t} (\cos 2t + i \sin 2t) \right\}$$

$$= e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t} \cos(2t)$$

$$= e^{t} \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$$

Now, the general solution is

$$\vec{X} = c_1 \vec{X}_R + c_2 \vec{X}_I = c_1 e^{t} \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + c_2 e^{t} \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$$

(b) Since e^{t} grows as $t \to \infty$, we have an <u>Unstable</u> <u>Spiral</u> point.

How to determine direction of spiral:-

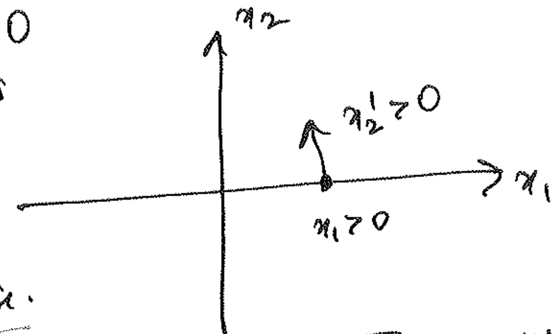
from the original equation, we have

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ 4x_1 - x_2 \end{bmatrix}$$

when $x_2 = 0$, then $x_2' = 4x_1 - x_2 = 4x_1$

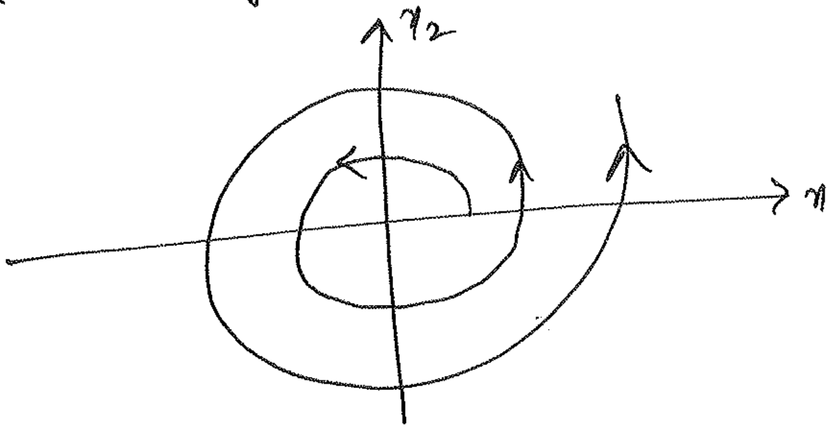
$\therefore$ If $x_1 > 0$, $x_2' > 0$

$\Rightarrow$ x_2 is increasing with time



$\Rightarrow$ counter-clockwise.

Both the spirals will be counter clockwise. They would look something like this:



Section 7.6 #5

$$\vec{X}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \vec{X}$$

(a) Put $\vec{X} = \vec{\xi} e^{\lambda t}$.

We get $(A - \lambda I)\vec{\xi} = 0$

$\Rightarrow |A - \lambda I| = 0 \Rightarrow (1-\lambda)(-3-\lambda) + 5 = 0$

So $\lambda^2 + 2\lambda + 2 = 0$

So $\lambda^2 + 2\lambda + 1 = -1 \Rightarrow (\lambda+1)^2 = -1$

$\longrightarrow \lambda + 1 = \pm i$

$\longrightarrow \lambda_\pm = -1 \pm i$

Eigenvectors: Now $(A - \lambda_+ I)\vec{\xi} = 0$

$$\Rightarrow \begin{bmatrix} 1-(-1+i) & -1 \\ 5 & -3-(-1+i) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus $(2-i)\xi_1 = \xi_2$

$5\xi_1 = (2+i)\xi_2 \longrightarrow$ Multiplying by $(2-i)$, this becomes

$5(2-i)\xi_1 = 5\xi_2 \Rightarrow (2-i)\cdot\xi_1 = \xi_2$

$\therefore (2-i)\xi_1 = \xi_2$. Let $\xi_1 = 1 \Rightarrow \xi_2 = 2-i$

$$\therefore \vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix} \longrightarrow \text{Complex eigen vector.}$$

Now $\vec{X} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix} e^{(-1+i)t} = e^{-t} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} e^{it}$

Real part: $\vec{X}_R = \text{Re}\left\{ e^{-t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) (\cos t + i \sin t) \right\}$

$$= e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos t + e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t$$

Imaginary part: $\vec{X}_I = \text{Im}\left\{ e^{-t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) (\cos t + i \sin t) \right\}$

$$= e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin t - e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t$$

$\therefore \vec{X}_R = e^{-t} \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix}$

$\vec{X}_I = e^{-t} \begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}$

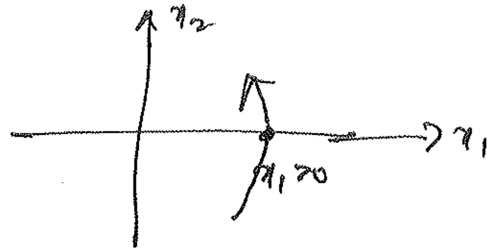
Both decay to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ with time.
$\Rightarrow$ Stable spiral point.

General solution:

$$\vec{X} = c_1 e^{-t} \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}$$

(b) We now have a stable spiral point. To determine the direction of rotation, we let $x_1 > 0$ & $x_2 = 0$, i.e. looking for direction when the spiral crosses the x_1-axis.

From the equation, we have

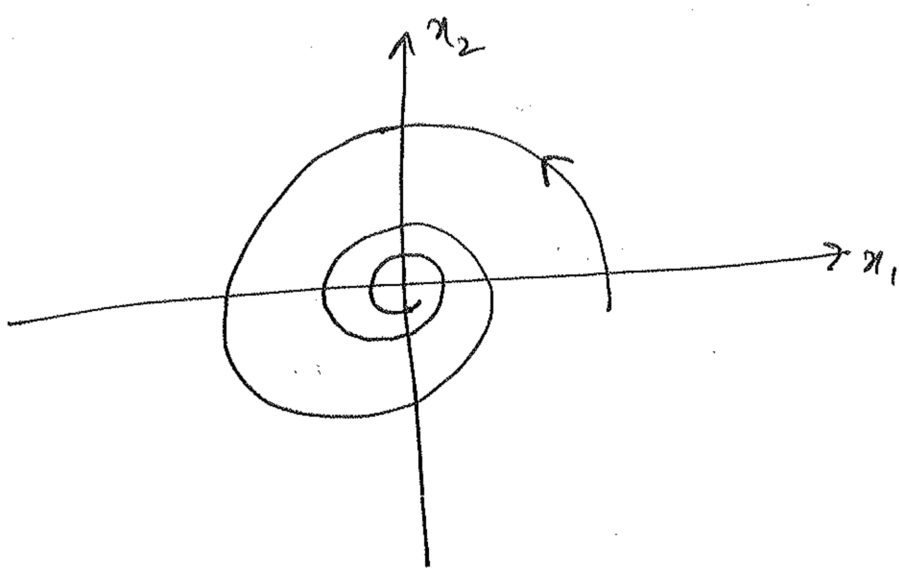


$$x_2' = 5x_1 - 3x_2$$

At $x_1 > 0$ & $x_2 = 0$, $\quad x_2' = 5x_1 > 0$

$\Rightarrow x_2$ is increasing with time

$\Rightarrow$ spiral is counterclockwise.



Section 7.6 #9

$$\vec{X}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \vec{X}, \qquad \vec{X}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Put $\vec{X} = \vec{\xi} e^{\lambda t}$. This gives us $(A - \lambda I)\vec{\xi} = 0$

Thus $|A - \lambda I| = 0 \Rightarrow (1-\lambda)(-3-\lambda) + 5 = 0$

$\Rightarrow \lambda^2 + 2\lambda + 2 = 0$

So $\lambda^2 + 2\lambda + 1 = -1 \Rightarrow (\lambda+1)^2 = -1 \rightarrow \lambda_{\pm} = -1 \pm i$

Eigenvectors:

• Now $(A-\lambda_+ I)\vec{\xi}=0 \longrightarrow \begin{bmatrix}2-i & -5\\ 1 & -2-i\end{bmatrix}\vec{\xi}=\vec{0}$

$$\Rightarrow \begin{cases}(2-i)\,\xi_1 - 5\xi_2 = 0\\ \xi_1-(2+i)\,\xi_2=0 \longrightarrow \text{Multiplying by } (2-i)\end{cases}$$

$$\Rightarrow (2-i)\,\xi_1 - 5\xi_2 = 0$$

Let $\xi_1 = 5$, then $\xi_2 = (2-i)$

$$\therefore \vec{\xi} = \begin{bmatrix}5\\2-i\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}$$: complex eigenvector.

$$\therefore \vec{X}^{(1)} = \left(\begin{bmatrix}5\\2\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}\right) e^{(-1+i)t} \longrightarrow e^{-t}(\cos t + i\sin t)$$

Real part: $$\vec{X}_R = \mathrm{Re}\left\{ e^{-t}\left(\begin{bmatrix}5\\2\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}\right)(\cos t + i\sin t)\right\}$$

$$= e^{-t}\begin{bmatrix}5\\2\end{bmatrix}\cos t + e^{-t}\begin{bmatrix}0\\1\end{bmatrix}\sin t$$

Imaginary part: $$\vec{X}_I = \mathrm{Im}\left\{ e^{-t}\left(\begin{bmatrix}5\\2\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}\right)(\cos t + i\sin t)\right\}$$

$$= e^{-t}\begin{bmatrix}5\\2\end{bmatrix}\cancel{\cos}\,\sin t - \begin{bmatrix}0\\1\end{bmatrix}e^{-t}\cos t$$

$$\therefore \vec{X}_R = e^{-t}\begin{bmatrix}5\cos t\\ 2\cos t + \sin t\end{bmatrix}$$

$$\vec{X}_I = e^{-t}\begin{bmatrix}5\sin t\\ 2\sin t - \cos t\end{bmatrix}$$

General Solution is

$$\vec{X} = c_1 \vec{X}_R + c_2 \vec{X}_I$$

$$= c_1 \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 5\sin t \\ 2\sin t - \cos t \end{bmatrix}$$

Now $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

So $\quad 5c_1 = 1 \longrightarrow c_1 = 1/5$

$\quad 2c_1 - c_2 = 1 \longrightarrow c_2 = -3/5$

$$\therefore \vec{X}(t) = \frac{1}{5} \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} e^{-t} - \frac{3}{5} \begin{bmatrix} 5\sin t \\ 2\sin t - \cos t \end{bmatrix} e^{-t}$$

$$= \begin{bmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{bmatrix} e^{-t}$$

As $t \to \infty$; $\quad \vec{X} \to 0.$