MATH 215/255: Elementary Differential Equations I
Homework - 1, Due: 14-Sep-2012

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Student number:

Note: *Write your answers clearly in the space provided.*

I. Classification

Ordinary Differential Equations (in short, ODE) are equations in which the unknown function (dependent variable), typically, $y(x)$ or $y(t)$, depends on just one independent variable, x or t. These are the kind that were discussed in class. For the following ODE's, determine the order of the equation and state whether the equation is linear or nonlinear. Write your answers in the space below each equation.

Problems 1-5:

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$,

   Order: 1
   Linear/Nonlinear: Nonlinear

2. $x^2 \dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 10x$,

   Order: 2
   Linear/Nonlinear: Linear

3. $a(x)\dfrac{d^2y}{dx^2} + b(x)\dfrac{dy}{dx} + c(x)y = d(x)$,

   Order: 2
   Linear/Nonlinear: Linear

4. $M(x,y)\dfrac{dy}{dx} + N(x,y) = 0$,

   Order: 1
   Linear/Nonlinear: Nonlinear

   If $M(x,y)$ is not a function of y, i.e; it is just $M(x)$, & $N(x,y)$ is linear in y, then we have a linear equation.

5. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - \alpha^2)y = 0, \quad \alpha = \text{constant}$: This is the famous Bessel's equation which appears in many branches of physics.

Order: 2
Linear/Nonlinear: Linear

Problem 6:

Let's now look at a slightly more complicated example. Newton's second law of motion relates the acceleration of particle of mass m to the external force, $\mathbf{F}$ acting on it. Often, the force on the particle depends on the position of the particle. In this case, Newton's second law can be written as

$$m\frac{d^2\mathbf{r}(t)}{dt^2} = \mathbf{F}(\mathbf{r}(t), t),$$

where $\mathbf{r}(t)$ is the vector position of the particle. This is clearly a second order equation. Determine the conditions in which this equation is linear and nonlinear. Write your answers in the space below.

Condition for linear equation: $F(\vec{r}(t), t)$ has to be a linear function of $\vec{r}(t)$

Condition for nonlinear equation: F has to be an nonlinear function of $\vec{r}$

The other class of differential equations are equations where the unknown function can depend on many independent variables. Instead of regular derivatives, we now have to use partial derivatives, hence these equations are called Partial Differential Equations (PDE's). Many real life problems are governed by PDE's. In this course, we'll not be studying them. For the equations below, again determine the order and linearity of the equations.

Problem 7-9:

7. $\dfrac{\partial T}{\partial t} = \alpha\dfrac{\partial^2 T}{\partial x^2}$: This is the Heat equation which describes the distribution of temperature T as a function of time, t, and space, x, and α is the thermal diffusivity (which is a constant here).

Order: 2
Linear/Nonlinear: Linear

8. $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$: This is called the Laplace equation.

Order: 2
Linear/Nonlinear: Linear

9. $\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}=\nu\dfrac{\partial^2 u}{\partial x^2}$: This is the Burger's equation which is used to model Shock Waves.

Order: 2
Linear/Nonlinear: Nonlinear

II. Radioactive decay

Consider a sample of some radioactive material (say Uranium-234) with N_0 being the initial mass of the sample. As radioactive radiation is emitted, the sample decays at a rate proportional to the amount currently present. If $N(t)$ is the size of the sample at any time t, then the above statement can be written mathematically as

$$\frac{d}{dt}N(t) \propto -N(t),$$

where the negative sign indicates that this is a decay process. If λ is the proportionality constant, called the decay constant, this equation becomes

$$\frac{d}{dt}N(t) = -\lambda N(t),$$

10 (i). Solve the above equation and determine the integration constant using the condition $N(t=0)=N_0$.
10 (ii). The time required for the sample size to reduce from N_0 to half its size, i.e., $N_0/2$ is called the half-life of the material, $t_{1/2}$. Obtain an expression for $t_{1/2}$ in terms of the decay rate, λ.
10 (iii). If the initial sample is $N_0 = 1$ gram and it decays to 0.6 grams in 100 seconds, determine the decay rate λ.

10 (i)

$$\frac{dN}{dt} = -\lambda N \qquad N(0) = N_0$$

$$\Big\} \quad \frac{dN}{N} = -\lambda\, dt$$

$$\Rightarrow \ln|N| = -\lambda t + c_1$$

$$\Rightarrow N = e^{-\lambda t + c_1} = e^{-\lambda t} e^{c_1}$$

$$= ce^{-\lambda t}$$

where $c = e^{c_1}$

$$\therefore \boxed{N(t) = ce^{-\lambda t}}$$

$N(0) = N_0 \Rightarrow N_0 = ce^{0} \Rightarrow c = N_0$

$$\therefore \boxed{N(t) = N_0 e^{-\lambda t}}$$

↳ Exponential decay

10) (ii) Given $N(t=0) = N_0$

$N(t = t_{1/2}) = \frac{N_0}{2}$

From the second condition, and using the solution from (i), we have

$$\frac{N_0}{2} = N_0 e^{-\lambda t_{1/2}} \Rightarrow e^{-\lambda t_{1/2}} = \ln(\tfrac{1}{2})$$

$$\Rightarrow -\lambda t_{1/2} = \ln(2^{-1}) \Rightarrow \boxed{t_{1/2} = \frac{\ln 2}{\lambda}}$$

10) (iii) Given that $N_0 = 1$ gm

and $N(t = 100 \text{ sec}) = 0.6$ gm.

Using the solution from (i) we have:

$$N(t=100 \text{ sec}) = N_0 e^{-\lambda \times 100} \Rightarrow 0.6 = 1 e^{-100\lambda}$$

$$\Rightarrow \ln(0.6) = -100\lambda \Rightarrow \lambda = -\frac{\ln(0.6)}{100} \simeq 0.005108 \text{ sec}^{-1}$$

Infact $t_{1/2} = \frac{\ln 2}{\lambda} \approx 135.69$ sec.

$\therefore$ The sample decays to half its size in 135.69 seconds.

III. Population growth

Now let's look at a problem which is inverse of the radioactivity problem. Consider a population, $p(t)$, of some lucky rabbits in a field without any predators. If the rate of growth of the population is proportional to the current population, i.e. $dp/dt = rp$, where r is the growth rate,

11(i). Determine the rate constant r if the population doubles in 30 days.

11(ii). Luck changes to misfortune. Now, the rabbits have contracted a disease and die at a constant rate of k rabbits per day. Now the equation becomes $dp/dt = rp - k$. If the initial population is $p(0) = 50$, growth rate $r = 0.5$ and death rate is $k = 30$, how long will it take for the rabbits to become extinct? (Note that r is now different and is unrelated to part (i)).

11) (i) $\frac{dp}{dt} = rp$

Given that when $t = 30$ days, $p = 2p_0$

Let $p(0) = p_0$

$\Rightarrow p(t=30) = 2p_0$

Solve the equation first:

$$\frac{dp}{p} = r\,dt \Rightarrow \ln|p| = rt + c_1$$

$$\Rightarrow p = ce^{rt} \quad (c = e^{c_1})$$

Now $p(0) = p_0 \Rightarrow p_0 = c$

$\therefore \boxed{p(t) = p_0 e^{rt}}$

Now $p(t=30) = 2p_0$

$$\Rightarrow 2\cancel{p_0} = \cancel{p_0} e^{30r} \Rightarrow e^{30r} = 2$$

$$\Rightarrow 30r = \ln 2 \Rightarrow \boxed{r = \frac{\ln 2}{30} \text{ day}^{-1}}$$

11)(ii) Now $\dfrac{dp}{dt} = rp - K$

$p(0) = 50$

$r = 0.5$

$K = 30$

$\Rightarrow \dfrac{dp}{dt} = \frac{1}{2}p - 30$

$\Rightarrow \dfrac{dp}{dt} = \dfrac{p-60}{2}$

$\Rightarrow 2\,dp = (p-60)\,dt \Rightarrow \dfrac{dp}{p-60} = \frac{1}{2}dt \Rightarrow \ln|p-60| = \frac{t}{2} + C_1$

$\Rightarrow p - 60 = ce^{t/2} \Rightarrow \boxed{p = 60 + ce^{t/2}}$

Now $p(0) = 50 \Rightarrow 50 = 60 + C \Rightarrow c = -10$

$\therefore \boxed{p(t) = 60 - 10e^{t/2}}$

When population becomes extinct, say at $t = t_{ext}$, $p(t) \to 0$

$\therefore p(t = t_{ext}) = 0 \Rightarrow 0 = 60 - 10e^{t_{ext}/2} \Rightarrow \frac{t_{ext}}{2} = \ln 6$

$\therefore t_{ext} = 2\ln 6 \approx 3.58$ days

III. Direction fields

12. Draw the direction field for the following problem and determine the behaviour of y as $t \to \infty$. (Note: y' is same as $\frac{dy}{dx}$).

$$y' = 3 - 2y.$$

A) First, lets determine the equilibrium solution, i.e; when $\frac{dy}{dx} = 0$.

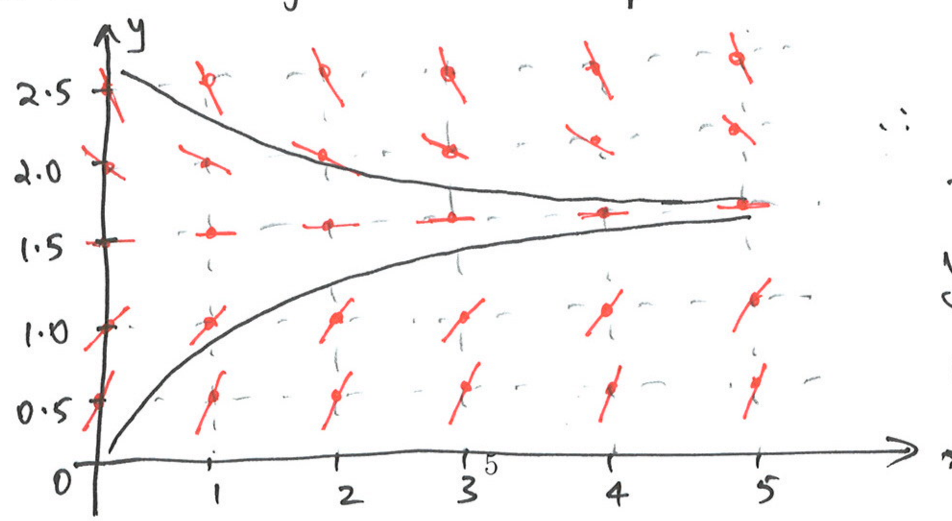
This happens when $3 - 2y = 0 \Rightarrow \boxed{y_{eq} = 1.5.}$

Now construct a grid with y_{eq} somewhere in the middle.

At $y = 1.5$; $y' = 0$

At $y = 1$; $y' = 3 - 2 = 1$

At $y = 2$; $y' = -1$



$\therefore$ As $t \to \infty$; $y \to 1.5$ for all initial conditions at $x = 0$.

13. Draw the direction field for the following problem. In this case, the solution as $t \to \infty$ will depend on the where you start at $t = 0$. Explain this dependence in words.

$$y' = y(y-3).$$

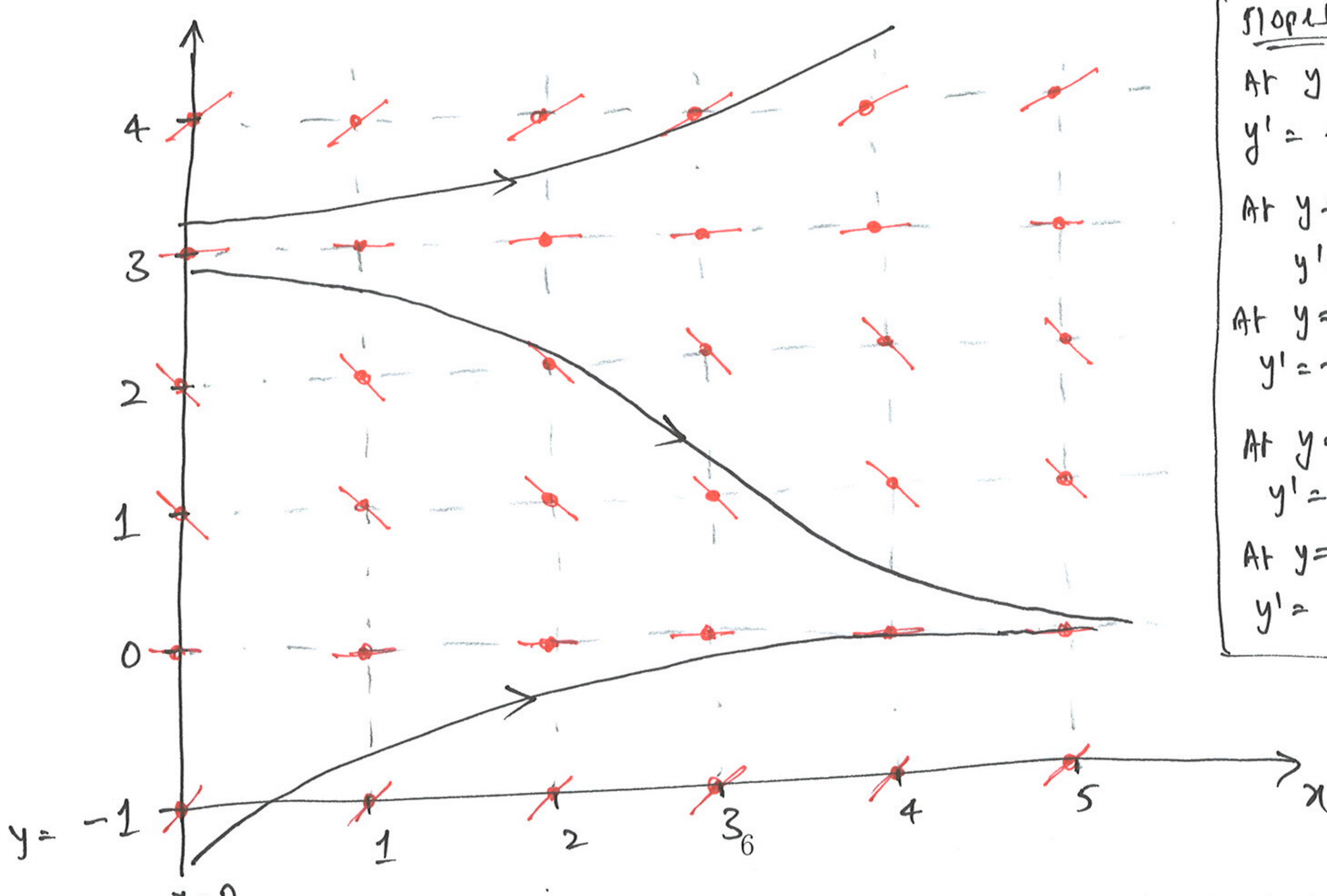
A) We first determine the equilibrium solutions.

$y'_{eq} = 0 \rightarrow y(y-3) = 0$

$\Rightarrow y_{eq} = 0$ and $y_{eq} = 3$

Now we can draw a grid with some region on either side of the equilibrium.

Let $y \in [-1, 4]$ and $x \in [0, 5]$



Slopes :

At $y = -1$; $y' = 4 > 0$

At $y = 0$; $y' = 0$

At $y = 2$, $y' = -2 < 0$

At $y = 3$; $y' = 0$

At $y = 4$, $y' = 4 > 0$

If $y(x=0) < 3$; $y(x \to \infty) \to 0$. ~~Else $y \to \infty$~~ If $y_0 > 3$; $y \to \infty$. If $y_0 = 3$; $y = 3$ always.