Separable Equations

We have already solved equations of the form

$$(*) \begin{cases} \dfrac{dy}{dx} + p(x)\,y = q(x) & \longrightarrow p \text{ \& } q \text{ are functions of } x \\ \text{and} & \\ \dfrac{dy}{dx} = ay + b & \longrightarrow a \text{ \& } b \text{ are constants.} \end{cases}$$

In general, the above equations can be written in the form

$$\frac{dy}{dx} = f(x,y) \quad \text{——} \quad (**)$$

To integrate equations of the type $(*)$, we used either a direct integration, or used integrating factors. If we can separate the x & y parts, we can use direct integration. Consider the equation in the form

$$\boxed{M(x) + N(y)\frac{dy}{dx} = 0}$$

This is a Separable Equation

We can write the above equation as

$$N(y)\,dy + M(x)\,dx = 0$$

or $\underbrace{N(y)\,dy}_{\text{contains only } y} = \underbrace{-M(x)\,dx}_{\text{contains only } x}$

Now we can integrate the two sides and find the solution.

Ex: $y' + y^2 \sin x = 0$

This is a first order non-linear equation.
Cannot use integrating factors.

$$\frac{dy}{dx} + y^2 \sin x = 0$$

$$\Rightarrow \frac{dy}{y^2} + \sin x \, dx = 0$$

$$\Rightarrow N(y)\,dy + M(x)\,dx = 0$$

$N(y) = \frac{1}{y^2}$

$M(x) = \sin x$

$$\Rightarrow \frac{dy}{y^2} = -\sin x \, dx$$

Integrate both sides

$$\int \frac{dy}{y^2} = -\int \sin x \, dx$$

$$\Rightarrow -\frac{1}{y} = -\cos x + C$$

$$\Rightarrow \frac{1}{y} = \cos x - C \Rightarrow \boxed{y = \frac{1}{\cos x - C}}$$

The other solution is $y=0$ everywhere.

When we have an equation of the form

$$\frac{dy}{dx} = f(x,y), \quad \text{~~sometimes~~}$$

we solve for y as a function of x.

But sometimes y does not have a solution for all x. This interval in which the solution exists is sometimes referred to as the Domain of Existence.

Problem 14

(Q) Solve $\quad y' = \dfrac{xy^3}{\sqrt{1+x^2}} \quad ; \quad y(0) = 1$

and determine the interval in which the solution exists.

A)
$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

$$\Rightarrow \underbrace{\frac{dy}{y^3}}_{\text{function of } y \text{ alone}} = \underbrace{\frac{x}{\sqrt{1+x^2}}\,dx}_{\text{function of } x \text{ alone}}$$

$$\int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}}\,dx \quad \Rightarrow \int \frac{dy}{y^3} = \int \frac{\frac{1}{2}\,d(1+x^2)}{(1+x^2)^{1/2}}$$

$$\Rightarrow \quad \frac{-1}{2}y^{-2} = \frac{1}{2}\,\frac{(1+x^2)^{\frac{1}{2}}}{1/2} + C$$

$$\Rightarrow \quad \frac{-1}{2y^2} = (1+x^2)^{1/2} + C$$

Now $\quad y(0) = 1 \Rightarrow \quad -\frac{1}{2} = 1 + C \Rightarrow C = \frac{-3}{2}$

$$\Rightarrow \quad \frac{-1}{2y^2} = (1+x^2)^{1/2} - \frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{y^2} = 3 - 2(1+x^2)^{1/2}$$

$$\Rightarrow \quad \boxed{y = \frac{1}{\left[3 - 2(1+x^2)^{1/2}\right]^{1/2}}}$$

y is not defined when the term inside the $\sqrt{\ }$ in the denominator becomes negative.

Therefore, we require $\quad 3 - 2(1+x^2)^{1/2} > 0$

$$\Rightarrow \quad 3 > 2(1+x^2)^{1/2}$$

$$\Rightarrow \quad 4(1+x^2) < 9$$

$$\Rightarrow \quad 1+x^2 < \frac{9}{4} \Rightarrow x^2 < \frac{5}{2}$$

$$\Rightarrow \quad \boxed{|x| < \frac{\sqrt{5}}{2}}$$

$\therefore$ Solution is defined for $\quad -\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$.

Outside this interval, we have no solution.

Can we transform a complicated equation into a form which we know how to solve?

(Section 2.2
Problem. 35)

Ex!

$$\frac{dy}{dx} = \frac{x+3y}{x-y}$$

We can't integrate this equation with any of our earlier methods. The x & y are mixed up.

Let's divide the right-hand side by x in the numerator & denominator.

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+3\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)}$$

~~Now, the equation of~~

Now the right side is a function of the ratio. $\left(\frac{y}{x}\right)$. Let us define this as a new variable.

Let $\quad v = \frac{y}{x}$

$$\Rightarrow \quad y = vx$$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{(Product rule)}$$

Substituting $\frac{dy}{dx}$ back into the diff. equation, we have

$$v + x\frac{dv}{dx} = \frac{1+3v}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+3v}{1-v} - v = \frac{1+3v-v+v^2}{1-v}$$

$$= \frac{v^2+2v+1}{1-v}$$

$$\Rightarrow \quad \underbrace{dv\,\frac{1-v}{(1+v)^2}}_{\text{Function of } v \text{ alone}} = \underbrace{\frac{dx}{x}}_{\text{Function of } x \text{ alone.}}$$

$$\int \frac{1-v}{(1+v)^2}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{-2}{1+v} - \ln(1+v) + C = \ln x$$

$$\Rightarrow \quad \frac{-2}{1+\left(\frac{y}{x}\right)} - \ln\left(1+\frac{y}{x}\right) + C = \ln x$$

$$\Rightarrow \quad \frac{-2x}{x+y} - \ln\left(\frac{x+y}{x}\right) + C = \ln x$$

$$\Rightarrow \quad \boxed{\frac{-2x}{x+y} - \ln(x+y) + C = 0}$$

In this case, we cannot express y as an explicit function of x. This can happen sometimes.

<u>Ex:</u> (Problem 33, Page 50) Solve $\dfrac{dy}{dx} = \dfrac{4y-3x}{2x-y}$

Section 2.6 Exact Equations

Consider a first order equation which is neither separable nor linear. Can we solve it?

Def: Consider an equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$ where

M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous. This equation is said to be EXACT if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

We can solve an Exact equation.

Key Point: The important point in solving an exact equation is to find a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and}$$

$$\frac{\partial \psi}{\partial y} = N(x,y).$$ Then the solution would be

$$\underline{\psi(x,y) = \text{constant}}$$

Sketchy Proof:

Let there be a function such that

$$\left.\begin{aligned} \cancel{\frac{\partial M(x,y)}{\partial x}}\ \frac{\partial \psi(x,y)}{\partial x} &= M(x,y) \\ \text{and} \quad \frac{\partial \psi(x,y)}{\partial y} &= N(x,y) \end{aligned}\right\} \star$$

Consider the derivative of $\psi(x,y)$ with respect to x. from calculus of multivariables (chain rule), we have

$$\frac{d}{dx}\psi(x, y(x)) = \frac{\partial \psi}{\partial x}(x, y(x)) + \frac{\partial \psi}{\partial y}(x, y(x))\frac{dy}{dx}$$

$$\star \Rightarrow \frac{d}{dx}\psi(x, y(x)) = \underbrace{M(x,y) + N(x,y)\frac{dy}{dx}}_{=0}$$

But the right side is the differential equation & is equal to zero.

$$\Rightarrow \frac{d}{dx}\psi(x, y(x)) = 0$$

$$\Rightarrow \boxed{\psi(x, y(x)) = \text{Constant}} \text{ is}$$

the solution.

The condition $\star$ is equivalent to:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This is because

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)$$

LHS RHS

$$\frac{\partial^2 \psi}{\partial x \partial y} \qquad \frac{\partial^2 \psi}{\partial x \partial y}$$

LHS = RHS since $\psi(x,y)$ is a continuous function of x & y.

Ex: (Problem 14, Page 100) Solve

$$(9x^2 + y - 1)\,dx - (4y - x)\,dy = 0$$

$$y(1) = 0$$

(A) Write the equation into standard form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Here
$$M(x,y) = 9x^2 + y - 1$$
$$N(x,y) = -4y + x$$

$$\frac{\partial M}{\partial y} = 1 \quad ; \quad \frac{\partial N}{\partial x} = 1$$

Therefore $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

$\Rightarrow$ The equation is exact.

Now we need to find a function ψ such that $\dfrac{\partial \psi}{\partial x} = M$ and $\dfrac{\partial \psi}{\partial y} = N$.

Consider $\dfrac{\partial \psi}{\partial x} = M$

$$\Rightarrow \frac{\partial \psi}{\partial x} = 9x^2 + y - 1$$

Integration, we have

$$\psi = \frac{9x^3}{3} + xy - x + h(y)$$

$$= 3x^3 + xy - x + h(y)$$

NOTE! We integrated a partial derivative here treating y as a constant. Therefore the constant of integration should be a function of y.

Consider the second condition ; $\dfrac{\partial \psi}{\partial y} = N$

$$\therefore \frac{\partial}{\partial y}\left(3x^3 + xy - x + h(y)\right) = -4y + x$$

$$\Rightarrow \cancel{x} + \frac{dh}{dy} = -4y + \cancel{x}$$

$$\Rightarrow \frac{dh}{dy} = -4y \Rightarrow h(y) = -2y^2 + C$$

A constant (not a function of x or y)

Also note that since h is a function of y, the right side of the above equation should not have x.

Substituting $h(y)$ back into Ψ, we have

$$\boxed{\Psi(x,y) = 3x^3 + xy - x - 2y^2 + C}$$

Eq

Generate solution: $\Psi(x,y) = \text{constant}$

$$\Rightarrow \boxed{3x^3 + xy - x - 2y^2 = C_2}$$

where C_2 is an unknown constant.

Initial condition:- Now $y(1) = 0$

$$\Rightarrow 3 - 1 = C_2 \Rightarrow C_2 = 2$$