Second Order Linear Equations

Complex Roots of Characteristic Equation

Consider the equation

$$ay'' + by' + cy = 0$$

where a, b, c are given real numbers.

Let's try $y = e^{rt}$

$\Rightarrow$ we get $\boxed{ar^2 + br + c = 0}$

$\quad$ ⌊ characteristic equation.

Roots: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

We now consider the case $b^2 - 4ac < 0$

The two roots r_1 & r_2 will be complex conjugates.

Let $r_1 = \lambda + i\mu$; $\mu \neq 0$

$r_2 = \lambda - i\mu$

$\therefore \quad y_1(t) = e^{(\lambda + i\mu)t}$; $y_2(t) = e^{(\lambda - i\mu)t}$

But what does $e^{(\lambda + i\mu)t}$ mean?

$\quad$ ⌊ exponential of a complex number.

$$e^{(\lambda+i\mu)t} = e^{\lambda t} \cdot e^{i\mu t}$$

We need to determine $e^{i\mu t}$

Recall that $i^2 = -1$; $i^3 = -i$; $i^4 = 1$, so on.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ;$$

$$\sin x = x - \frac{x^3}{3!} + \cdots ;$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\therefore e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \cdots\right)$$

$$= \cos x + i \sin x$$

$$\therefore \boxed{e^{ix} = \cos x + i \sin x}$$

This is called the Euler's formula.

Replacing x by μt, we have

$$e^{i\mu t} = \cos(\mu t) + i \sin(\mu t)$$

Therefore $e^{(\lambda + i\mu)t} = e^{\lambda t}\left(\cos \mu t + i \sin(\mu t)\right)$

Now, our two solutions become

$$y_1 = e^{\lambda t}(\cos \mu t + i \sin \mu t)$$
$$y_2 = e^{\lambda t}(\cos \mu t - i \sin \mu t)$$

These are complex valued functions. It is easier to deal with real valued functions. Using the "Principle of Superposition", we know that $(y_1 + y_2)$ and $(y_1 - y_2)$ will also be solutions.

Therefore, $y_1(t) + y_2(t) = 2e^{\lambda t}\cos(\mu t) = y_I$

and $y_1(t) - y_2(t) = 2i\, e^{\lambda t} \sin(\mu t) = y_{II}$

General solution: $y(t) = c_1[y_1 + y_2] + c_2[y_1 - y_2]$

$$= c_1 \cdot 2e^{\lambda t}\cos \mu t + c_2 \cdot 2i e^{\lambda t} \sin \mu t.$$

Defining the constants $A = 2c_1$, $B = 2i\, c_2$, we have

$$\boxed{y(t) = e^{\lambda t}(A \cos \mu t + B \sin \mu t)}$$: Real valued function.

A, B can be found from the initial conditions.

Are the solutions y_I & y_{II} fundamental solutions?

$$W = \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \frac{d}{dt}\left(e^{\lambda t}\cos\mu t\right) & \frac{d}{dt}\left(e^{\lambda t}\sin\mu t\right) \end{vmatrix}$$

Recall,

$$W = \begin{vmatrix} y_I & y_{II} \\ y_I' & y_{II}' \end{vmatrix}$$

$$= e^{\lambda t}\cos(\mu t)\left\{\lambda e^{\lambda t}\sin\mu t + e^{\lambda t}\cdot\mu\cos\mu t\right\} - e^{\lambda t}\sin\mu t\left\{\lambda e^{\lambda t}\cos\mu t - e^{\lambda t}\cdot\mu\sin\mu t\right\}$$

$$= e^{2\lambda t}\cdot\mu\cos^2(\mu t) + \mu\cdot e^{\lambda t}\sin^2(\mu t)$$

$$= \mu e^{2\lambda t}$$

Since $\mu \neq 0$; $W \neq 0$

$\Rightarrow$ y_I and y_{II} ~~are~~ indeed form a fundamental set.

Consider the IVP

Ex: ~~Let~~ $y'' + 9y = 0$

$$y(0) = 1$$
$$y'(0) = 3$$

Let $y = e^{rt} \Rightarrow r^2 + 9 = 0$

$\Rightarrow r = \pm 3i$

Now $\lambda = 0$; $\mu = 3$

Complex valued solutions : $y_1 = e^{i3t}$; $y_2 = e^{-i3t}$

Using the Euler formula,

$$y_1 = \cos(3t) + i\sin(3t)$$
$$y_2 = \cos(3t) - i\sin(3t)$$

$$y_1(t) + y_2(t) = 2\cos(3t)$$
$$y_1(t) - y_2(t) = 2i\sin(3t)$$

$\therefore$ $y_I = \cos(3t)$

$y_{II} = \sin(3t)$

$\therefore$ General solution (real valued function) :

$$y(t) = A\cos(3t) + B\sin(3t)$$

$$W[y_I, y_{II}](t) = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3$$

$\therefore$ y_I and y_{II} are fundamental solutions.

$\therefore$ $y(t) = A\cos(3t) + B\sin(3t)$

$y'(t) = -3A\sin(3t) + 3B\cos(3t)$

Now, $y(0) = 1 \Rightarrow 1 = A + B\times 0 \Rightarrow A = 1$

$y'(0) = 3 \Rightarrow 3 = -3A\times 0 + 3B \Rightarrow B = 1$

$\therefore$ Exact solution : $\boxed{y(t) = \cos(3t) + \sin(3t)}$

Notice that the real part of the eigenvalue (root) is equal to zero, ie; $\lambda = 0$. Therefore, the general solution does not have an exponential term.