Second Order Linear Equations

A general second order linear equation can be written in the form

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = g(t)$$

where $y' = \frac{dy}{dt}$

$y'' = \frac{d^2y}{dt^2}$

If $g(t) = 0$, the equation is said to be Homogeneous.

If $g(t) \neq 0$, the equation is said to be Nonhomogeneous.

↳ Nonhomogeneous term

Constant Coefficients: Let's start with the simpler case first.

Assume $p(t) = a$, $q(t) = b$, $r(t) = c$.

Equation becomes $ay'' + by' + cy = g(t)$

In this course, we are only concerned with constant coefficient case.

Overview of this chapter:- Sections 3.1 - 3.4 : Equations of the form $ay'' + by' + cy = 0$ (Homogeneous)

Sections 3.5 - 3.6 : Equations of the form $ay'' + by' + cy = g(t)$ (Nonhomogeneous)

Section 3.7 : Applications

also satisfies the equation.

We will show later that this is indeed our general solution. To summarise, the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$
$$= c_1 e^{t} + c_2 e^{-t}$$

How to find c_1 & c_2 ?

We need two initial conditions.

$$y'(t) = c_1 e^{t} - c_2 e^{-t}$$

Let $y(0) = 1$

$y'(0) = 2$

$$\Rightarrow \quad 1 = c_1 + c_2$$
$$2 = c_1 - c_2$$

$$\Rightarrow \quad c_1 = \frac{3}{2} \quad ; \quad c_2 = \frac{-1}{2}$$

$$\therefore \quad y(t) = \frac{3}{2} e^{t} - \frac{1}{2} e^{-t}$$

Can we extend what we learnt above to a general linear equation of the form

$$ay'' + by' + cy = 0$$

, $a \neq 0$ and a, b, c are given constants. (real)

here try an exponential type solution again.

(Problem #9, pg. 344)

Ex:

$$y'' + y' - 2y = 0,$$

$$y(0) = 1 \quad ; \quad y'(0) = 1$$

Let $y = e^{rt}$. $\Rightarrow$ $r^2 e^{rt} + re^{rt} - 2e^{rt} = 0$

Characteristic equation: $r^2 + r - 2 = 0$

$$\Rightarrow (r+2)(r-1) = 0$$

$$\Rightarrow r_1 = -2 \; ; \; r_2 = 1$$

$$\therefore y(t) = c_1 e^{-2t} + c_2 e^{t}$$

Find c_1 & c_2 using the initial conditions:-

Section 3.2 Existence and Uniqueness Theorem

Just like in the first order equations, we have an Existence and Uniqueness theorem for linear second order equations.

Theorem: Consider the IVP (Initial Value Problem)

$$y'' + p(t)\, y' + q(t)\, y = g(t)$$
$$y(t_0) = y_0 \;; \quad y'(t_0) = y_0' .$$

If p, q, g are ~~const~~ continuous on an interval containing the initial condition, then there exists a solution to the IVP and it is unique in this interval.

The above theorem only ~~guarantees~~ gives the smallest interval of existence. Actual interval of existence may be larger.

Ex: $ty'' + 3y = t$; $y(1) = 1$; $y'(1) = 2$

Equation can be written as $y'' + \frac{3}{t} y = 1$

$$p(t) = 0 \;; \quad q(t) = \frac{3}{t} \;; \quad g(t) = 1$$

The only point of discontinuity of the coefficients are at $t=0$. Since the initial condition is at $t=1$,

Operator & L :

As a short hand notation, the differential equation can also be described in terms of a differential operator, L, defined as

$$L[y] = y'' + p(t)y' + q(t)y$$

$\therefore$ The equation is $L[y] = 0$. If $y_1(t)$ and $y_2(t)$ are solutions, then $L[y_1] = 0$ and $L[y_2] = 0$. The superposition principle can be written as

$$L[c_1 y_1 + c_2 y_2] = 0$$

where c_1 and c_2 are arbitrary constants.

WRONSKIAN :- In order to call

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

as the general solution, we need to verify that all solutions of the differential equation are contained in this solution. Indeed $y(t) = c_1 y_1 + c_2 y_2$ represents an infinite family of solutions. The constants c_1 and c_2 should be determined from the initial conditions.

If $W(y_1, y_2)(t_0) = 0$, then c_1 and c_2 cannot be found. This gives us the next theorem.

__Theorem:__ If $y_1(t)$ and $y_2(t)$ are solutions of

$$L[y] = y'' + p(t)\, y' + q(t)\, y = 0, \quad \text{with}$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0',$$

then it is always possible to determine the constants c_1 and c_2 such that

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and the initial conditions if and only if

$$W[y_1, y_2](t_0) = y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0) \neq 0.$$

__Theorem:__ __General Solution.__

If $W(y_1, y_2) \neq 0$ for some t, then the linear combination $y(t) = c_1 y_1(t) + c_2 y_2(t)$ gives us all the solutions of $L[y] = 0$.

In summary, $y(t) = c_1 y_1 + c_2 y_2$ contains "all" possible solutions if and only if $W(y_1, y_2) \neq 0$. If $W(y_1, y_2) = 0$ everywhere, then the above linear combination does not contain all the solutions.

Abel's Theorem If y_1 & y_2 are solutions of

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0,$$

with p & q continuous on an interval $I : (\alpha, \beta)$, then

$$W(y_1, y_2)(t) = c \exp\left\{ -\int p(t)\,dt \right\},$$

where c depends on y_1 & y_2.

What does this theorem tell us?

Since $\exp\{\ \} \neq 0$, $W = 0$ everywhere on I with $c = 0$ or $W \neq 0$ everywhere on I.

Proof: Since y_1 & y_2 are solutions, we have

$$y_1'' + p(t)\,y_1' + q(t)\,y_1 = 0;$$

$$y_2'' + p(t)\,y_2' + q(t)\,y_2 = 0.$$

Eliminating $q(t)$, we have

$$(y_1 y_2'' - y_2 y_1'') + p(t)\,(y_1 y_2' - y_1' y_2) = 0$$

Since $W = y_1 y_2' - y_2 y_1'$, you can check that

$$W' = y_1 y_2'' - y_2 y_1''.$$

$\therefore$ We have $W' + p(t)\,W = 0 \implies \boxed{W = c \exp\left\{ -\int p(t)\,dt \right\}}$